

## SOME CROSSING NUMBERS OF PRODUCTS OF CYCLES

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### Abstract

The exact values of crossing numbers of the Cartesian products of four special graphs of order five with cycles are given and, in addition, all known crossing numbers of Cartesian products of cycles with connected graphs on five vertices are summarized.

**Keywords:** graph, drawing, crossing number, cycle, Cartesian product.

**2000 Mathematics Subject Classification:** 05C10, 05C38.






















### 1. INTRODUCTION

The *crossing number*  $cr(G)$  of a graph  $G$  is the minimum number of crossings among all drawings of the graph in the plane. All drawings considered herein are *good* drawings, meaning that no edge crosses itself, no two edges cross more than once, no two edges incident with the same vertex cross, no more than two edges cross at a point of the plane, and no edge meets a vertex, which is not its endpoint. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a good drawing. Let  $D$  be a good drawing of the graph  $G$ . We denote the number of crossings in  $D$  by  $cr_D(G)$ . Let  $G_i$  and  $G_j$  be edge-disjoint subgraphs of  $G$ . We denote by  $cr_D(G_i, G_j)$  the number of crossings between edges of  $G_i$  and edges of  $G_j$ , and by  $cr_D(G_i)$  the number of crossings among edges of  $G_i$  in  $D$ .

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Let  $C_n$  be the cycle with  $n$  vertices. Determining the crossing numbers of Cartesian products (for a definition of Cartesian product see [6]) of two cycles or of cycles and small graphs have received a good deal of attention. Harary et al. [7] conjectured that the crossing number of  $C_m \times C_n$  is  $(m - 2)n$ , for all  $m, n$  satisfying  $3 \leq m \leq n$ . This has been proved only for  $m, n$  satisfying  $n \geq m$ ,  $m \leq 6$ , [1, 3, 4, 14, 15, 17], and for the special case  $m = n = 7$  [2]. Recently, Glebsky and Salazar [5] proved that this conjecture holds for values of  $n$  sufficiently large compared to  $m$  (roughly, for  $n \geq m^2$ ). The general conjecture remains open.

$G_i$	$cr(G_i \times C_n)$		$G_i$	$cr(G_i \times C_n)$	
	0			$2n$	
	$2n$ ( $n > 5$ ) [9]	$2$ ( $n=3$ ) $4$ ( $n=4$ ) $8$ ( $n=5$ ) [9]		$3n$ ( $n > 3$ ) [*]	$7$ ( $n=3$ ) [*]
	$n$ ( $n > 5$ ) [*]	$1$ ( $n=3$ ) $2$ ( $n=4$ ) $4$ ( $n=5$ ) [*]		$3n$ [*]	
	$n$				
	$n$			$3n$ (even $n$ ) $3n+1$ (odd $n$ ) [10]	
	$2n$ ( $n > 5$ ) [*]	$4$ ( $n=3$ ) $6$ ( $n=4$ ) $9$ ( $n=5$ ) [*]			
	$2n$ ( $n > 3$ ) [*]	$4$ ( $n=3$ ) [*]			
	$3n$ ( $n > 4$ ) [14]	$5$ ( $n=3$ ) $10$ ( $n=4$ ) [17] [3]			
	$2n$ [*]				
		$9$ ( $n=3$ ) [12]			
	$3n$ [13]	$7$ ( $n=3$ ) [13]			

The crossing numbers of the Cartesian products of cycles and all graphs of order four are determined in [3, 8]. It thus seems natural to inquire about the crossing numbers of the products of 5-vertex graphs with cycles. Except the above-mentioned result  $cr(C_5 \times C_n) = 3n$ , [14, 16], in [9] it is shown that  $cr(K_{1,4} \times C_3) = 2$ ,  $cr(K_{1,4} \times C_4) = 4$ ,  $cr(K_{1,4} \times C_5) = 8$ , and  $cr(K_{1,4} \times C_n) = 2n$  for  $n \geq 6$ . Let  $G$  be the 5-vertex graph obtained from  $K_5$  by removing three edges incident with a common vertex. It is shown in [10] that the crossing number of  $G \times C_n$  is  $3n$  for even  $n$  or  $3n + 1$  if  $n$  is odd. In [12] the upper bound of  $4n$  for the crossing number of  $K_{2,3} \times C_n$  for any  $n \geq 4$  is given and it is proved that the crossing number of  $K_{2,3} \times C_3$  is 9. The table shows the summary of known crossing numbers for Cartesian products of cycles and connected graphs of order five. The results without citation in the lower right corners of table entries are obtained by finding a subgraph with the same crossing number as shown in the table and also finding a suitable drawing of the graph with the same number of crossings ( $G_1, G_4, G_5$  and  $G_{12}$ ). The graph  $G_9$  consists of two edge-disjoint subgraphs  $C_3 \times C_n$  with crossing number  $n$ , and therefore  $cr(G_9 \times C_n) \geq 2n$  for  $n \geq 3$ . As for  $n \geq 3$  there is a drawing of  $G_9 \times C_n$  with exactly  $2n$  crossings,  $cr(G_9 \times C_n) = 2n$ . It is the purpose of this paper to give the exact values of crossing numbers for graphs  $G_j \times C_n$  for  $j = 3, 6, 7, 13$  and 14. These results are denoted by  $[\star]$  in the lower right corners of the table entries.

2. THE GRAPHS  $G_3$  AND  $G_7$

It is not difficult to state exact values of crossing numbers for the graphs  $G_3 \times C_n$  for  $n = 3, 4, 5$ , and for the graph  $G_7 \times C_n$  for  $n = 3$ . First we show that  $cr(G_7 \times C_3) = 4$ . Figure 1 shows the drawing of the graph  $G_7 \times C_3$  with

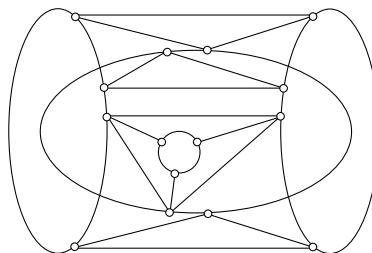


Figure 1

four crossings. Thus,  $cr(G_7 \times C_3) \leq 4$ . On the other hand,  $cr(G_7 \times C_3) \geq 4$ , because the graph  $G_7 \times C_3$  contains the graph  $C_4 \times C_3$  as a subgraph and its crossing number is four. As the graph  $G_7 \times C_n$  contains the graph  $C_4 \times C_n$  as a subgraph and  $cr(C_4 \times C_n) = 2n$ , the crossing number of the graph  $G_7 \times C_n$  is at least  $2n$ . One can easily find the drawing of  $G_7 \times C_n$  with  $2n$  crossings, hence  $cr(G_7 \times C_n) = 2n$ .

Figure 2 in the next section shows the graph  $G_6 \times C_n$  for  $n = 3, 4$  and  $5$ . For  $n = 3, 4$  and  $5$ , the graph  $G_3 \times C_n$  can be obtained by deleting three, four and five edges, respectively, from the graph  $G_6 \times C_n$ . As, in Figure 2, every of these considered edges is crossed exactly once, we have  $cr(G_3 \times C_3) \leq 1$ ,  $cr(G_3 \times C_4) \leq 2$ , and  $cr(G_3 \times C_5) \leq 4$ . On the other hand, the graph  $G_3 \times C_n$  contains the graph  $K_{1,3} \times C_n$  as a subgraph and therefore,  $cr(G_3 \times C_3) \geq 1$ ,  $cr(G_3 \times C_4) \geq 2$ , and  $cr(G_3 \times C_5) \geq 4$ , because  $cr(K_{1,3} \times C_3) = 1$ ,  $cr(K_{1,3} \times C_4) = 2$ , and  $cr(K_{1,3} \times C_5) = 4$  (see [8]). Thus, for  $n = 3, 4$  and  $5$  the crossing number of the graph  $G_3 \times C_n$  is 1, 2 and 4, respectively. We can generalize this idea for  $n \geq 6$  and, using the fact that  $cr(K_{1,3} \times C_n) = n$ , state that  $cr(G_3 \times C_n) = n$ .

### 3. THE GRAPH $G_6$

We assume  $n \geq 3$  and find it convenient to consider the graph  $G_6 \times C_n$  in the following way. It has  $5n$  vertices, which we denote  $x_i$  for  $x = a, b, c, d, e$  and  $i = 1, 2, \dots, n$ , and  $10n$  edges that are the edges of the  $n$  copies  $G_6^i$  and the five cycles  $C_n^x$ . For  $i = 1, 2, \dots, n$ , let  $a_i$  and  $b_i$  be the vertices of  $G_6^i$  of degree two, let  $c_i$  be the vertex of degree four, and let  $d_i$  and  $e_i$  be the vertices of  $G_6^i$  of degree one.

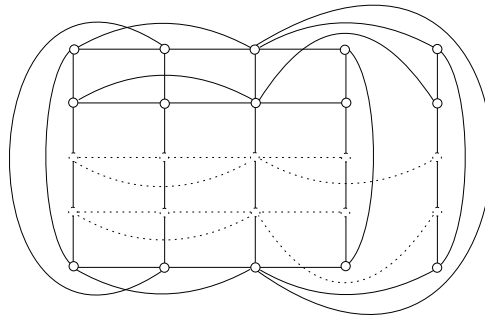


Figure 2

In Figure 2 there is the drawing of the graph  $G_6 \times C_3$  and the dotted lines show how to obtain drawings for  $n \geq 4$ . It is not difficult to redraw Figure 2 in such a way that every copy of  $G_6^i$  is crossed exactly two times and two different  $G_6^i$  and  $G_6^j$  do not cross each other. This obviously generalizes to show that the crossing number of  $G_6 \times C_n$  is at most  $2n$ . The graph  $G_6 \times C_n$  contains the graph  $K_{1,4} \times C_n$  as a subgraph. As  $cr(K_{1,4} \times C_n) = 2n$  for all  $n \geq 6$  (see [9]), we have  $cr(G_6 \times C_n) = 2n$  for all  $n \geq 6$ . Theorem 1 states the crossing numbers for  $G_6 \times C_n$  for the remaining values of  $n$ .

In the proofs of the paper, we will often use the term “region” also in nonplanar drawings. In this case, crossings are considered to be vertices of the “map”.

**Theorem 1.**  $cr(G_6 \times C_3) = 4$ ,  $cr(G_6 \times C_4) = 6$  and  $cr(G_6 \times C_5) = 9$ .

*Proof.* Consider the subgraph  $G'$  of the graph  $G_6 \times C_n$  induced on the vertices of three cycles  $C_n^a$ ,  $C_n^b$ , and  $C_n^c$ . This subgraph  $G'$  is isomorphic to the graph  $C_3 \times C_n$  with crossing number  $n$  for all  $n \geq 3$  (see [17]). Let  $T^x$ ,  $x \in \{a, b, d, e\}$ , denote the subgraph of  $G_6 \times C_n$  induced on the edges incident with the vertices of the cycle  $C_n^x$ . Thus,  $G_6 \times C_n = G' \cup T^d \cup T^e$ . From the Figure 2 (not regarding the dotted lines) it follows that  $cr(G_6 \times C_3) \leq 4$ . In the same figure, considering one copy of  $G_6$  with two crossings drawn by dotted lines or both copies of  $G_6$  drawn by dotted lines, we can see that  $cr(G_6 \times C_4) \leq 6$  and  $cr(G_6 \times C_5) \leq 9$ , respectively. To prove Theorem 1, it is necessary to show that  $cr(G_6 \times C_3) \geq 4$ ,  $cr(G_6 \times C_4) \geq 6$ , and  $cr(G_6 \times C_5) \geq 9$ .

Assume that there is a drawing of the graph  $G_6 \times C_3$  with less than four crossings and let  $D$  be such a drawing. Then the subdrawing  $D'$  of the drawing  $D$  induced by the subgraph  $G'$  isomorphic to  $C_3 \times C_3$  contains three crossings and, in  $D$ , there is no crossing on the edges of  $T^d \cup T^e$ . As the planar subdrawing  $D''$  of the graph  $T^d \cup T^e$  is unique up to the isomorphism, Figure 3(a) shows that in this case there is no region with all three vertices of  $C_3^c$  (the vertices of degree two in  $D''$ ) on the boundary of the same region. Since, in  $D$ , the cycle  $C_3^a$  does not cross the edges of  $D''$ , one can easily see that  $C_3^a$  lies in one region of the planar subdrawing  $D''$ . But in this case the edges joining  $C_3^a$  with the vertices of  $C_3^c$  cross the edges of  $T^d \cup T^e$ , which contradicts our assumption that  $D$  has fewer than four crossings. Thus,  $cr(G_6 \times C_3) = 4$ .

Note that for  $n \geq 4$  there is no drawing of the subgraph  $T^d \cup T^e$  with one crossing. In fact, if any two edges not incident with the same vertex

cross each other, then one can find in  $T^d \cup T^e$  two vertex-disjoint cycles in such a way that every of these cycles contains exactly one of the considered edges. As two vertex disjoint cycles cannot cross each other exactly once, in the drawing there is at least one additional crossing.

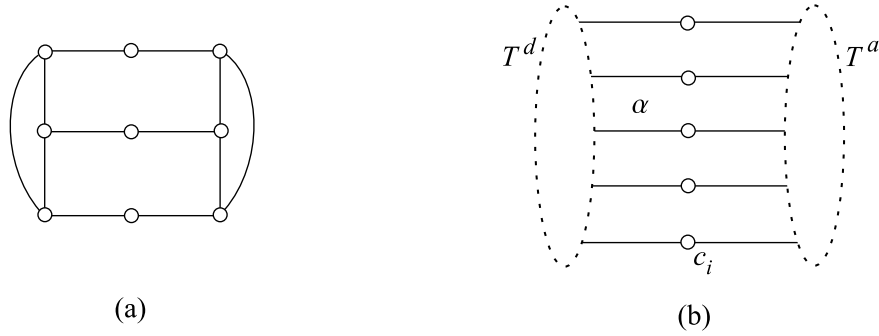


Figure 3

Assume now that there is a drawing of the graph  $G_6 \times C_4$  with less than six crossings and let  $D$  be such a drawing. As  $cr(C_3 \times C_4) = 4$ , there are at least four crossings in the subdrawing  $D'$  induced from  $D$  by the subgraph  $G'$ . In this case, in  $D$  there is at most one crossing on the edges of  $T^d \cup T^e$ . Since there exists no drawing of  $T^d \cup T^e$  with exactly one crossing, the subdrawing  $D''$  of  $T^d \cup T^e$  is planar and unique up to the isomorphism. It divides the plane into six regions such that any of them has at most two vertices of  $C_4^c$  on its boundary. The cycle  $C_4^a$  cannot lie in  $D$  in more than one region in the view of the subdrawing  $D''$ , otherwise it crosses  $T^d \cup T^e$  more than once. In  $D''$  there are at most two vertices of  $C_4^c$  on the boundary of one region and therefore the edges of  $T^a$  cross in  $D$  the edges of  $T^d \cup T^e$  at least two times, a contradiction.

For  $n = 5$ , assume that there is a drawing of the graph  $G_6 \times C_5$  with less than nine crossings and let  $D$  be such a drawing. As  $cr(C_3 \times C_5) = 5$ , there are at least five crossings in the subdrawing  $D'$  induced from  $D$  by the subgraph  $G'$ . In this case, in  $D$  there are at most three crossings on the edges of  $T^d \cup T^e$ . The subdrawing  $D''$  induced from  $D$  by the subgraph  $T^d \cup T^e$  without crossings divides the plane into seven regions with at most two vertices of  $C_5^c$  on the boundary of a region. Then the cycle  $C_5^a$  crosses in  $D$  the edges of  $T^d \cup T^e$  more than once or the edges of  $T^a$  cross the edges of  $T^d \cup T^e$  more than once if  $C_5^a$  lies in one region in the view of  $D''$ . The same arguments hold for the subgraph  $T^b$ ; hence,  $D''$  has in  $D$  more than three

crossings on its edges, a contradiction. As there is no drawing of  $T^d \cup T^e$  with exactly one crossing, assume that there is the subdrawing  $D''$  with two or three crossings. In this case at least one of the subgraphs  $T^a$  and  $T^b$  does not cross in  $D$  the edges of  $T^d \cup T^e$ . Without loss of generality, let  $cr_D(T^a, T^d \cup T^e) = 0$ . Then, regardless of whether or not the edges of  $T^a$  cross each other or the edges of  $T^d$  cross each other, the subdrawing  $D'''$  of  $D$  induced by  $T^d \cup T^a$  divides the plane in such a way that on the boundary of a region there are at most two vertices of  $C_5^c$  and no two regions with common boundary contain more than three vertices of  $C_5^c$  on their boundaries, see Figure 3(b). As  $cr_D(T^a, T^e) = 0$ , the edges of  $T^e$  can cross in  $D$  only the edges of  $T^d$ . Since the subgraph  $T^d \cup T^a$  is two connected, the vertices of  $C_5^c$  cannot lie in  $D$  in two non-neighbouring regions in the view of the subdrawing  $D'''$ , otherwise the edges of  $C_5^c$  cross the edges of  $T^d$  more than three times. If  $C_5^c$  lies in two neighbouring regions in the view of  $D'''$ , then  $cr_D(T^e, T^d) \geq 2$  and at least two vertices of  $C_5^c$  are not on the boundaries of these regions. Thus,  $T^e$  crosses  $T^d$  more than three times again. The last possibility is that  $C_5^c$  lies in one region of  $D'''$ . If there are less than two vertices of  $C_5^c$  on the boundary of this region, then  $cr_D(T^e, T^d) \geq 4$ , a contradiction. If  $C_5^c$  lies in the region  $\alpha$ , see Figure 3(b), with two vertices of  $C_5^c$  on its boundary, then the vertex  $c_i$  of  $C_5^c$  is not on the boundary of any region neighbouring with the region  $\alpha$ . In this case the edge of  $T^e$  joining  $C_5^c$  with the vertex  $c_i$  of  $C_5^c$  crosses  $T^d$  at least two times and other two edges of  $T^e$  joining  $C_5^c$  with the vertices of  $C_5^c$  that do not belong to the boundary of the region  $\alpha$  also cross  $T^d$ . This contradiction completes the proof. ■

#### 4. THE GRAPH $G_{13}$

We assume  $n \geq 3$  and find it convenient to consider the graph  $G_{13} \times C_n$  in the following way. It has  $5n$  vertices denoted  $x_i$  for  $x = a, b, c, p, q$  and  $i = 1, 2, \dots, n$ , and  $11n$  edges that are the edges in the  $n$  copies  $G_{13}^i$  and the five cycles  $C_n^x$ . For  $i = 1, 2, \dots, n$ , let  $a_i$  and  $b_i$  be two adjacent vertices of  $G_{13}^i$  of degree two, let  $p_i$  and  $q_i$  be the vertices of  $G_{13}^i$  of degree three adjacent with  $a_i$  and  $b_i$ , respectively, and let  $c_i$  be the vertex of degree two adjacent with  $p_i$  and  $q_i$ .

For  $n \geq 3$ , it is not difficult to find a drawing of the graph  $G_{13} \times C_n$  with  $3n$  crossings. Figure 4(a) shows the case when the  $n$ -cycles cross every copy of  $G_{13}^i$  exactly three times. Hence,  $cr(G_{13} \times C_n) \leq 3n$ . Figure 4(b) shows that  $3n$  is not the best upper bound for every  $n$ . For  $n = 3$  we have

the drawing of the considered graph with 7 crossings. For  $n \geq 5$ , the graph  $G_{13} \times C_n$  contains the graph  $C_5 \times C_n$  as a subgraph. As  $cr(C_5 \times C_n) = 3n$ , see [14], we have  $cr(G_{13} \times C_n) \geq 3n$ , and therefore, for  $n \geq 5$ ,  $cr(G_{13} \times C_n) = 3n$ . Theorem 2 gives the crossing numbers of the graph  $G_{13} \times C_n$  for  $n = 3$  and  $n = 4$ .

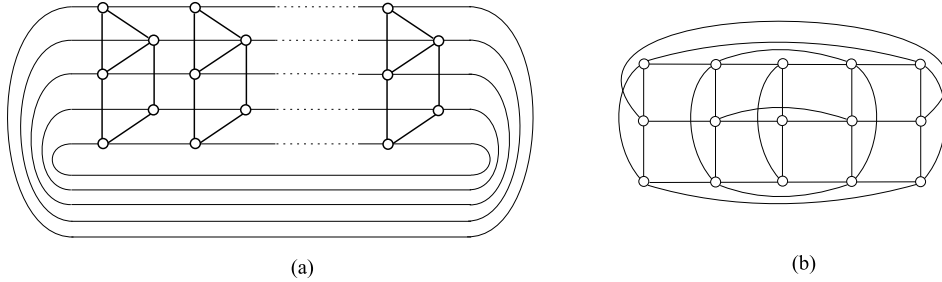


Figure 4

**Theorem 2.**  $cr(G_{13} \times C_3) = 7$  and  $cr(G_{13} \times C_4) = 12$ .

**Proof.** Denote by  $T^c$  the subgraph of the graph  $G_{13} \times C_n$  induced on the edges incident with the vertices of the cycle  $C_n^c$  and by  $I$  the subgraph consisting of the vertices of  $C_n^p$  and  $C_n^q$  and the edges joining the cycles  $C_n^p$  and  $C_n^q$ . Thus, the subgraph  $T^c \cup I \cup C_n^p \cup C_n^q$  is isomorphic to  $C_3 \times C_n$ .

First we show that  $cr(G_{13} \times C_3) = 7$ . Since  $cr(G_{13} \times C_3) \leq 7$ , see Figure 4(b), assume that there is a drawing of the graph  $G_{13} \times C_3$  with less than seven crossings and let  $D$  be such a drawing. By deleting the edges of  $C_3^a$  or the edges of  $C_3^b$  from the graph  $G_{13} \times C_3$  we obtain the subgraph homeomorphic to the graph  $(K_4 - e) \times C_3$  with crossing number six, see [11]. So, the cycles  $C_3^a$  and  $C_3^b$  are not crossed in  $D$ . Deleting the edges of  $I$  from  $G_{13} \times C_3$  results in the graph  $C_5 \times C_3$ . Thus, in  $D$  there is at most one crossing on the edges of  $I$ . As  $G_{13} \times C_3$  contains several subgraphs isomorphic or homeomorphic to the graph  $C_4 \times C_3$  with crossing number four, we have the next restrictions: in  $D$  there are at most two crossings on the edges of  $T^c$ , at most two crossings on the edges of  $C_3^p \cup I$ , and at most two crossings on the edges of  $C_3^q \cup I$ .

Consider now the subdrawing  $D'$  of the drawing  $D$  induced by the vertices of  $C_3^p$  and  $C_3^q$ . By Lemma 2 in [12] (which states that if in a good drawing of  $C_m \times C_n$  two  $m$ -cycles ( $n$ -cycles) cross each other, then at least one of them has at least three crossings on its edges), we have that the cycles



$C_3^p$  and  $C_3^q$  do not cross each other. As the subgraph  $T^c \cup I \cup C_3^p \cup C_3^q$  is isomorphic to  $C_3 \times C_3$  and on the edges of  $T^c$  there are at most two crossings, there is at least one crossing in  $D'$ . Suppose that two edges of  $I$ , say  $\{p_i, q_i\}$  and  $\{p_j, q_j\}$  cross each other. In this case in  $D$  — as there is no other crossing on the edges of  $I$  — the 3-cycle  $p_i c_i q_i p_i$  separates the vertices  $p_j$  and  $q_j$  and the path  $p_i c_i q_i$  is crossed by the paths  $p_j c_j q_j$  and  $p_j a_j b_j q_j$ . A similar consideration shows that the path  $p_i a_i b_i q_i$  is crossed by both paths  $p_j c_j q_j$  and  $p_j a_j b_j q_j$ . Thus, in this case the subgraphs  $G_{13}^i$  and  $G_{13}^j$  cross each other at least five times and on the edges of  $T^c$  there are at least three crossings. Since this contradicts the restriction that the edges of  $T^c$  are crossed at most two times, two edges of  $I$  do not cross each other. So, the only possibility for the subdrawing  $D'$  is that one edge of  $I$  crosses one of the cycles  $C_3^p$  or  $C_3^q$ . Such a subdrawing  $D'$  with one crossing is unique up to the isomorphism and divides the plane into six regions in such a way that there are two vertices  $p_i$  and  $q_i$  which are not on the boundary of the same region. But in this case, in  $D$ , one of the cycles  $C_3^p$  and  $C_3^q$  is crossed by both paths  $p_i c_i q_i$  and  $p_i a_i b_i q_i$ , which contradicts the restriction that none of the subgraphs  $C_3^p \cup I$  and  $C_3^q \cup I$  has more than two crossings on its edges. Hence, there is no drawing of  $G_{13} \times C_3$  with less than seven crossings and  $cr(G_{13} \times C_3) = 7$ .

Assume now that there is a drawing of the graph  $G_{13} \times C_4$  with less than twelve crossings and let  $D$  be such a drawing. In  $D$  there is at most one crossing on the edges of  $I$ , otherwise deleting the edges of  $I$  results in the graph  $C_5 \times C_4$  with less than ten crossings. For every  $i = 1, 2, 3, 4$ , in  $D$  there are at most four crossings on the edges of  $G_{13}^i$ , otherwise, by deleting the edges of  $G_{13}^i$ , we obtain the graph homeomorphic to  $G_{13} \times C_3$  with less than seven crossings. Thus, two edges of  $I$  do not cross each other in  $D$ , otherwise, as shown above, two different  $G_{13}^i$  and  $G_{13}^j$  cross each other more than four times. As  $G_{13} \times C_4$  contains several subgraphs isomorphic or homeomorphic to the graph  $C_4 \times C_4$  with crossing number eight, we have the next restrictions: in  $D$  there are at most three crossings on the edges of  $T^c$ , at most three crossings on the edges of  $C_4^p \cup I$ , and at most three crossings on the edges of  $C_4^q \cup I$ . We show that in  $D$  the cycles  $C_4^p$  and  $C_4^q$  do not cross each other. If the cycles  $C_4^p$  and  $C_4^q$  cross each other, then they cross at least twice. Moreover, at least one of them, regardless of whether or not its edges cross each other, separates two vertices of the other cycle, otherwise the drawing is not good. Without loss of generality, let  $C_4^p$  separates the vertices  $q_i$  and  $q_j$ . Then the paths  $q_i c_i c_j q_j$  and  $q_i b_i b_j q_j$  cross the cycle  $C_4^p$ ,

and this contradicts the restriction that there are at most three crossings on the edges of  $C_4^p \cup I$ .

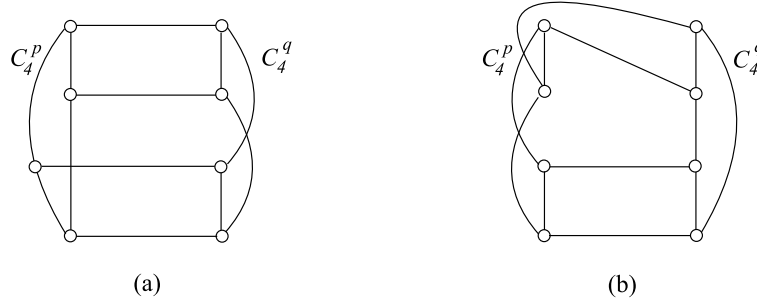


Figure 5

Consider now the subdrawing  $D'$  of the drawing  $D$  induced by the vertices of  $C_4^p$  and  $C_4^q$ . As the subgraph  $T^c \cup I \cup C_4^p \cup C_4^q$  is isomorphic to  $C_4 \times C_3$  and  $T^c$  has at most three crossings on its edges, there is at least one crossing in  $D'$ . Suppose that in  $D'$  an edge of  $I$  is crossed. Since two different edges of  $I$  do not cross mutually, the unique possibility of the subdrawing  $D'$  is that one edge of  $I$  crosses one of the cycles  $C_4^p$  or  $C_4^q$ . Without loss of generality, let an edge of  $I$  cross an edge of  $C_4^p$ . These two edges are in two vertex-disjoint cycles in the subdrawing  $D'$  and, since two cycles cannot cross once, there is one more crossing in  $D'$ . By our restriction, this crossing can appear in  $D'$  only as crossing of  $C_4^p$  or as crossing of  $C_4^q$ . Since in a good drawing no 4-cycle can have more than one internal crossing, one can easily see that, up to the isomorphism, there are only two possible drawings  $D'$  shown in Figure 5. If  $cr_D(C_4^p) = 0$  (see Figure 5(a)), in  $D$  there is at most one another crossing on the edges of  $C_4^q$  and at most two other crossings on the edges of  $C_4^p$ . If  $cr_D(C_4^p) = 1$  (see Figure 5(b)), in  $D$  there is at most one another crossing on the edges of  $C_4^p$  and at most two other crossings on the edges of  $C_4^q$ . In both cases, in  $D$  there are at most three other crossings on the edges of the subdrawing  $D'$ . Since in  $D'$  there are no more than five vertices on the boundary of a region, the cycle  $C_4^c$  crosses in  $D$  the edges of  $D'$  at least twice or, if it lies in one region in the view of the subdrawing  $D'$ , the edges joining  $C_4^c$  to  $C_4^q$  and  $C_4^p$  cross  $D'$  at least three times. The same arguments as for  $C_4^c$  hold for the subgraph induced on the vertices of  $C_4^a$  and  $C_4^b$  and, in  $D$ , the edges not belonging to  $D'$  cross the edges of  $D'$  more than three times. This contradicts with our assumptions. The similar contradictions with the restrictions are obtained, when in  $D'$  the edges of  $I$  are not crossed. This completes the proof. ■

5. THE GRAPH  $G_{14}$ 

For  $n \geq 3$ , one can obtain the graph  $G_{14} \times C_n$  by adding the edges  $\{a_i, q_i\}$  for  $i = 1, 2, \dots, n$  to the graph  $G_{13} \times C_n$ . In Figure 4(a) this is possible to do without crossings on the edges  $\{a_i, q_i\}$ , so  $cr(G_{14} \times C_n) \leq 3n$ . On the other hand,  $G_{14} \times C_n$  contains  $G_{13} \times C_n$  as a subgraph and this implies that  $cr(G_{14} \times C_n) \geq cr(G_{13} \times C_n)$ . Hence, for  $n \geq 4$ , the crossing number of the graph  $G_{14} \times C_n$  is  $3n$ .

**Theorem 3.**  $cr(G_{14} \times C_3) = 9$ .

*Proof.* Let us use the same notations as in the proof of Theorem 2 and moreover, denote by  $T^b$  the subgraph of  $G_{14} \times C_n$  induced on the edges incident with the vertices of the cycle  $C_3^b$  and by  $J$  the subgraph consisting of the vertices of  $C_3^q$  and  $C_3^a$  and the edges joining the cycles  $C_3^q$  and  $C_3^a$ . As shown above,  $cr(G_{14} \times C_3) \leq 9$  and, to prove Theorem 3, it is necessary to show that  $cr(G_{14} \times C_3) \geq 9$ .

Assume that there is a drawing of the graph  $G_{14} \times C_3$  with less than nine crossings and let  $D$  be such a drawing. In  $D$  there is at most one crossing on the edges of  $I$ , because deleting the edges of  $I$  results in the graph  $G_{13} \times C_3$  with crossing number seven. The same argument gives that there is at most one crossing on the edges of  $J$ . As  $G_{14} \times C_3$  contains several subgraphs isomorphic or homeomorphic to the graph  $(K_4 - e) \times C_3$  with crossing number six, we have the next restrictions: every of the subgraphs  $T^b$ ,  $T^c$ ,  $C_3^p \cup I$  and  $C_3^a \cup J$  has in  $D$  at most two crossings on its edges. Moreover, since deleting the edges of  $C_3^q \cup I \cup J$  results in the graph homeomorphic to  $C_4 \times C_3$  with crossing number four, on the edges of  $C_3^q \cup I \cup J$  there are at most four crossings. Using the same arguments as in the proof of Theorem 2, one can easily see that  $cr_D(I) = cr_D(J) = 0$ , otherwise  $T^b$  or  $T^c$  has its edges crossed more than two times. The cycles  $C_3^a$  and  $C_3^p$  do not cross each other, otherwise, as we noted in the proof of Theorem 2, at least one of them is crossed more than two times.

We show that  $cr_D(C_3^p, C_3^q) = cr_D(C_3^a, C_3^q) = 0$ . Without loss of generality, let the cycles  $C_3^p$  and  $C_3^q$  cross each other. In the good drawing  $D$ , at least one of the considered cycles separates the vertices of the other. Since the edges of  $C_3^p$  are crossed at most two times,  $C_3^p$  cannot separate two vertices of  $C_3^q$ . Thus, the cycle  $C_3^q$  separates two vertices, say  $p_i$  and  $p_j$ , of the cycle  $C_3^p$ . In this case, both paths  $p_i c_i c_j p_j$  and  $p_i a_i a_j p_j$  cross

$C_3^q$ , and  $C_3^q$  has at least four crossings. Thus,  $cr_D(C_3^a, C_3^q) = 0$ . The subgraph induced on the vertices of  $C_3^a$ ,  $C_3^q$  and  $C_3^b$  is isomorphic to the graph  $C_3 \times C_3$  with crossing number three. As  $T^b$  has in  $D$  at most two crossings,  $cr_D(C_3^a \cup J \cup C_3^q) \geq 1$ . But in this case, since  $cr_D(C_3^a, C_3^q) = 0$ , an edge of  $J$  crosses an edge of  $C_3^a \cup C_3^q$ . This contradiction with the property that on the edges of  $C_3^q \cup I \cup J$  there are at most four crossings implies that  $cr_D(C_3^p, C_3^q) = 0$  and, by the symmetric of the graph  $G_{14}$ ,  $cr_D(C_3^a, C_3^q) = 0$  also.

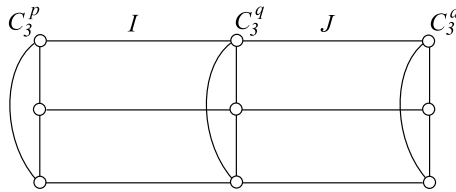


Figure 6

Consider now the subdrawing  $D'$  of the drawing  $D$  induced by the subgraph  $C_3^p \cup I \cup C_3^q \cup J \cup C_3^a$ . As shown above,  $cr_D(C_3^a \cup J \cup C_3^q) \geq 1$  and an edge of  $J$  crosses an edge of  $C_3^a \cup C_3^q$ . Using the same arguments for the subgraph induced on the vertices of  $C_3^p$ ,  $C_3^q$  and  $C_3^c$ , we have that an edge of  $I$  crosses an edge of  $C_3^p \cup C_3^q$ . Thus, in  $D'$  there are exactly two crossings. Suppose that both crossings appear on the edges of  $C_3^q$ . Then deleting the edges of  $C_3^q$  from  $D'$  results in the subdrawing without crossings as shown in Figure 3(a), and one can easily verify that the edges of  $C_3^q$  cannot cross the edges of  $I \cup J$  two times. In  $D'$  the cycle  $C_3^q$  does not separate  $C_3^p$  and  $C_3^a$ , otherwise  $T^c$  crosses  $C_3^q$  more than two times. Hence, in  $D'$  the cycle  $C_3^q$  crosses  $I$  and  $C_3^a$  crosses  $J$  or  $C_3^q$  crosses  $J$  and  $C_3^p$  crosses  $I$ . There is no other crossing in  $D'$ . Without loss of generality, assume the first situation. The unique drawing up to the isomorphism is shown in Figure 6. As in  $D$  the cycle  $C_3^c$  cannot cross an edge of the subdrawing  $D'$ , it lies in  $D$  in one of regions in the view of the subdrawing  $D'$ . If  $C_3^c$  lies in a region with less than four vertices of  $C_3^p \cup C_3^q$  on its boundary, then  $T^c$  crosses  $D'$  more than two times, a contradiction. If  $C_3^c$  lies in the unique region of  $D'$  with more than three vertices of  $C_3^p \cup C_3^q$  on its boundary — the unbounded region in Figure 6 — the edge  $\{c_i, q_i\}$  crosses the cycle  $C_3^p$  at least two times or it crosses the cycle  $C_3^a$  at least two times. This contradiction with the restriction that

every of  $C_3^p \cup I$  and  $C_3^a \cup J$  has at most two crossings on its edges completes the proof. ■

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