COMBINATORIAL LEMMAS FOR POLYHEDRONS

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Abstract

We formulate general boundary conditions for a labelling to assure the existence of a balanced \( n \)-simplex in a triangulated polyhedron. Furthermore we prove a Knaster-Kuratowski-Mazurkiewicz type theorem for polyhedrons and generalize some theorems of Ichiishi and Idzik. We also formulate a necessary condition for a continuous function defined on a polyhedron to be an onto function.

Keywords: KKM covering, labelling, primoid, pseudomanifold, simplicial complex, Sperner lemma.

2000 Mathematics Subject Classification: 05B30, 47H10, 52A20, 54H25.
1. Preliminaries

By $N$ and $R$ we denote the set of natural numbers and reals, respectively. Let $n \in N$ and $V$ be a finite set of cardinality at least $n + 1$. $P(V)$ is the family of all subsets of $V$ and $P_n(V)$ is the family of all subsets of $V$ of cardinality $n + 1$. For $A \subset R^n$ $co A$ is the \textit{convex hull} of $A$ and $af A$ is the \textit{affine hull} of $A$ (the minimal affine subspace containing $A$). Let $ri Z$ and $bd Z$ be the \textit{relative interior} and the \textit{boundary} of a set $Z \subset R^n$, respectively.

By a polyhedron we understand the convex hull of a finite set of vertices of the polyhedron. Let $af P$ be a polyhedron of dimension $n$. Let $ri Z$ and $bd Z$ be the \textit{relative interior} and the \textit{boundary} of a set $Z \subset R^n$, respectively. The relative interior of the set $Z$ is considered with respect to the affine hull of $Z$. \textit{Dimension} of a set $A \subset R^n$ is the dimension of $af A$. If for some $A \subset R^n$ the dimension of $af A$ is $n - 1$, then $af A$ is called a \textit{hyperplane}. And if for a finite set $A = \{a_0, \ldots, a_m\} \subset R^n \ (m \in \{0, \ldots, n\})$ the dimension of $af A$ is equal to $m$, then $co A$ is called a \textit{simplex} (precisely an \textit{m-simplex}).

2. Polyhedrons

By a polyhedron we understand the convex hull of a finite set of $R^n$. Let $P \subset R^n$ be a polyhedron of dimension $n$. A \textit{face} of the polyhedron $P$ is the intersection of $P$ with some of its supporting hyperplane. Denote the set of all $k$-dimensional faces of the polyhedron $P$ by $F_k(P) \ (k \leq n)$ and the set of all vertices of the polyhedron $P$ by $V(P) \ (V(P) = F_0(P))$. The maximal dimension proper faces of the polyhedron $P$ are called \textit{facets}. Let $Tr_n$ be a family of $n$-simplexes such that $P = \bigcup_{\delta \in Tr_n} \delta$ and for any $\delta_1, \delta_2 \in Tr_n$, $\delta_1 \cap \delta_2$ is the empty set or their common face. A \textit{triangulation} of the polyhedron $P$ (we denote it by $Tr$) is a family of simplexes containing $Tr_n$ and fulfilling the following condition: any face of any simplex of $Tr$ also belongs to $Tr$. Let $Tr_n \ (m \in \{0, \ldots, n\})$ denote the family of $m$-simplexes belonging to a triangulation $Tr$. Hence $Tr = \bigcup_{n=0}^n Tr_n$. Let $V = Tr_0$ be the set of vertices of all simplexes of $Tr$. Notice, that $Tr_0 = \bigcup_{\delta \in Tr_n} V(\delta)$. An $(n - 1)$-simplex of $Tr_{n-1}$ is a \textit{boundary $(n - 1)$-simplex} if it is a facet of exactly one $n$-simplex of $Tr_n$.

Let $U$ be a finite set. An \textit{n-primoid} $L^U_n$ over $U$ is a nonempty family of subsets of $U$ of cardinality $n + 1$ fulfilling the following condition: for every set $T \in L^U_n$ and for any $u \in U$ there exists exactly one $u' \in T$ such that a set $T \setminus \{u'\} \cup \{u\} \in L^U_n$.

Each function $l : V \rightarrow U$ is called a \textit{labelling}. An $n$-simplex $\delta \in Tr_n$ is \textit{completely labelled} if $l(V(\delta)) \in L^U_n$ and an $(n - 1)$-simplex $\delta \in Tr_{n-1}$ is $x$-\textit{labelled} ($x \in U$) if $l(V(\delta)) \cup \{x\} \in L^U_n$. 
The following theorem is a special case of the theorem of Idzik and Junosza-Szaniawski formulated for geometric complexes. This theorem generalizes the well known Sperner lemma [9].

Theorem 2.1 (Theorem 6.1 in [3]). Let $\text{Tr}$ be a triangulation of an $n$-dimensional polyhedron $P \subset \mathbb{R}^n$, $V = \text{Tr}_0$, $L_U^n$ be an $n$-primoid over a set $U$ and $x \in U$ be a fixed element. Let $l : V \to U$ be a labelling. Then the number of completely labelled $n$-simplexes in $\text{Tr}$ is congruent to the number of boundary $x$-labelled $(n - 1)$-simplexes in $\text{Tr}$ modulo 2.

Let $U \subset \mathbb{R}^n$ be a finite set containing $V(P)$ and let $b \in \text{ri } P$ be a point, which is not a convex combination of fewer than $n + 1$ points of the set $U$. The family $L^b_n = \{ T \subset U : |T| = n + 1, \ b \in \text{co } T \}$ is a primoid over the set $U$ (see Example 3.6 in [3]). We say a $b$-balanced $n$-simplex instead of a completely labelled $n$-simplex if $L^b_U = L^b_n$. In the case $b = 0$ a $b$-balanced $n$-simplex is called a balanced $n$-simplex.

3. Main Theorem

Theorem 3.1. Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension $n$, $\text{Tr}$ be a triangulation of the polyhedron $P$, $V = \text{Tr}_0$. Let $U \subset \mathbb{R}^n$ be a finite set containing $V(P)$, let $b \in \text{ri } P$ be a point which is not a convex combination of fewer than $n + 1$ points of $U$ and let $l : V \to U$ be a labelling. If for every facet $F_{n-1}$ of the polyhedron $P$ we have $l(V \cap F_{n-1}) \subset F_{n-1}$, then the number of $b$-balanced $n$-simplexes in the triangulation $\text{Tr}$ is odd.

Remark 3.2. Notice that the condition $l(V \cap F_{n-1}) \subset F_{n-1}$ implies that for each lower dimensional face $F$ we have $l(V \cap F) \subset \bigcap_{F \subset F_{n-1} \in \mathcal{F}_{n-1}(P)} F_{n-1} = F$.

Proof of Theorem 3.1. We apply the induction with respect to dimension of the polyhedron $P$. If dimension of $P$ is equal to 1, then the theorem is obvious. Assume that the theorem is true for all polyhedrons of dimension $k$ ($k \in \mathbb{N}$). Consider a polyhedron $P$ of dimension $k + 1$. Choose a vertex of $P$ and denote it by $x$. Let $b'$ be a point different from $x$, lying on the boundary of $P$ and on the straight line passing through points $b$ and $x$. Let $F_{b'}$ be a face of $P$ containing $b'$. Observe that dimension of $F_{b'}$ is equal to $k$, because otherwise the point $b$ would be a convex combination of fewer than $(k + 1)+1$ points of $V(P)$. 

Let us count $x$-labeled $k$-simplexes on $bd P$. For any facet $F$ different from $F'_v$ there is no $x$-labeled $k$-simplex contained in $F$ since for all $\delta \in Tr^k \cap F$ $co(l(V(\delta))) \subset F$ and $b \notin co(\{x\} \cup V(F))$. Hence all $x$-labeled $k$-simplexes are contained in $F'_v$. Notice that a $k$-simplex $\delta \in Tr^k \cap F'_v$ is the $x$-labelled $k$-simplex if and only if $\delta$ is a $b'$-balanced $k$-simplex. Because of Remark 3.2 we may apply the induction assumption for $F'_v$ ($F'_v$ is considered as a subset of af $F'_v$) and the point $b'$. Therefore the number of $b'$-balanced $k$-simplexes on $F'_v$ is odd. Thus the number of boundary $x$-labeled $k$-simplexes in $Tr$ is odd and by Theorem the number of the $b$-balanced $(k + 1)$-simplexes in $Tr$ is odd.

Observe that for any polyhedron $Q$, triangulation $Tr'$ of $bd Q$ and a point $c \in ri Q$ the family $Tr = \{co(\{c\} \cup V(\delta)) : \delta \in Tr'\} \cup Tr' \cup \{c\}$ is a triangulation of the polyhedron $Q$.

For any $(n - 1)$-dimensional hyperplane $h^F_b$ containing the point $b$ and disjoint with a facet $F$ of the polyhedron $P$ let $H^F_b$ denote the open halfspace containing $F$ and such that $h^F_b$ is its boundary.

**Theorem 3.3.** Let $P \subset R^n$ be a polyhedron of dimension $n$, $Tr$ be a triangulation of the polyhedron $P$, $V = Tr_0$. Let $U \subset R^n$ be a finite set containing $V(P)$, let $b \in ri P$ be a point which is not a convex combination of fewer than $n + 1$ points of $U$ and let $l : V \to U$ be a labelling. If for every facet $F_{n-1}$ of the polyhedron $P$ there exists an $(n - 1)$-dimensional hyperplane $h^{F\cap_{n-1}}_b$ containing the point $b$ and disjoint with $F_{n-1}$ such that $l(V \cap F_{n-1}) \subset H^{F\cap_{n-1}}_b$, then the number of $b$-balanced $n$-simplexes in the triangulation $Tr P$ is odd.

**Proof.** For $n = 1$ the theorem is obvious, so we consider $n > 1$. Let $V(P) = \{a_0, \ldots, a_k\}$ ($k \geq n$). Let $a'_i = 2a_i - b$ for $i \in \{0, \ldots, k\}$ and let $P' = co\{a'_0, \ldots, a'_k\}$. Notice that $P \subset P'$.

Now we define a triangulation of $P'$, which is an extension of the triangulation $Tr$ on $P$. We will define a triangulation of $P' \setminus ri P$.

For every face $F = co\{a_{i(0)}, \ldots, a_{i(l)}\}$ ($\{a_{i(0)}, \ldots, a_{i(l)}\} \subset V(P)$) of the polyhedron $P$ we denote $F' = co\{a'_{i(0)}, \ldots, a'_{i(l)}\}$. Every face $F$ of $P$ has one-to-one correspondence to the face $F'$ of $P'$.

Let us denote $FF' = co\{F \cup F'\}$. Thus $P' \setminus ri P = \bigcup_{F \in F_{n-1}} FF'$.

For $n = 1$ the triangulation of $P'$ is trivial, so we may assume $n > 1$.

For any face $F_1 \in F_1(P)$ we choose a point $v_{F'_1} \in ri F'_1$ in such a way that the point $b$ is not a convex hull of less than $n + 1$ points of $U \cup \{v_{F'_1} :
$F_1 \in \mathcal{F}_1(P)$. We join $v_{F'_1}$ with every vertex of the face $F'_1$. Thus we receive triangulation of $F'_1$. We choose a point $v_{F'_1} \in \mathcal{F}_1(P)$ in such a way that the point $b$ is not a convex hull of less than $n + 1$ points of $U \cup \{v_{F'_1} : F \in \mathcal{F}_1(P)\}$. We join $v_{F'_1}$ with every vertex of the face $F'_1$, with the point $v_{F'_1}$ and with every vertex of $V \cap F_1$. Thus we receive triangulation of $F_1 F'_1$.

Now we apply the induction for $k \in \{2, \cdots, n - 1\}$: For any face $F_k \in \mathcal{F}_k(P)$ we choose a point $v_{F_k} \in \mathcal{F}_k F'_k$ in such a way that the point $b$ is not a convex hull of less than $n + 1$ points of $U \cup \bigcup_{i=1}^{k} \{v_{F'_i} : F \in \mathcal{F}_i(P)\} \cup \bigcup_{i=1}^{k-1} \{v_{FF'_i} : F \in \mathcal{F}_i(P)\}$. We join $v_{F_k}$ with every vertex of $F'_k$ and every point of the set $\bigcup_{F \subset F'_k} \{v_{FF'_i}\}$. Thus we get a triangulation of the face $F'_k$.

We choose a point $v_{F_k F'_k} \in \mathcal{F}_k F'_k$ in such a way that the point $b$ is not a convex hull of less than $n + 1$ points of $U \cup \bigcup_{i=1}^{k} \{v_{F'_i}, v_{FF'_i} : F \in \mathcal{F}_i(P)\}$. For each $F_k \in \mathcal{F}_k(P)$ we join the vertex $v_{F_k F'_k}$ with the vertex $v_{F'_i}$, with all the vertices of $V \cap F_k$, vertices of $F'_k$ and with the vertices of the set $\bigcup_{F \subset F'_k} \{v_{F'_i}, v_{FF'_i}\}$.

We get the triangulation of $P' \setminus \mathcal{R} P$ and we denote it by $Tr''$. Hence $Tr' = Tr \cup Tr''$ is a triangulation of $P'$, which is an extension of the triangulation $Tr$ on $P$.

Let $U' = U \cup \bigcup_{i=1}^{n-1} \{v_{F'_i}, v_{FF'_i} : F \in \mathcal{F}_i(P)\}$. Let $V' = Tr'$. We define a labelling $l' : V' \to U'$. Let $l'(v) = l(v)$ for $v \in V$ and $l(v) = v$ for $v \in V' \setminus V$. Notice that the labelling $l'$ satisfies conditions of Theorem 3.1. Thus there exists an odd number of $b$-balanced $n$-simplexes in $Tr'$. All $b$-balanced $n$-simplexes belong to $Tr$ since for any facet $F$ of $P$ we have $l'(V' \cap FF'_i) \subset H^F_b$, where $H^F_b$ is an open halfspace such that the point $b$ is on its boundary. 

In the proof of Theorems 3.1, 3.3 the condition: $b \in \mathcal{R} P$ is a point which is not a convex combination of fewer than $n + 1$ elements of $l(V)$ is essential. If we omit this condition we may still prove that there exists at least one $b$-balanced $n$-simplex (not necessarily an odd number of such $n$-simplexes). Related results were obtained by van der Laan, Talman and Yang [6, 7].

**Theorem 3.4.** Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension $n$, $Tr$ be a triangulation of the polyhedron $P$, $V = Tr_0$. Let $U \subset \mathbb{R}^n$ be a finite set, let $b \in \mathcal{R} P$ and let $l : V \to U$ be a labelling. If for every facet $F$ of the polyhedron $P$ there exists an $(n - 1)$-dimensional hyperplane $h^F_b$ containing the point $b$ and disjoint with $F$ such that $l(V \cap F) \subset H^F_b$, then there exists a $b$-balanced $n$-simplex in the triangulation $Tr$. 
Proof. Take a sequence of points \( b_k \), which converges to the point \( b \) and \( b_k \) is not a convex combination of fewer than \( n + 1 \) elements of \( l(V) \) for any \( k \in N \). For sufficiently large \( k \) we may assume that \( H_{b_k}^F \cap l(V \cap F) = H_{b_k}^F \cap l(V \cap F) \) for some \( (n - 1) \)-dimensional hyperplane \( H_{b_k}^F \) and every facet \( F \) of \( P \) and apply Theorem 3.3 to \( b_k \). Thus there exists a \( b_k \)-balanced \( n \)-simplex in \( T_r \) of the polyhedron \( P \). Since the points \( b_k \) converge to the point \( b \) and the set \( U \) is finite, then there exists at least one \( b \)-balanced \( n \)-simplex in \( T_r \).

Theorem 3.4 applied to the \( n \)-dimensional cube implies the Poincaré-Miranda theorem [5].

Theorem 3.5. Let \( P \) be an \( n \)-dimensional polyhedron, \( b \in ri P \) and \( U \subset R^n \) be a finite set containing \( V(P) \). Let \( \{ C_u \subset R^n : u \in U \} \) be a family of closed sets such that \( P \subset \bigcup_{u \in U} C_u \) and for every facet \( F_{n-1} \) of the polyhedron \( P \) there exists a hyperplane \( H_b^{F_{n-1}} \) containing \( b \) and disjoint with \( F_{n-1} \) such that for every face \( F \) of \( P \) we have \( F \subset \bigcup_{u \in U \cap H_b^{F_{n-1}}} C_u \), where \( H_b^F = \bigcap_{F \subset F_{n-1} \subset \bigcup_{u \in U \cap H_b^{F_{n-1}}}} H_b^{F_{n-1}} \). Then there exists \( T \subset U \), \( |T| = n + 1 \), such that \( b \in co T \) and \( \bigcap_{u \in T} C_u \neq \emptyset \).

Proof. Let \( T_r^k \) be a sequence of triangulations of \( P \) with the diameter of simplexes tending to zero, when \( k \) tends to infinity. Denote \( V_k = T_r^k \). We define a labelling \( l_k \) on the vertices \( V_k \) \( (k \in N) \) in the following way: for \( v \in V_k \) let \( l_k(v) = u \) for some \( u \in C_u \) and furthermore if \( v \in bd P \), then \( u \in \bigcap_{F_{n-1} \in F_{n-1} \subset \bigcup_{u \in U \cap H_b^{F_{n-1}}}} H_b^{F_{n-1}} \).

Since \( P \subset \bigcup_{u \in U} C_u \) and \( F \subset \bigcup_{u \in H_b^{F_{n-1}}} C_u \), then the labelling \( l_k \) is well defined and it satisfies the conditions of Theorem 3.4. Thus there exists a \( b \)-balanced \( n \)-simplex \( \delta_k \in T_r^k \). Let \( V(\delta_k) = \{ v_0^k, \ldots, v_n^k \} \). Hence for \( i \in \{0, \ldots, n\} \) \( v_i^k \in C_{l_k(v_i^k)} \). Because the diameter of simplexes of \( T_r^k \) tends to zero, there exists \( z \in P \) and a subsequence of \( v_i^k \) which converges to \( z \) for each \( i \in N \). Since \( C_u \) is a closed set for \( u \in U \) and \( U \) is a finite set, then \( z \in C_{l_i} \) for \( i \in \{0, \ldots, n\} \) and \( T = \{ t_0, \ldots, t_n \}, |T| = n + 1, b \in co T \) and thus \( \bigcap_{u \in T} C_u \neq \emptyset \).

Theorem 3.5 is a generalization of an earlier result of Ichiiishi and Idzik.

Theorem 3.6 (Theorem 3.1 in [1]). Let \( P \) be an \( n \)-dimensional polyhedron, \( b \in ri P \) and \( U \subset R^n \) be a finite set containing \( V(P) \). Let \( \{ C_u \subset R^n : u \in U \} \) be a family of closed sets such that \( P \subset \bigcup_{u \in U} C_u \) and \( F \subset \bigcup_{u \in U \cap F} C_u \) for every face \( F \) of the polyhedron \( P \). Then there exists \( T \subset U \), \( |T| = n + 1 \), such that \( b \in co T \) and \( \bigcap_{u \in T} C_u \neq \emptyset \).
Notice that the theorem of Ichiishi and Idzik is more general than the Knaster-Kuratowski-Mazurkiewicz covering lemma [4] and Shapley’s covering lemma (Theorem 7.3 in [8]).

The theorem below is related to Corollary 4.2 in [2].

**Theorem 3.7.** Let $P \subset \mathbb{R}^n$ be an $n$-dimensional polyhedron and $f : P \to \mathbb{R}^n$ be a continuous function. If for every facet $F$ of the polyhedron $P$ the set $f(F)$ is in the closed halfspace $H^F$, such that $\text{bd } H^F = \text{af } F$ and $P$ is not contained in $H^F$, then $P \subset f(P)$.

**Proof.** Let $b \in \text{ri } P$ be a fixed point. Let $Tr^k$ be a triangulation of the polyhedron $P$ with the diameter of simplexes tending to zero and with a set of vertices denoted by $V_k$ ($k \in \mathbb{N}$). We define a labelling $l_k : V_k \to \mathbb{R}^n$ by putting $l_k(v) = f(v)$ ($v \in V_k$, $k \in \mathbb{N}$). Notice that the labelling $l_k$ satisfies the conditions of Theorem 3.4 and there exists a $b$-balanced $n$-simplex in $Tr^k$. Denote this $n$-simplex by $\delta_k$. Without loss of generality we may assume that there exists $x \in P$ such that $x = \lim_{k \to \infty} x_k$ for every $x_k \in \delta_k$. Because $f$ is a continuous function and $b \in \text{co } f(V(\delta_k))$ we have $f(x) = b$.

We have proved that $\text{ri } P \subset f(P)$. Since the set $f(P)$ is closed, we have $P \subset f(P)$. \qed

**Acknowledgement**

We are indebted to the referee for many valuable comments.

**References**


Received 3 November 2003
Revised 21 March 2005