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PLANAR RAMSEY NUMBERS

Izolda Gorgol

Department of Applied Mathematics
Lublin University of Technology
Nadbystrzycka 38, 20–618 Lublin, Poland

e-mail: I.Gorgol@pollub.pl

Abstract

The planar Ramsey number $PR(G, H)$ is defined as the smallest integer n for which any 2-colouring of edges of K_n with red and blue, where red edges induce a planar graph, leads to either a red copy of G, or a blue H. In this note we study the weak induced version of the planar Ramsey number in the case when the second graph is complete.

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1. Introduction

The 2-colouring (say red and blue) of edges of any graph is said to be *planar* if the graph induced by the first (red) colour is planar. Let *the planar Ramsey number* $PR(G, H)$ be the smallest integer n such that any planar 2-colouring of K_n guarantees a red copy of G or a blue copy of H. This is the usual definition of the Ramsey number with the restriction to the set of allowed colourings. The planar Ramsey numbers were introduced independently by Walker [14] and Steinberg and Tovey [13]. They calculated all planar Ramsey numbers for pairs of complete graphs, and showed that they increase only linearly with the number of vertices.

Theorem 1 [13]**.**

(i) $PR(K_2, K_n) = n$; $PR(K_k, K_2) = k$, *for* $k \leq 4$,

(ii) *$PR(K_3, K_n) = 3n - 3$,*

(iii) *$PR(K_k, K_n) = 4n - 3$, for $k \geq 4$ and $(k, n) \neq (4, 2)$.*

We remark that to prove the above theorem the authors used very strong tools, namely the Four Colour Theorem [1, 2, 11] and the generalization of Grötzsch's Theorem [9] known as Grünbaum's Theorem [10]. Each of them describes deep structural properties of planar graphs. As an easy collorary from Theorem 1 we can formulate the following observation.

Proposition 1. *If $|V(G)| \geq 5$ and G is connected, then $PR(G, K_n) = 4n-3$.*

Proof. The upper bound follows from Theorem 1(iii). To get the lower bound we consider the graph K_{4n-4}, colour the edges of $(n-1)K_4$ red and the remaining edges blue. ■

2. Induced Planar Ramsey Numbers

The induced Ramsey number $IR(G, H)$ is the least n such that there exists a graph F on n vertices with the property that any 2-colouring of its edges with red and blue results in either a red copy of G induced in F, or an induced blue H. The existence of $IR(G, H)$ for each pair of graphs G and H was proved independently by Deuber [3], Erdős, Hajnal and Pósa [4] and Rödl [12]. One of the few known exact values of the induced Ramsey number is the following one.

Theorem 2 [5]**.** *For arbitrary $k \geq 1$ and $n \geq 2$ we have*

$$IR(K_{1,k}, K_n) = (k-1)\frac{n(n-1)}{2} + n.$$

A modification of this number was introduced in [7]. Consider an arbitrary 2-colouring of edges of a certain graph F. It partitions graph F into two monochromatic subgraphs: red F_r and blue F_b. If a graph G is induced in F_r then we say that G is *induced in red*. Similarly, if G is induced in F_b, we say that G is *induced in blue*. *The weak induced Ramsey number $IR_w(G, H)$* is the smallest integer n for which there exists a graph F_w on n vertices such that any 2-colouring of its edges with red and blue leads to either a copy of G induced in red, or a copy of H induced in blue. The existence of a graph F_w is a consequence of the fact that if a given monochromatic copy of a graph is induced in the graph then it is induced in its colour as well. Typically, the values of induced Ramsey numbers are very hard to find. Similarly as in the non-induced case we consider their planar versions. *The induced planar Ramsey number* $IPR(G, H)$ [*the weak induced planar Ramsey number* $IPR_w(G, H)$] is defined in the same way as $IR(G, H)$ [$IR_w(G, H)$], but in this case we allow only 2-colourings for which the subgraph induced by the first (red) colour is planar.

We show here that for each graph G containing a connected non-complete induced subgraph on at least three vertices we have $IPR_w(G, K_n) = 4n - 3$.

Theorem 3. *For arbitrary graph G and for arbitrary $n \geq 2$ we have* $IPR_w(G, K_n) \leq 4n - 3$.

Proof. The assertion is a strightforward consequence of the Four Colour Theorem. The complement of an arbitrary planar graph on $4n - 3$ vertices contains a complete graph on $\lceil \frac{4n-3}{4} \rceil = n$ vertices. So K_{4n-3} is the graph from the definition of the weak induced planar Ramsey number. ■

To show the opposite inequality we need some definitions and lemmas. Each of the graphs K_4, $K_3 \cup K_1$, $2K_2$, $K_2 \cup 2K_1$, $\overline{K}_4$ we call *a pseudoclique*. By *covering the graph G with pseudocliques* we mean a division of the vertex-set of the graph G into pairwise disjoint subsets $V_1, V_2, \ldots, V_t$ such that $V(G) = V_1 \cup V_2 \cup \cdots \cup V_t$ and $G[V_i]$ is a pseudoclique for $i = 1, 2, \ldots, t$.

Lemma 1. *Each graph on $4m$, $m \geq 1$, vertices containing a clique K_{3m+1} can be covered by a union of m disjoint pseudocliques.*

Proof. We use induction on m. The assertion is trivial for $m = 1$. Consider an arbitrary graph G on $4m$, $m \geq 2$, vertices containing a clique $K_{3m+1} = K$. Note that each vertex of $G \setminus K$ forms a pseudoclique K_4 or $K_3 \cup K_1$ together with certain three vertices of K. Fix a pseuduclique K^* isomorphic to K_4 consisting of a vertex of $G \setminus K$ and any three vertices of K. The graph $G \setminus K^*$ satisfies the induction hypothesis and so it can be covered by a union of $m - 1$ disjoint pseudocliques. This covering together with K^* gives the required covering of G. ■

Lemma 2. *Each graph on at least 18 vertices contains a pseudoclique.*

Proof. The assertion follows from the fact that $R(K_4, K_4) = 18$ [8]. ■

Lemma 3. *Each graph on $4n$, $n \geq 51$, vertices containing a clique K_{n+1} can be covered by a union of n disjoint pseudocliques.*

Proof. Consider an arbitrary graph G on $4n$, $n \geq 51$, vertices containing a clique $K_{n+1} = K$. Let $H = G \setminus K$. By Lemma 2 all but at most 17 vertices of H can be covered with disjoint pseudocliques. Let S, $|S| \leq 17$, be the set of vertices of H which are not covered and let F be the subgraph induced in G by $V(K) \cup S$. Certainly $|V(F)| = n + 1 + |S|$ is divisible by 4, so $|V(F)| = 4m$ for a certain integer m. If $n \geq 54 \geq 3|S| + 3$ then $n + 1 \geq 3m + 1$. It can be checked by hand that also for $n = 51, 52, 53$ the last inequality holds ($|S| = 16, 15, 14$ respectively). Therefore F fulfils the assumptions of Lemma 1, so it can be covered with disjoint pseudocliques. ∎

Theorem 4. *Let G be a graph containing a connected non-complete induced subgraph on at least three vertices. Then $IPR_w(G, K_n) = 4n - 3$ for $n \geq 52$.*

Proof. The upper bound follows from Theorem 3. Let F be an arbitrary graph on $4n - 4$ vertices. If F does not contain any clique K_n then we colour all edges of F blue, otherwise by Lemma 3, we can cover F with $n-1$ disjoint pseudocliques, colour the edges of them red and all the remaining edges blue. There is no G induced in red and no blue clique K_n in such a colouring, so $IPR_w(G, K_n) > 4n - 4$. ∎

It occurs that in most cases we can improve the bound $n \geq 52$ to $n \geq 3$. We need, however, the following lemma.

Lemma 4. *Let G be one of the graphs $K_4 - e$, $K_4 - P_3$, C_4, P_4, $K_{1,3}$. Then $IPR_w(G, K_3) > 8$.*

The proof of the lemma is somewhat technical and not very exciting so we refer the reader to [6].

Theorem 5. *Let G be a graph containing a connected non-complete induced subgraph on at least four vertices. Then $IPR_w(G, K_n) = 4n - 3$ for $n \geq 3$.*

Proof. Let F be an arbitrary graph on $4n-4$ vertices. We can assume that F contains K_4, otherwise we could colour the whole graph blue. We colour this clique K_4 red. Now we can assume that the rest of the graph contains K_4 ($n > 4$), otherwise we could use the blue colour on the uncoloured edges.

Analogously we can assume that F contains $(n-3)K_4$ and we colour all these cliques K_4 red. Now there are 8 vertices with all incident edges uncoloured. If G contains a component on at least 5 vertices then we colour red any two disjoint subgraphs on 4 vertices and the remaining edges blue. In other cases the assertion follows from Lemma 4. ■

In the above proof we actually reduce the colouring of a graph on $4n-4$ vertices to an appropriate colouring of a graph on 8 vertices. This method fails for the smallest non-complete graph, i.e., for the star $K_{1,2}$. From Theorem 2 it follows that $IR(K_{1,2}, K_3) = 6$. This implies that we could not be able to colour the remaining eight-vertex graph with no star $K_{1,2}$ induced in red and with no blue triangle. Theorem 2 gives an upper bound which is better than $4n-3$ for small n, i.e., $IPR_w(K_{1,2}, K_n) \leq \frac{n(n+1)}{2}$ for $n \leq 6$. It is easy to observe that actually $IPR_w(K_{1,2}, K_n) = \frac{n(n+1)}{2}$ for $n = 2, 3, 4$.

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References

[1] K. Appel and W. Haken, *Every planar map is four colourable. Part* I. *Discharging*, Illinois J. Math. **21** (1977) 429–490.

[2] K. Appel, W. Haken, and J. Koch, *Every planar map is four colourable. Part* II. *Reducibility*, Illinois J. Math. **21** (1977) 491–567.

[3] W. Deuber, *A generalization of Ramsey's theorem*, in: R. Rado, A. Hajnal and V. Sós, eds., Infinite and finite sets, vol. 10 (North-Holland, 1975) 323–332.

[4] P. Erdős, A. Hajnal and L. Pósa, *Strong embeddings of graphs into colored graphs*, in: R. Rado, A. Hajnal and V. Sós, eds., Infinite and finite sets, vol. 10 (North-Holland, 1975) 585–595.

[5] I. Gorgol, *A note on a triangle-free — complete graph induced Ramsey number*, Discrete Math. **235** (2001) 159–163.

[6] I. Gorgol, Planar and induced Ramsey numbers (Ph.D. thesis (in Polish), Adam Mickiewicz University Poznań, Poland, 2000) 51–57.

[7] I. Gorgol and T. Łuczak, *On induced Ramsey numbers*, Discrete Math. **251** (2002) 87–96.

[8] R.E. Greenwood and A.M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955) 1–7.

[9] H. Grötzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math. Natur. Reihe **8** (1958/1959) 109–120.

[10] B. Grünbaum, *Grötzsch's theorem on* 3*-colorings*, Michigan Math. J. **10** (1963) 303–310.

[11] N. Robertson, D. Sanders, P.D. Seymour and R. Thomas, *The four-colour theorem*, J. Combin. Theory (B) **70** (1997) 145–161.

[12] V. Rödl, A generalization of Ramsey theorem (Ph.D. thesis, Charles University, Prague, Czech Republic, 1973) 211–220.

[13] R. Steinberg and C.A. Tovey, *Planar Ramsey number*, J. Combin. Theory (B) **59** (1993) 288–296.

[14] K. Walker, *The analog of Ramsey numbers for planar graphs*, Bull. London Math. Soc. **1** (1969) 187–190.

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