ON DOMINATION IN GRAPHS

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Abstract

For a finite undirected graph $G$ on $n$ vertices two continuous optimization problems taken over the $n$-dimensional cube are presented and it is proved that their optimum values equal the domination number $\gamma$ of $G$. An efficient approximation method is developed and known upper bounds on $\gamma$ are slightly improved.

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1. Introduction and Results

For terminology and notation not defined here we refer to [3]. Let $V = V(G) = \{1, \ldots, n\}$ be the vertex set of an undirected graph $G$, and for $i \in V$, $N(i)$ be the neighbourhood of $i$ in $G$, $N_2(i) = \{k \in V \mid k \in \bigcup_{j \in N(i)} N(j) \setminus (\{i\} \cup N(i))\}$, $d_i = |N(i)|$, $t_i = |N_2(i)|$, $\delta = \min_{i \in V} d_i$, and $\Delta = \max_{i \in V} d_i$.

A set $D \subseteq V(G)$ is a dominating set of $G$ if $(\{i\} \cup N(i)) \cap D \neq \emptyset$ for every $i \in V$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma$. 

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number $\gamma$ of $G$. In [7] $\gamma = \min_{x_1, \ldots, x_n \in [0, 1]} \sum_{i \in V} (x_i + (1 - x_i) \prod_{j \in N(i)} (1 - x_j))$ was proved. With $x_1 = \ldots = x_n = x$ we have $\gamma \leq (x + (1 - x)\delta + 1)n \leq (x + e^{-(\delta + 1)x})n$ for every $x \in [0, 1]$. Minimizing $x + (1 - x)\delta + 1$ and $x + e^{-(\delta + 1)x}$, the well-known inequalities $\gamma \leq (1 - \frac{1}{(\delta + 1)\pi} + \frac{1}{(\delta + 1)\pi})n \leq \frac{1 + \ln(\delta + 1)}{\delta + 1}n$ (see [4, 8]) follow. Obviously, it is easily checked whether $\gamma = 1$ or not. Thus, we will assume $G \in \Gamma$ in the sequel, where $\Gamma$ is the set of graphs $G$ such that each component of $G$ has domination number greater than 1. Without mentioning in each case, we will use $d_i, t_i \geq 1$ for $i = 1, \ldots, n$ if $G \in \Gamma$. For $x_1, \ldots, x_n \in [0, 1]$ let

$$f_G(x_1, \ldots, x_n) = \sum_{i \in V} \left( x_i \left( 1 - \left( \prod_{j \in N(i)} x_j \right) \left( 1 - \prod_{k \in N_2(i)} x_k \right) \right) + (1 - x_i) \prod_{j \in N(i)} (1 - x_j) \right)$$

$$g_G(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n) - \sum_{i \in V} \left( \frac{1}{1 + d_i} (1 - x_i) \left( \prod_{j \in N(i)} (1 - x_j) \right) \prod_{k \in N_2(i)} (1 - x_k) \right).$$

**Theorem 1.** If $G \in \Gamma$ then

$$\gamma = \min_{x_1, \ldots, x_n \in [0, 1]} f_G(x_1, \ldots, x_n) = \min_{x_1, \ldots, x_n \in [0, 1]} g_G(x_1, \ldots, x_n)$$

$$\leq \min_{x \in [0, 1]} \sum_{i \in V} \left( x \left( 1 - x^{d_i} (1 - x^{t_i}) \right) + (1 - x)^{d_i + 1} \left( 1 - \frac{1}{1 + d_i} (1 - x)^{t_i} \right) \right)$$

$$\leq \min_{x \in [0, 1]} \left( x \left( 1 - x^{\Delta} (1 - x) \right) + (1 - x)^{\delta + 1} \left( 1 - \frac{1}{1 + \Delta} (1 - x)^{\Delta - 1} \right) \right)n.$$  

Since DOMINATING SET is an NP-complete decision problem ([5]), it is difficult to solve the continuous optimization problem $P$ :

$$\min_{x_1, \ldots, x_n \in [0, 1]} g_G(x_1, \ldots, x_n).$$

However, if $(x_1, \ldots, x_n)$ is the solution of any approximation method for $P$, then (see Theorem 2) we can easily find a dominating set of $G$ of cardinality at most $g_G(x_1, \ldots, x_n)$. 

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Theorem 2. Given a graph $G \in \Gamma$ on $V = \{1, \ldots, n\}$ with maximum degree $\Delta$, $x_1, \ldots, x_n \in [0, 1]$, there is an $O(\Delta^4 n)$-algorithm finding a dominating set $D$ of $G$ with $|D| \leq g_G(x_1, \ldots, x_n)$.

2. Proofs

Proof of Theorem 1. For events $A$ and $B$ and for a random variable $Z$ of an arbitrary random space, $P(A)$, $P(A|B)$, and $E(Z)$ denote the probability of $A$, the conditional probability of $A$ given $B$, and the expectation of $Z$, respectively. Let $\overline{A}$ be the complementary event of $A$. We will use the well-known facts that $P(B)P(A|B) = P(A \cap B) = P(B) - P(\overline{A} \cap B) = P(B)(1 - P(\overline{A}|B))$ and $E(|S'|) = \sum_{s \in S} P(s \in S')$ for a random subset $S'$ of a given finite set $S$. $I \subset V$ is an independent set if $N(i) \cap I = \emptyset$ for all $i \in I$. Consider fixed $x_1, \ldots, x_n \in [0, 1]$. $X \subseteq V$ is formed by random and independent choice of $i \in V$, where $P(i \in X) = x_i$. Let $X' = \{i \in X \mid N(i) \subseteq X\}$, $X'' = \{i \in X' \mid N(i) \cap (X \setminus X') \neq \emptyset\}$, $Y = \{i \in V \mid i \notin X \land N(i) \cap X = \emptyset\}$, $Y' = \{i \in Y \mid N(i) \cap Y \neq \emptyset\}$, and $I$ be a maximum independent set of the subgraph of $G$ induced by $Y'$.

Lemma 3. $(X \setminus X'') \cup (Y \setminus I)$ is a dominating set of $G$.

Proof. Obviously, $X'' \subseteq X' \subseteq X$ and $(X \setminus X') \subseteq (X \setminus X'')$. If $i \in V \setminus (X \cup Y)$ then $N(i) \cap (X \setminus X') \neq \emptyset$, if $i \in X''$ then again $N(i) \cap (X \setminus X') \neq \emptyset$, and if $i \in I$ then $N(i) \cap (Y \setminus I) \neq \emptyset$.

Lemma 4. $\gamma \leq E(|X|) - E(|X''|) + E(|Y|) - E(|I|)$.

Proof. Let $D$ be a random dominating set of $G$. Because of the property of the expectation to be an average value we have $\gamma \leq E(|D|)$. With Lemma 3 and linearity of the expectation, $\gamma \leq E(|(X \setminus X'') \cup (Y \setminus I)|) = E(|X| - |X''| + |Y| - |I|) = E(|X|) - E(|X''|) + E(|Y|) - E(|I|)$ since $(X \setminus X'') \cap (Y \setminus I) = \emptyset$.

Lemma 5. $E(|X|) = \sum_{i \in V} x_i$, $E(|X''|) = \sum_{i \in V} x_i \left( \prod_{j \in N(i)} x_j \right) \left( 1 - \prod_{k \in N_2(i)} x_k \right)$, $E(|Y|) = \sum_{i \in V} (1 - x_i) \left( \prod_{j \in N(i)} (1 - x_j) \right)$, and $E(|I|) \geq \sum_{i \in V} \frac{1}{1 + d_i} (1 - x_i) \left( \prod_{j \in N(i)} (1 - x_j) \right) \left( \prod_{k \in N_2(i)} (1 - x_k) \right)$. 

Proof. \[ E(|X|) = \sum_{i \in V} P(i \in X) = \sum_{i \in V} x_i. \]

\[
E(|X'|) = \sum_{i \in V} P(i \in X') = \sum_{i \in V} P(i \in X \land N(i) \subseteq X \land N(i) \cap (X \setminus X') \neq \emptyset)
\]

\[
= \sum_{i \in V} P(i \in X)P(N(i) \subseteq X)P(N(i) \cap (X \setminus X') \neq \emptyset | i \in X \land N(i) \subseteq X)
\]

\[
= \sum_{i \in V} x_i \left( \prod_{j \in N(i)} x_j \right) (1 - P(N(i) \subseteq X' | i \in X \land N(i) \subseteq X))
\]

\[
= \sum_{i \in V} x_i \left( \prod_{j \in N(i)} x_j \right) (1 - P(N_2(i) \subseteq X)) = \sum_{i \in V} x_i \left( \prod_{j \in N(i)} x_j \right) \left( 1 - \prod_{k \in N_2(i)} x_k \right).
\]

\[ E(|Y|) = \sum_{i \in V} P(i \in Y) = \sum_{i \in V} P(i \notin X)P(N(i) \cap X = \emptyset)
\]

\[
= \sum_{i \in V} (1 - x_i) \prod_{j \in N(i)} (1 - x_j).
\]

A lower bound on \(|I|\) (see \([1, 9, 2, 6]\)) is given by the following inequality
\[
|I| \geq \sum_{i \in Y'} \frac{1}{1 + d_i}.
\]

For \(i \in V(G)\) define the random variable \(Z_i\) with \(Z_i = \frac{1}{1 + d_i}\) if \(i \in Y'\) and \(Z_i = 0\) if \(i \notin Y'\). Hence,

\[
E(|I|) \geq E \left( \sum_{i \in V} Z_i \right) = \sum_{i \in V} E(Z_i) = \sum_{i \in V} \frac{1}{1 + d_i} P(i \in Y')
\]

\[
= \sum_{i \in V} \frac{1}{1 + d_i} P(i \notin X \land N(i) \cap X = \emptyset \land N(i) \cap Y \neq \emptyset).
\]

Because \(d_i \geq 1\), \(N_2(i) \cap X = \emptyset\) implies \(N(i) \cap Y \neq \emptyset\). Hence,

\[
E(|I|) \geq \sum_{i \in V} \frac{1}{1 + d_i} P(i \notin X \land N(i) \cap X = \emptyset \land N_2(i) \cap X = \emptyset)
\]

\[
= \sum_{i \in V} \frac{1}{1 + d_i} P(i \notin X)P(N(i) \cap X = \emptyset)P(N_2(i) \cap X = \emptyset)
\]

\[
= \sum_{i \in V} \frac{1}{1 + d_i} (1 - x_i) \left( \prod_{j \in N(i)} (1 - x_j) \right) \left( \prod_{k \in N_2(i)} (1 - x_k) \right).
\]
From Lemma 4 and Lemma 5 we have $\gamma \leq g_G(x_1, \ldots, x_n) \leq f_G(x_1, \ldots, x_n)$. Let $D^*$ be a minimum dominating set of $G$ and let $y_i = 1$ if $i \in D^*$ and $y_i = 0$ if $i \notin D^*$. Then $y_i \prod_{j \in N(i)} y_j = 0$ and $(1 - y_i) \prod_{j \in N(i)} (1 - y_j) = 0$ for every $i \in V$. $\gamma = |D^*| = \sum_{i \in V} y_i = g_G(y_1, \ldots, y_n)$, and $f_G(y_1, \ldots, y_n)$, and the proof of Theorem 1 is complete.

\textbf{Proof of Theorem 2.} Given a graph $H$ on $n_H$ vertices with $m_H$ edges, there is an $O(n_H + m_H)$-algorithm $A$ finding an independent set of $H$ with cardinality at least $\sum_{y \in V(H)} \frac{1}{1 + d_H(y)}$, where $d_H(y)$ is the degree of $y \in V(H)$ (see [2]).

First we present an algorithm that constructs a set $D \subseteq V$.

\textbf{Algorithm}

INPUT: a graph $G \in \Gamma$ on $V = \{1, \ldots, n\}$, $x_1, \ldots, x_n \in [0,1]$

OUTPUT: $D$

(1) For $l = 1, \ldots, n$ do if $\frac{\partial g_G(x_1, \ldots, x_n)}{\partial x_l} \geq 0$ then $x_l := 0$ else $x_l := 1$.

(2) $X := \{l \in \{1, \ldots, n\} \mid x_l = 1\}$. Calculate $X'', Y, Y'$, and $I$ using $A$.

(3) $D := (X \setminus X'') \cup (Y \setminus I)$.

END

Let $g^* = g_G(x_1, \ldots, x_n)$, where $(x_1, \ldots, x_n)$ is the input vector. Note that the function $g_G$ is linear in each variable. Thus, in step (1), for fixed $x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n$ we always choose $x_l$ in such a way that the value of $g_G(x_1, \ldots, x_n)$ is not increased. Hence, $x_l \in \{0, 1\}$ for $l = 1, \ldots, n$ and $g_G(x_1, \ldots, x_n) \leq g^*$ after step (1) of the algorithm. With Lemma 3, $D$ is a dominating set, and with $|S| = E(|S|)$ for a deterministic set $S$ and Lemma 5, $|D| \leq g^*$. It is easy to see that $\frac{\partial g_G(x_1, \ldots, x_n)}{\partial x_l}$ can be calculated in $O(\Delta^4)$ time. Since $G$ has $O(\Delta n)$ edges, the algorithm runs in $O(\Delta^4 n)$ time.

\textbf{References}


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