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TOTAL DOMINATION IN CATEGORICAL PRODUCTS OF GRAPHS

Douglas F. Rall

Department of Mathematics
Furman University
Greenville, South Carolina 29613, USA

e-mail: drall@herky.furman.edu

Abstract

Several of the best known problems and conjectures in graph theory arise in studying the behavior of a graphical invariant on a graph product. Examples of this are Vizing's conjecture, Hedetniemi's conjecture and the calculation of the Shannon capacity of graphs, where the invariants are the domination number, the chromatic number and the independence number on the Cartesian, categorical and strong product, respectively. In this paper we begin an investigation of the total domination number on the categorical product of graphs. In particular, we show that the total domination number of the categorical product of a nontrivial tree and any graph without isolated vertices is equal to the product of their total domination numbers. In the process we establish a packing and covering equality for trees analogous to the well-known result of Meir and Moon. Specifically, we prove equality between the total domination number and the open packing number of any tree of order at least two.

Keywords: categorical product, open packing, total domination, submultiplicative, supermultiplicative.

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1. Introduction

For notation and graph theory terminology we follow [3] and for graph products we refer the reader to [8]. Specifically, let $G = (V, E)$ be a finite, simple undirected graph. The *open neighborhood* of a vertex v is $N_G(v) = \{ u \in V \mid uv \in E \}$, while its *closed neighborhood* is $N_G[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. If the graph is clear from the context, then we omit the subscript on these neighborhood names. For a set $S \subseteq V$, the subgraph of G induced by S is denoted by $\langle S \rangle_G$ or simply $\langle S \rangle$. A family $\{S_k\}_{k \in I}$ of subsets of vertices in G is a *cover* of (or *covers*) G if $V(G) = \cup_{k \in I} S_k$. The family is a *packing* if the subsets are pairwise disjoint. The maximum order of a complete subgraph is denoted $\omega(G)$, and the *vertex independence number* of G, $\beta_0(G)$, is the cardinality of a largest independent set of vertices in G.

A subset D of vertices is a *dominating set* of G if every vertex x in V either belongs to D or is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Hence, $\gamma(G)$ is the minimum cardinality of a set D of vertices such that the family $\{N[x]\}_{x \in D}$ of closed neighborhoods covers G. The set D is a *total dominating set* of the graph G if every vertex x in V is adjacent to a vertex of D. Equivalently, a set D is a total dominating set if the collection of open neighborhoods $\{N(x)\}_{x \in D}$ covers G. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. If G has minimum degree at least one, then G has a total dominating set and it is clear that $\gamma(G) \le \gamma_t(G)$. Unless specifically stated otherwise, all graphs considered in this paper are assumed not to have isolated vertices, and hence the total domination number is defined. The set $A \subseteq V(G)$ is called a 2-*packing* if the family of closed neighborhoods $\{N[u]\}_{u \in A}$ is a packing. The 2-*packing number* $\rho(G)$ is the maximum cardinality of a 2-packing. For any graph G, $\rho(G) \le \gamma(G)$ since every dominating set of G must intersect each closed neighborhood. In 1975 Meir and Moon established the following result that in the language of packings and coverings says the minimum number of closed neighborhoods in a covering of a tree is the same as the maximum number of closed neighborhoods in a packing of that tree. This result has proved to be useful in studying domination — especially in Cartesian products.

Theorem 1 ([10]). *For any tree T, $\gamma(T) = \rho(T)$.*

A subset A of vertices in G is called an *open packing* if $\{N_G(x)\}_{x \in A}$ is a packing. As in [3], the maximum cardinality, $\rho^o(G)$, of an open packing in G is the *open packing number* of G. It can easily be verified that S is an open packing if and only if $|N(x) \cap S| \leq 1$ for every vertex $x \in V(G)$. The open packing number was studied in [5] and [7]. In this paper we establish a result similar to Theorem 1 for total domination and open packings and use this to investigate total domination in categorical products.

2. Graphical Invariants on Graph Products

Let G and H be finite, simple graphs. By a graph product $G \otimes H$ with G and H as factors we mean the graph that has vertex set $V(G \otimes H) = V(G) \times V(H)$ (the Cartesian product of the vertex sets of G and H) and edge set that is determined entirely by the adjacency relations of G and H. See [11]. For a given graphical invariant σ and given graph product $\otimes$ it is natural to investigate the behavior of σ on $\otimes$. It is often the case that the value $\sigma(G \otimes H)$ depends directly on the two values $\sigma(G)$ and $\sigma(H)$ for all pairs of graphs G and H. We say that σ is *supermultiplicative* (respectively, *submultiplicative*) on $\otimes$ if $\sigma(G \otimes H) \geq \sigma(G)\sigma(H)$ (respectively, $\sigma(G \otimes H) \leq \sigma(G)\sigma(H)$) for all pairs G and H. A class of graphs $\mathcal{C}$ is called a *universal multiplicative class for* σ *on* $\otimes$ if for every graph H it follows that $\sigma(G \otimes H) = \sigma(G)\sigma(H)$ whenever G is from the class $\mathcal{C}$. (We only require the given inequality or equality to hold for those graphs for which the invariant is defined.)

Nowakowski and Rall studied the "multiplicative" behavior of twelve graphical invariants related to domination and coloring on nine of the ten associative graph products whose edge sets depend on the adjacency relation of both factors. In [11] they cited many of the known multiplicative relationships and established several new ones. Perhaps the most well-known, outstanding problem in this area is the nearly four-decade-old conjecture of Vizing, stated here in multiplicative language.

Conjecture 2 (V.G. Vizing)**.** *The domination number γ is supermultiplicative on the Cartesian graph product.*

We are concerned primarily with the *categorical* product. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in the categorical product $G \times H$ if and only if u_1u_2 is an edge of G and v_1v_2 is an edge of H. For $u \in V(G)$ we let H_u denote the set of vertices $\{(u, v) \mid v \in V(H)\}$ in the product graph $G \otimes H$. The set G_v is defined similarly. Note that H_u and G_v are independent sets in $G \times H$.

Nowakowski and Rall observed that if $\otimes$ is any graph product such that the categorial product $G \times H$ is a spanning subgraph of $G \otimes H$ for all graphs G and H, then the (set) Cartesian product of total dominating sets of G and H is a total dominating set of $G \otimes H$. An immediate consequence of this is the following result that established that γ_t is submultiplicative on the categorical product.

Theorem 3 [11]**.** *If G and H have no isolated vertices, then $\gamma_t(G \times H) \leq \gamma_t(G)\gamma_t(H)$.*

The total domination number of the categorical product can be strictly smaller than the product of the total domination numbers. The smallest such example is $\gamma_t(K_3 \times K_3) = 3 < \gamma_t(K_3)\gamma_t(K_3)$. The existence of a nontrivial universal multiplicative class for an invariant on a product is unusual. In [2] the authors show that if G is either a tree having a vertex adjacent to at least two leaves or a path of order a multiple of three, then $\gamma(G \square H) > \gamma(G)\gamma(H)$ for any graph H of order at least two. An immediate consequence is that the only universal multiplicative class for γ on the Cartesian product is the trivial class $\mathcal{C} = \{K_1\}$. In this paper we use open packings to establish a lower bound for $\gamma_t(G \times H)$. In doing so we show that the class of nontrivial trees is a universal multiplicative class for γ_t on $\times$. That is, we shall prove the following theorem.

Theorem 4. *If T is any tree of order at least two and H is a graph without isolated vertices, then $\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H)$.*

3. Total Domination and Open Packings

Any dominating set in a graph must have a nonempty intersection with every closed neighborhood. Total domination is defined in terms of open neighborhoods. Hence we have the following result.

Lemma 5. *If G has no isolated vertices, then $\gamma_t(G) \geq \rho^o(G)$.*

Proof. Let B be an open packing of cardinality $\rho^{\mathrm{o}}(G)$ in G, and let D be a minimum total dominating set of G. Then, D contains at least one vertex from every open neighborhood. Since there are $|B|$ pairwise disjoint open neighborhoods, it follows that $\gamma_t(G) = |D| \geq |B| = \rho^{\mathrm{o}}(G)$. ■

Fix a vertex v in H and let D be a total dominating set of the graph $G \times H$ of minimum cardinality. For $u \in V(G)$ the open neighborhood of (u,v) in $G \times H$ is given by $N((u,v)) = N_G(u) \times N_H(v)$. Then, D must contain a vertex (x,y) adjacent to (u,v) and $(x,y) \in \cup_{w \in N(v)} G_w$. Let $D_v = \{x \mid (x,y) \in D \cap (\cup_{w\in N(v)} G_w)\}$. It follows that D_v is a total dominating set of G. Let S be an open packing in H such that $|S| = \rho^{\mathrm{o}}(H)$. Then,

$$D \supseteq \bigcup_{v \in S} \left[D \cap \left(\bigcup_{w \in N(v)} G_w \right) \right],$$

and so

$$\gamma_t(G \times H) = |D| \geq \sum_{v \in S} \left| D \cap \left(\bigcup_{w \in N(v)} G_w \right) \right| \geq \sum_{v \in S} |D_v| \geq \rho^{\mathrm{o}}(H)\gamma_t(G).$$

Interchanging the roles of G and H yields the following result.

Lemma 6. *For any graphs G and H with no isolated vertices,*

$$\gamma_t(G \times H) \geq \max\{\rho^{\mathrm{o}}(G)\gamma_t(H), \rho^{\mathrm{o}}(H)\gamma_t(G)\}.$$

By combining Lemma 6 with Theorem 3 we obtain the following result.

Theorem 7. *Let G be a graph such that $\gamma_t(G) = \rho^{\mathrm{o}}(G)$. For any graph H that has no isolated vertices, $\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$.*

For example, for any pair of positive integers m and n, $\gamma_t(K_{m,n}) = 2 = \rho^{\mathrm{o}}(K_{m,n})$. To construct an infinite class of graphs satisfying the condition $\gamma_t = \rho^{\mathrm{o}}$ in Theorem 7, one can start with any connected graph F and attach at least one vertex of degree one (a *leaf*) to each vertex of F. In the resulting graph G, the set $V(F)$ is a minimum total dominating set. Any subset of vertices consisting of precisely one leaf from the neighborhood of each vertex of F is an open packing, and so it is clear that $\gamma_t(G) = \rho^{\mathrm{o}}(G)$. Hence we have proved the following result.

Theorem 8. *The class of connected graphs for which each vertex is either a leaf or is adjacent to at least one leaf is a universal multiplicative class for* γ_t *on* $\times$.

For any graph G without isolated vertices let $N_o(G)$ be the *open neighborhood intersection graph* of G. That is, $N_o(G)$ is the graph with the same vertex set as G such that distinct vertices x and y are adjacent in $N_o(G)$ if and only if $N_G(x) \cap N_G(y) \neq \emptyset$. Independent sets and complete subgraphs in $N_o(G)$ are related to total domination of G as the following proposition shows. Part (ii) of Proposition 9 is a special case of a result of Acharya [1]. We prove the weaker form here for the sake of completeness.

Proposition 9. *Let G be any graph without isolated vertices.*

(i) $\rho^o(G) = \beta_0(N_o(G))$.

(ii) (*Acharya* [1]) *For any complete subgraph M of $N_o(G)$ there exists a vertex a of G such that $M \subseteq N_G(a)$ if G has girth at least* 7.

(iii) *If G has girth at least* 7, *then $\gamma_t(G)$ is the minimum number of complete subgraphs of $N_o(G)$ that cover $N_o(G)$.*

Proof. By the definition of $N_o(G)$ it follows immediately that a subset A of $V(G)$ is an open packing of G exactly when A is independent in $N_o(G)$. Consequently, the open packing number of G is equal to $\beta_0(N_o(G))$, the vertex independence number of $N_o(G)$ and (i) follows. Assume now that G has no cycles of order less than 7 and let M be a complete subgraph of $N_o(G)$. If $|M| = 2$, say $M = \{v_1, v_2\}$, then the conclusion follows directly from the definition of adjacency in $N_o(G)$. Assume then that $M = \{v_1, v_2, \ldots, v_r\}$ for $r \geq 3$. Let i, j and k be three distinct indices from $\{1, 2, \ldots, r\}$. Since M is complete there exist vertices $a, b, c \in V(G)$ such that $a \in N_G(v_i) \cap N_G(v_j)$, $b \in N_G(v_j) \cap N_G(v_k)$, and $c \in N_G(v_k) \cap N_G(v_i)$. Assume $a \neq b$. Then G has the path v_i, a, v_j, b, v_k. If $c \neq a$ and $c \neq b$, then G contains the cycle $v_i, a, v_j, b, v_k, c, v_i$ contradicting the assumption that G has girth at least 7. Therefore, $c = a$ or $c = b$. In both cases G contains a 4-cycle, contradicting the girth condition. Hence, $a = b$. Since i and j were arbitrary, every vertex in M is adjacent in G to a, and as a result $M \subseteq N_G(a)$. Hence (ii) is established. For any G without isolated vertices the total domination number is the same as the minimum number of open neighborhoods that cover G. But by (ii), when the girth of G is at least 7 the complete subgraphs of $N_o(G)$ are subsets of open neighborhoods of G. In addition, every open neighborhood of G is a complete subgraph of $N_{\mathrm{o}}(G)$. Consequently, (iii) holds. ■

The bound on girth in parts (ii) and (iii) of Proposition 9 is sharp, but the girth restriction is not a necessary condition. In fact, $N_{\mathrm{o}}(C_6) = 2C_3$ and $\gamma_t(C_6) = 4$, whereas $\gamma_t(C_5) = 3$ and three complete subgraphs of $N_{\mathrm{o}}(C_5)$ are needed to cover $N_{\mathrm{o}}(C_5) = C_5$.

We now prove Theorem 4 by establishing the following lemma, which is perhaps interesting in its own right, because of its similarity to Theorem 1.

Lemma 10. *If T is any tree of order at least two, then $\gamma_t(T) = \rho^{\mathrm{o}}(T)$.*

Proof. Let $T = (V, E)$ have order at least two, and let $V_1 \cup V_2$ be the unique bipartition of V. Note that since T is bipartite, the condition for uv to be an edge in $N_{\mathrm{o}}(T)$ is equivalent to requiring the distance between u and v in T to be exactly two. So, in particular, if u and v belong to different parts of the bipartition, then $\{u, v\}$ is independent in $N_{\mathrm{o}}(T)$. As a result, $N_{\mathrm{o}}(T)$ is a disconnected graph with two components whose vertex sets are V_1 and V_2. Let $C : v_1, v_2, \ldots, v_n, v_1$, where $n \geq 4$, be a cycle in $N_{\mathrm{o}}(T)$, say in $\langle V_1 \rangle$. Assume C has no chords. For each $1 \leq i \leq n$ there exists a vertex $w_i \in N_T(v_i) \cap N_T(v_{i+1})$, where the subscripts are computed modulo n. Since C has no chords, $N_T(w_i) \cap (\{v_1, \ldots, v_n\} - \{v_i, v_{i+1}\}) = \emptyset$, for each i. But then

$$v_1, w_1, v_2, w_2, \ldots, v_{n-1}, w_{n-1}, v_n, w_n, v_1$$

is a cycle in the tree T. This contradiction shows that the graph $N_{\mathrm{o}}(T)$ is a chordal graph and hence is a perfect graph. By the perfect graph theorem of Lovász [9], the complement of $N_{\mathrm{o}}(T)$, $\overline{N_{\mathrm{o}}(T)}$, is also perfect. Consequently,

$$\rho^{\mathrm{o}}(T) = \beta_0(N_{\mathrm{o}}(T)) = \omega(\overline{N_{\mathrm{o}}(T)}) = \chi(\overline{N_{\mathrm{o}}(T)}).$$

The chromatic number of the complement of $N_{\mathrm{o}}(T)$ is equal to the smallest number of complete subgraphs of $N_{\mathrm{o}}(T)$ that cover $V(N_{\mathrm{o}}(T)) = V(T)$. By Proposition 9 it follows that $\gamma_t(T) = \rho^{\mathrm{o}}(T)$. ■

***Proof of Theorem* 4.** Let T be any tree of order at least two and let H be a graph without isolated vertices. By Lemma 10 $\gamma_t(T) = \rho^{\mathrm{o}}(T)$, and so by Theorem 7 $\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H)$.

For any graph G of girth at least seven that has the property that the chromatic number of the complement of $N_o(G)$ is the same as its maximum clique size it follows as in the proof of Lemma 10 that $\gamma_t(G) = \rho^o(G)$. Therefore, for any such graph G, Theorem 7 implies that

$$\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$$

for any H with no isolates.

For example, $N_o(C_{4n}) = 2C_{2n}$, which is bipartite and hence is a perfect graph. This verifies the following corollary.

Corollary 11. *The class of cycles whose order is a multiple of four is a universal multiplicative class for γ_t on $\times$. For any positive integer n and any H with no isolated vertices,*

$$\gamma_t(C_{4n} \times H) = \gamma_t(C_{4n})\gamma_t(H) = 2n\gamma_t(H).$$

In contrast to the domination number or the total domination number of the Cartesian product of two paths, the total domination number of the categorical product of paths is easily computed. The graph $P_n \times P_m$ has two connected components, each of which is a subgraph of the Cartesian product $P_n \Box P_m$. If n is a multiple of 4, then $\gamma_t(P_n) = \frac{n}{2}$; otherwise $\gamma_t(P_n) = \lceil \frac{n+1}{2} \rceil$. Using this and Theorem 4 we can now compute the total domination number of "categorical grids" of any dimension. The exact value is not given here because of the large number of cases involved.

Corollary 12. *For any collection of positive integers $n_1, n_2, \ldots, n_k$ each at least two,*

$$\gamma_t(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = \prod_i \gamma_t(P_{n_i}).$$

4. Domination in Categorical Products

In contrast to the situation with total domination, the domination invariant γ is neither submultiplicative nor supermultiplicitive on categorical products. For example, $\gamma(K_3 \times K_3) = 3 > \gamma(K_3)\gamma(K_3)$, while if G is a complete graph of even order at least six with a perfect matching removed, then $\gamma(G \times G) = 3 < \gamma(G)\gamma(G)$. However, using Theorem 3 and the fact that the total domination number of a graph without isolates is no larger than twice its domination number, it is easy to verify the following bound for the domination number of the categorical product.

Theorem 13. *For any graph G and H without isolated vertices,*

$$\gamma(G \times H) \leq 4\gamma(G)\gamma(H).$$

Next we show, using an argument similar to that given in [11], that dominating a categorical product of two graphs with no isolates is actually related to the total domination number of these graphs.

Lemma 14. *Let G and H be graphs with no isolated vertices. Then*

$$\gamma(G \times H) \geq \max\{\rho(G)\gamma_t(H), \rho(H)\gamma_t(G)\}.$$

Proof. Let x be any vertex of H and let S be any subset of $V(G \times H)$ such that S dominates G_x. Assume first that $S \cap G_x = \emptyset$. Then, for every $u \in V(G)$ the vertex (u, x) must be adjacent to some vertex (v, y) in S such that $v \in N_G(u)$ and $y \in N_H(x)$. Let $S' = \{v \mid (v, y) \in S \text{ for some } y \in N_H(x)\}$. It is clear that S' is a total dominating set of G, so

$$\gamma_t(G) \leq |S'| \leq |S \cap (V(G) \times N_H(x))| \leq |S \cap (V(G) \times N_H[x])|.$$

Now assume that $S \cap G_x \neq \emptyset$. Replace each $(u, x) \in S$ by a vertex (w, y) for any $w \in N_G(u)$ and any $y \in N_H(x)$. This modified set dominates G_x and does not intersect G_x, so by the previous case it once again follows that $\gamma_t(G) \leq |S \cap (V(G) \times N_H[x])|$.

If A is any maximum 2-packing of H and D is any minimum dominating set of $G \times H$, it follows from above that $|D| \geq |A|\gamma_t(G) = \rho(H)\gamma_t(G)$. Interchanging the roles of G and H establishes the lemma. ■

Corollary 15. *Let H be any graph with no isolated vertices and let T be any tree.*

- *Then $\gamma(T \times H) \geq \gamma(T)\gamma_t(H) \geq \gamma(T)\gamma(H)$.*
- *If $\gamma_t(T) = \gamma(T)$, then $\gamma(T \times H) = \gamma_t(T)\gamma_t(H) = \gamma_t(T \times H)$.*

Proof. The first statement follows from Lemma 14, Theorem 1 and the fact that $\gamma_t(H) \geq \gamma(H)$. If $\gamma_t(T) = \gamma(T)$ then applying Theorem 4 we obtain

$$\gamma_t(T)\gamma_t(H) = \gamma_t(T \times H) \geq \gamma(T \times H) \geq \gamma(T)\gamma_t(H) = \gamma_t(T)\gamma_t(H),$$

and the result follows. ∎

The conclusion of Corollary 15 also holds for graphs more general than trees as long as the 2-packing number and the domination number are equal. For example, any connected graph G with the property that every vertex is either a leaf or is adjacent to a leaf and to a non-leaf vertex will in fact satisfy $\gamma(G \times H) = \gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$.

References

[1] B.D. Acharya, *Graphs whose r-neighbourhoods form conformal hypergraphs*, Indian J. Pure Appl. Math. **16** (5) (1985) 461–464.

[2] B.L. Hartnell and D.F. Rall, *Lower bounds for dominating Cartesian products*, J. Combin. Math. Combin. Comput. **31** (1999) 219–226.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds), Fundamentals of Domination in Graphs (Marcel Dekker, Inc. New York, 1998).

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds), Domination in Graphs: Advanced Topics (Marcel Dekker, Inc. New York, 1998).

[5] M.A. Henning, *Packing in trees*, Discrete Math. **186** (1998) 145–155.

[6] M.A. Henning and D.F. Rall, *On the total domination number of Cartesian products of graphs*, Graphs and Combinatorics, to appear.

[7] M.A. Henning and P.J. Slater, *Open packing in graphs*, J. Combin. Math. Combin. Comput. **29** (1999) 3–16.

[8] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition (John Wiley & Sons, Inc. New York, 2000).

[9] L. Lovász, *Normal hypergraphs and the perfect graph conjecture*, Discrete Math. **2** (1972) 253–267.

[10] A. Meir and J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975) 225–233.

[11] R.J. Nowakowski and D.F. Rall, *Associative graph products and their independence, domination and coloring numbers*, Discuss. Math. Graph Theory **16** (1996) 53–79.

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