Discussiones Mathematicae
Graph Theory 25 (2005) 121–128

GRAPH DOMINATION IN DISTANCE TWO *

Gábor Bacsó[1]

Attila Tálos[2]

Zsolt Tuza[1,3]

[1] *Computer and Automation Institute*
Hungarian Academy of Sciences
H–1111 Budapest, Kende u. 13–17, Hungary

[2] *Eötvös Lóránd University*
H–1088 Budapest, Múzeum krt. 6–8, Hungary

[3] *Department of Computer Science*
University of Veszprém
H–8200 Veszprém, Egyetem u. 10, Hungary

Abstract

Let $G = (V, E)$ be a graph, and $k \geq 1$ an integer. A subgraph D is said to be *k-dominating* in G if every vertex of $G - D$ is at distance at most k from some vertex of D. For a given class $\mathcal{D}$ of graphs, $Dom_k\mathcal{D}$ is the set of those graphs G in which every connected induced subgraph H has some k-dominating induced subgraph $D \in \mathcal{D}$ which is also connected. In our notation, $Dom\mathcal{D}$ coincides with $Dom_1\mathcal{D}$. In this paper we prove that $DomDom\mathcal{D}_u = Dom_2\mathcal{D}_u$ holds for $\mathcal{D}_u = \{$all connected graphs without induced $P_u\}$ $(u \geq 2)$. (In particular, $\mathcal{D}_2 = \{K_1\}$ and $\mathcal{D}_3 = \{$all complete graphs$\}$.) Some negative examples are also given.

Keywords: graph, dominating set, connected domination, distance domination, forbidden induced subgraph.

2000 Mathematics Subject Classification: 05C69, 05C75, 05C12.

*Research supported in part by the OTKA research fund, grant T-032969.

1. Introduction

Though domination is a relatively young subfield of graph theory, it already has an extensive literature. It is also impressive how many other areas are related to it. For a detailed account on the subject, we refer to the recent book [8] and the earlier edited volume [10].

In several papers, e.g. [2, 3] and [5], we have dealt with the graph class $Dom\mathcal{D}$, connected-hereditarily dominated by the members of a given family $\mathcal{D}$ (see formal definition in the Abstract, or below). Generally we look for the characterization of $Dom\mathcal{D}$ in terms of forbidden induced subgraphs. In this setting, an already classic result of Wolk [9] on "trivially perfect" graphs (i.e., those having a dominating vertex in each connected induced subgraph) can be formulated as the following equation between graph classes:

$$Dom\{K_1\} = Forb\,(P_4, C_4)\,. \tag{1}$$

(As usual, P_k and C_k denote the path and the cycle on k vertices, respectively. For the formal definition of $Forb$, see the next subsection.)

Cozzens and Kelleher [6] and, independently and simultaneously, two of the present authors [2] characterized the existence of dominating cliques as follows:

$$Dom\{K_t \mid t \geq 1\} = Forb\,(P_5, C_5)\,. \tag{2}$$

Further equations of this form are proved in [4] and [5].

As regards domination in distance k, in [1] we showed the existence of special k-dominating sets in some classes of graphs, continuing the work begun in [7].

In [3] we proved the following identity, which contains both (1) and (2) as particular cases, and will also be applied in the proof of the main result of this paper:

$$Dom\,Forb\,(P_k) = Forb\,(P_{k+2}, C_{k+2}) \tag{3}$$

for every $k \geq 2$.

Here we investigate the iterated application of the operator 'Dom' and its relation to the operator 'Dom_2' acting at distance 2. We introduce the *Property* (*) as follows. We say that a class $\mathcal{D}$ of graphs has *Property* (*) if

$$DomDom\mathcal{D} = Dom_2\mathcal{D}.$$

We shall prove that this equality does hold for infinitely many classes of graphs $\mathcal{D}$, namely for $\mathcal{D} = \mathcal{D}_u = Forb(P_u)$, for every $u \geq 2$.

Though Dom_2 is closely related to the iteration of *Dom*, *Property* (*) does not hold for every family $\mathcal{D}$. An infinite sequence of negative examples is presented in the last section, with two further positive examples. They lead to the following questions that remain open:

Problem 1. Characterize the families $\mathcal{D}$ satisfying *Property* (*).

Problem 2. More generally, characterize those classes $\mathcal{D}$ for which the k times iterated $Dom \cdots Dom\mathcal{D}$ coincides with $Dom_k\mathcal{D}$.

Definitions and notation

All graphs considered in this paper are assumed to be finite.

Let $k \geq 1$ be an integer. The set X of vertices — and the subgraph induced by X — is k-*dominating* in the graph $G = (V, E)$ if for every vertex $y \in V - X$ there exists some $x \in X$ such that the distance of x and y in G is at most k. For a given class $\mathcal{D}$ of graphs, $Dom_k\mathcal{D}$ denotes the set of those graphs G in which every connected induced subgraph H has some k-dominating induced subgraph $D \in \mathcal{D}$ which is also connected.

Remarks

1. The latter condition on connectedness was not necessary in other works on the *Dom* operator. *Dom* also generates disconnected graphs, however, and those would make the iteration *DomDom* meaningless without the additional assumption.

2. The term 'k-dominating set' is sometimes used in a different meaning in the literature. In our context it is a shorthand for the more complicated phrase 'set dominating at distance k.'

3. In our notation, $Dom\mathcal{D}$ coincides with $Dom_1\mathcal{D}$.

4. Every class $\mathcal{D}$ of graphs satisfies

$$DomDom\mathcal{D} \subseteq Dom_2\mathcal{D},$$

because every connected $G \in DomDom\mathcal{D}$ contains a dominating $H \in Dom\mathcal{D}$ which is connected, and this H is dominated by some connected $J \in \mathcal{D}$; i.e., this $J \in \mathcal{D}$ is a 2-dominating subgraph of G.

This relation extends to k-times iteration, too: $Dom \cdots Dom\mathcal{D} \subseteq Dom_k\mathcal{D}$ for every $\mathcal{D}$ and every $k \geq 2$.
We say that G is *H-free* if G does not contain H as an *induced* subgraph. For a family $\mathcal{H}$ of graphs, $Forb(\mathcal{H})$ will denote the class of graphs that are H-free for all $H \in \mathcal{H}$. If the members of $\mathcal{H} = \{H_1, H_2, \ldots\}$ are explicitly given, we write $Forb(H_1, H_2, \ldots)$ instead of $Forb(\{H_1, H_2, \ldots\})$.

We shall need two more definitions. For two disjoint sets X and Y of vertices in a graph $G = (V, E)$, X *dominates* Y if for every vertex $y \in Y$, there exists some $x \in X$ with $xy \in E(G)$. If x is the unique neighbor of y in X, we say that y is a *private neighbor* of x with respect to X.

2. Two Useful Lemmas

The following simple statement was already applied in [3].

Lemma 1. *Suppose that G is a C_t-free and P_t-free graph. If D is a dominating connected induced subgraph of G such that D is minimal under these properties, then D is P_{t-2}-free.*

The next assertion will play an important role in the proofs later.

Lemma 2. *Let S and T be connected graphs, and let T have minimum degree at least 2. Let G be a connected, $F(T)$-free graph in which every minimal connected dominating induced subgraph is S-free. Then G has a dominating connected induced subgraph which is S-free and T-free.*

Proof. We take a minimal connected dominating subgraph D which has as few induced subgraphs isomorphic to T as possible. This D is S-free by the conditions, and if it has no T subgraph, then we are done. Suppose for a contradiction that it has an induced T with non-cutting points $v_1, v_2, \ldots, v_t$. We assume that in this sequence the non-cutting vertices of D are listed first; i.e., for some $0 \leq s \leq t$, $D - v_i$ is connected if $1 \leq i \leq s$ and disconnected if $s < i \leq t$. We claim that if there is a subgraph $\Delta_i = D \cup \{u_j \mid 1 \leq j \leq i\}$ of G, with some $i < s$, such that

— for all $1 \leq j \leq i$, the vertex u_j is not in D,

— the only neighbor of u_j in Δ_i is v_j, and

— Δ_i contains the minimum number of copies of T among all dominating connected induced subgraphs of G, then there exists a vertex u_{i+1} such that the same properties hold for subscript $i+1$ instead of i in the subgraph Δ_{i+1} induced by $\Delta_i \cup \{u_{i+1}\}$.

(Actually, the last condition on the number of T subgraphs holds automatically, because the insertion of pendant vertices cannot create new subgraphs of minimum degree greater than one. The condition is listed here to make it more transparent that the procedure works indeed.)

The construction is as follows:

Since $i < s$, v_{i+1} is a non-cutting vertex of D, neither of Δ_i. Then the graph $\Pi = \Delta_i - v_{i+1}$ is connected and contains fewer copies of T than Δ_i does. Hence, by the conditions above, Π is not dominating in G. Consequently, v_{i+1} has a private neighbor, namely there exists a vertex $u_{i+1} \notin \Delta_i$ such that the only neighbor of this vertex in Δ_i is v_{i+1}. That is, we have found the structure required.

Starting this process with $i = 0$ and $\Delta_0 = T$, after s steps an induced subgraph Δ_s is constructed. We now observe that all the vertices v_j with $s < j \leq t$ are cutpoints of Δ_s, and the entire $T - v_j$ (which is connected) together with all the u_i $(i \leq s)$ belongs to the same component of $\Delta_s - v_j$. For each such j, we denote by u_j an arbitrarily selected neighbor of v_j in a connected component of $\Delta_s - v_j$ not containing $T - v_j$.

It only remains to observe that the subgraph of G induced by $T \cup \{u_i \mid 1 \leq i \leq t\}$ is isomorphic to $F(T)$, and this contradiction proves the Lemma. ■

3. Dominating Subgraphs Without Long Induced Paths

Let the class of graphs $\mathcal{D}_u$ consist of all the P_u-free graphs $(u \geq 2)$.

Theorem 1. *The class $\mathcal{D}_u$ satisfies Property $(*)$.*

Proof. We will use the notation $s := u + 2$, $t := u + 4$. By the remark after the definition of *Property* $(*)$, we need to show that $Dom_2\mathcal{D}_u \subseteq DomDom\mathcal{D}_u$. The class $Dom_2\mathcal{D}_u$ is closed under induced subgraphs, thus it is enough to prove for every $G \in Dom_2\mathcal{D}_u$ that G has a dominating connected induced subgraph being in $Dom\mathcal{D}_u$. Using the notation above, we see that G is $F(C_s)$-free, C_t-free, and P_t-free, since in these graphs there is no 2-dominating subgraph being in $\mathcal{D}_u$. Let us recall here Equation (3) from above:

$$Dom\mathcal{D}_u = Forb\,(P_s, C_s)\,. \tag{3}$$

Based on this equality, it will suffice to prove that G has a dominating connected induced subgraph being in $Forb(P_s, C_s)$.
By Lemma 1, every minimal dominating connected induced subgraph of G is P_s-free. Since G is $F(C_s)$-free, the conditions of Lemma 2 are fulfilled for $S = P_s$ and $T = C_s$. Thus, G has a dominating connected induced subgraph $H \in Forb(P_s, C_s)$, and the theorem follows. ■

Let us mention two interesting particular cases. If we apply Theorem 1 for $u = 2$, we obtain the following:

Corollary 1. *The one-element set* $\{K_1\}$ *satisfies Property* $(*)$.

For $u = 3$, we get

Corollary 2. *The set of all cliques has Property* $(*)$.

From the proofs above, the following characterization is also obtained:

Theorem 2. *For every* $u \geq 2$,
$Dom_2\mathcal{D}_u = DomDom\mathcal{D}_u = Forb(P_{u+4}, C_{u+4}, F(C_{u+2}))$.

Proof. Denoting $\mathcal{H} = \{P_{u+4}, C_{u+4}, F(C_{u+2})\}$, the following sequence of graph class containments can be extracted from Remark 4 and from the proof of Theorem 1:

$$Forb\,(\mathcal{H}) \subseteq DomDom\mathcal{D}_u \subseteq Dom_2\mathcal{D}_u \subseteq Forb\,(\mathcal{H})\ .$$

Thus, equality must hold throughout. ■

It is worth mentioning that the following characterization (concerning the 'standard' domination at distance one) has also been derived along the way:

Theorem 3. *For the family* $\mathcal{D} = \mathcal{D}(t) = Forb(P_t, C_t)$,

$$Dom\mathcal{D} = Forb\,(P_{t+2}, C_{t+2}, F(C_t))$$

for each $t \geq 3$. ■

4. Dominating Subgraphs of Bounded Diameter

Let $k \geq 0$ be an integer, and let $\mathcal{D}'_k$ denote the family of all graphs of diameter at most k. This $\mathcal{D}'_k$ is of interest because, depending on the value of k, it provides both positive and negative examples for *Property* $(*)$.

Proposition 1. *For $k \geq 5$, the family $\mathcal{D}'_k$ does not have Property $(*)$.*

Proof. The path P_{k+4} is not dominated by any subgraph of diameter at most k, i.e., no member of $Dom\mathcal{D}'_k$ may contain an induced P_{k+4}. However, every dominating connected subgraph of C_{k+6} contains an induced P_{k+4}, thus, $C_{k+6} \notin DomDom\mathcal{D}'_k$.

On the other hand, if $k \geq 5$, then the cycle C_{k+6} has diameter $\lfloor k/2 \rfloor + 3 \leq k$. Hence, $C_{k+6} \in Dom_2\mathcal{D}'_k \setminus DomDom\mathcal{D}'_k$ for every $k \geq 5$, therefore the two graph classes cannot be the same. ■

For k small, just the opposite is true:

Theorem 4. *The family $\mathcal{D}'_k$ has Property $(*)$ for every $k \leq 4$.*

Proof. If $k = 0$ or $k = 1$, then within the class of *connected* graphs, the conditions 'having diameter k' and 'not containing an induced P_{k+2}' are equivalent; that is, on applying Theorem 1 we obtain

$$DomDom\mathcal{D}'_k = DomDom\mathcal{D}_{k+2} = Dom_2\mathcal{D}_{k+2} = Dom_2\mathcal{D}'_k \qquad \text{for } k = 0, 1\,.$$

Consider next $k = 2$. Clearly, $DomDom\mathcal{D}'_2 \subseteq Dom_2\mathcal{D}'_2 \subseteq Forb(P_8, C_8, F(C_6))$. Conversely, we have already seen that $Dom\mathcal{D}'_2 = Dom(Forb\, P_4) = Forb(P_6, C_6)$, from which $DomDom\mathcal{D}'_2 = Forb(P_8, C_8, F(C_6))$ follows by Theorem 3.

Finally, let $k = 3$ or $k = 4$. It has been proved in [4] that, for every $k \geq 3$, $Dom\mathcal{D}'_k$ coincides with $Forb(P_{k+4})$. On applying Equation (3), from this we obtain

$$(4) \qquad DomDom\mathcal{D}'_k = Forb\,(P_{k+6}, C_{k+6}) \qquad \text{for all } k \geq 3$$

Now, the inclusion relation in Remark 4, the fact that P_{k+6} and C_{k+6} are not 2-dominated by any subgraph of diameter at most k, and the characterization exhibited in Equation (4), together yield:

$$DomDom\mathcal{D}'_k \subseteq Dom_2\mathcal{D}'_k \subseteq Forb\,(P_{k+6}, C_{k+6}) = DomDom\mathcal{D}'_k \;\; \text{for } k = 3, 4$$

Consequently, equality must hold throughout. ■

In fact, our results allow us to go one step further:

Theorem 5. *For $k = 3$ and $k = 4$, $DomDomDom\mathcal{D}'_k = Dom_3\mathcal{D}'_k = Forb(P_{k+8}, C_{k+8}, F(C_{k+6}))$.*

Proof. Combine Remark 4, Equation (4), Theorem 3, and the fact that the graphs P_{k+8}, C_{k+8}, and $F(C_{k+6})$ do not belong to $Dom_3\mathcal{D}'_k$. ■

Note added in Proof. From some recent results of the authors, the equation $DomDom\mathcal{D} = Dom_2\mathcal{D}$ can be derived for further classes of graphs. Details will be given in a forthcoming paper.

References

[1] G. Bacsó and Zs. Tuza, *A characterization of graphs without long induced paths*, J. Graph Theory **14** (1990) 455–464.

[2] G. Bacsó and Zs. Tuza, *Dominating cliques in P_5-free graphs*, Periodica Math. Hungar. **21** (1990) 303–308.

[3] G. Bacsó and Zs. Tuza, *Domination properties and induced subgraphs*, Discrete Math. **1** (1993) 37–40.

[4] G. Bacsó and Zs. Tuza, *Dominating subgraphs of small diameter*, J. Combin. Inf. Syst. Sci. **22** (1997) 51–62.

[5] G. Bacsó and Zs. Tuza, *Structural domination in graphs*, Ars Combinatoria **63** (2002) 235–256.

[6] M.B. Cozzens and L.L. Kelleher, *Dominating cliques in graphs*, pp. 101–116 in [10].

[7] P. Erdős, M. Saks and V.T. Sós *Maximum induced trees in graphs*, J. Combin. Theory (B) **41** (1986) 61–79.

[8] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, N.Y., 1998).

[9] E.S. Wolk, *The comparability graph of a tree*, Proc. Amer. Nath. Soc. **3** (1962) 789–795.

[10] – Topics on Domination (R. Laskar and S. Hedetniemi, eds.), Annals of Discrete Math. **86** (1990).

Received 3 November 2003
Revised 17 November 2004