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ON DOMINATING THE CARTESIAN PRODUCT OF A GRAPH AND K_2

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Abstract

In this paper we consider the Cartesian product of an arbitrary graph and a complete graph of order two. Although an upper and lower bound for the domination number of this product follow easily from known results, we are interested in the graphs that actually attain these bounds. In each case, we provide an infinite class of graphs to show that the bound is sharp. The graphs that achieve the lower bound are of particular interest given the special nature of their dominating sets and are investigated further.

Keywords: domination; 2-packing, Cartesian product.

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1. Introduction

The study of domination in graphs, which apparently began in the 1800's with the problem of finding the minimum number of queens needed to cover a chessboard, has expanded in many directions. While it is difficult to compute the domination number of an arbitrary graph, a number of general bounds are known. See Chapter 2 of [6] for a summary. In most of these results a bound for the domination number is given in terms of other, more easily computed, invariants of the graph, such as its minimum degree, maximum degree, size, diameter, or girth, to name a few. Other efforts have been directed at finding, or at least bounding, the domination number when the graph is a product (Cartesian, categorical, etc.) of two graphs. The reader is referred to [8], especially Chapter 8 and Appendix A, and to Chapter 7 of [7].

A conjecture made by V.G. Vizing in 1968 has been the motivating force for much of the study of domination of Cartesian products.

Conjecture 1 ([12]). *For every pair of graphs G and H, the domination number of the Cartesian product of G and H is at least as large as the product of their domination numbers.*

In this paper we consider the Cartesian product of an arbitrary graph and a complete graph of order two. An upper and a lower bound for the domination number of this Cartesian product follow easily from previous work. We are interested in the graphs which assume either of these bounds, and we provide an infinite class of graphs to show that each bound is sharp. The graphs which achieve the lower bound turn out to be interesting in their own right.

2. Notation and Definitions

We consider only finite, simple, undirected graphs. The vertex set of a graph G will be denoted by $V(G)$ and its edge set by $E(G)$. The order of G, denoted by $|G|$, is the cardinality of $V(G)$. For a subset A of $V(G)$, $\langle A\rangle$ is the subgraph of G induced by A. The *open neighborhood* of $v \in V(G)$ is $N(v) = \{u \mid uv \in E(G)\}$, and the *open neighborhood* of a subset D of vertices is $N(D) = \bigcup_{v\in D} N(v)$. The respective *closed neighborhoods* are $N[v] = N(v) \cup \{v\}$ and $N[D] = N(D) \cup D$. For $X, Y \subseteq V(G)$ we say that X *dominates* Y if $Y \subseteq N[X]$. The set D is called a *dominating set* for G if D dominates $V(G)$. The minimum cardinality of a dominating set for G is the *domination number* of G and is denoted by $\gamma(G)$. We will refer to any dominating set of G having cardinality $\gamma(G)$ as a $\gamma(G)$-*set* or simply as a γ-set if the graph is clear from context. A set $S \subset V(G)$ is a 2-*packing* of G if $N[x] \cap N[y] = \emptyset$ for every pair of distinct vertices x and y in S. The 2-*packing number* of G, denoted by $P_2(G)$, is the maximum cardinality of a

2-packing in G. An equivalent way of defining a dominating set in G is that it must contain at least one vertex from every closed neighborhood of G. It follows immediately that $\gamma(G) \geq P_2(G)$.

The *Cartesian product* of two graphs G and H is the graph $G\Box H$ whose vertex set is the Cartesian product of the sets $V(G)$ and $V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G\Box H$ precisely when either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. If $u \in V(G)$, then the subgraph of $G\Box H$ induced by $\{(u, v) \mid v \in V(H)\}$ will be denoted by H_u. It is clear from the definition of the Cartesian product that $H_u \simeq H$. Similarly, G_v is the subgraph induced by $\{(u, v) \mid u \in V(G)\}$; it is isomorphic to G. We assume throughout that the vertex set of the complete graph K_n is $\{1, 2, \ldots, n\}$.

See [6] and [8] for any notation not defined here.

3. Bounds

Let G and H be arbitrary graphs. The following is a sample of some of the bounds that have been shown for $\gamma(G\Box H)$.

- (Vizing [11]) $\gamma(G\Box H) \leq \min\{\gamma(G)|H|, \gamma(H)|G|\}$;
- (Jacobson and Kinch [9]) $\gamma(G\Box H) \geq \max\{\frac{|H|}{\Delta(H)+1}\gamma(G), \frac{|G|}{\Delta(G)+1}\gamma(H)\}$;
- (Jacobson and Kinch [10]) $\gamma(G\Box H) \geq \max\{\gamma(G)P_2(H), \gamma(H)P_2(G)\}$;
- (El-Zahar and Pareek [2]) $\gamma(G\Box H) \geq \min\{|G|, |H|\}$;
- (Clark and Suen [1]) $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$.

In general, upper or lower bounds for $\gamma(G\Box H)$ that hold for every pair of graphs G and H seem to be difficult to derive. When conditions are imposed on one or both of the two graphs it is sometimes possible to establish bounds on $\gamma(G\Box H)$ that improve upon one of those listed above. For example, in [4] Hartnell and Rall showed that when one of G or H has a 2-packing with certain characteristics, then the domination number of the Cartesian product $G\Box H$ is actually larger than that conjectured by Vizing. In [5] upper and lower bounds for $\gamma(G\Box H)$ were proved when additional conditions were imposed on both graphs. For example, although Vizing's conjecture is known to hold if one of the graphs involved is a tree, in [5] the authors proved the following stronger lower bound in a special case.

Theorem 1 ([5])**.** *If T is any tree, then*

$$\gamma(T\Box T) \geq \gamma(T)\gamma(T) + (|T| - 2\gamma(T)).$$

We consider here the Cartesian product of an arbitrary graph and a complete graph of order two. The bounds obtained follow immediately from the first and third in the above list. Indeed, for any graph G, the lower bound of Jacobson and Kinch [10] yields

$$\gamma(G\Box K_2) \geq \gamma(G)P_2(K_2) = \gamma(G),$$

and Vizing's upper bound from [11] implies

$$\gamma(G\Box K_2) \leq \gamma(G)|K_2| = 2\gamma(G).$$

As an example of a graph that achieves the upper bound from above, let G be the graph in Figure 1. We will show that $\gamma(G\Box K_2) = 2\gamma(G)$. The domination number of G is three, and there are three cliques $C_1 = \{a, b, c\}$, $C_2 = \{d, e, u, v, w\}$ and $C_3 = \{x, y, z\}$ that partition $V(G)$.

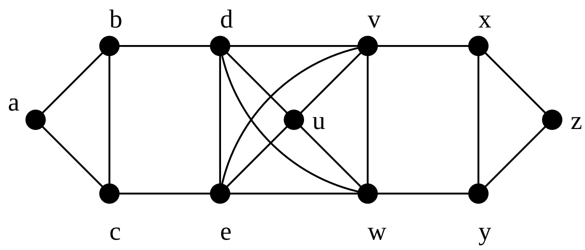

Figure 1: An example graph

Let D be any dominating set for $G\Box K_2$. If each of $|D\cap(C_i\times\{1,2\})|$, for $1 \leq i \leq 3$, is at least 2, then $|D| \geq 6$. So assume first that $|D\cap(C_1\times\{1,2\})| < 2$. Without loss of generality we may assume that $D\cap(C_1\times\{2\}) = \emptyset$. This implies that $(a,1) \in D$ and $(d,2),(e,2) \in D$. If, in addition, $|D\cap(C_3\times\{1,2\})| < 2$, then following the same reasoning as above we get $|D| \geq 6$. Therefore, assume that $|D\cap(C_3\times\{1,2\})| \geq 2$. But now, since D dominates $(u,1)$, it follows that either $(u,2) \in D$ or else $D\cap(C_2\times\{1\}) \neq \emptyset$, and hence $|D| \geq 6$.

Therefore, we assume that at least two vertices from $C_1 \times \{1,2\}$ and at least two vertices from $C_3 \times \{1,2\}$ belong to D. The set D dominates $\{(u,1),(u,2)\}$. If both of $(u,1)$ and $(u,2)$ belong to D, then $|D| \geq 6$. If only one of them, say $(u,2)$, is in D, then either $D\cap(C_2\times\{1\}) \neq \emptyset$, which implies $|D| \geq 6$, or $\{(b,1),(c,1),(x,1),(y,1)\} \subseteq D$. But $(a,2) \in N[D]$, and so in the latter case it follows that $|D \cap (C_1 \times \{1\})| \geq 3$ or $D \cap (C_1 \times \{2\}) \neq \emptyset$. Finally, if neither of $(u,1)$ nor $(u,2)$ is in D, then $|D \cap (C_2 \times \{1,2\})| \geq 2$. Hence $|D| \geq 6$ and we have shown that $\gamma(G \Box K_2) \geq 6$.

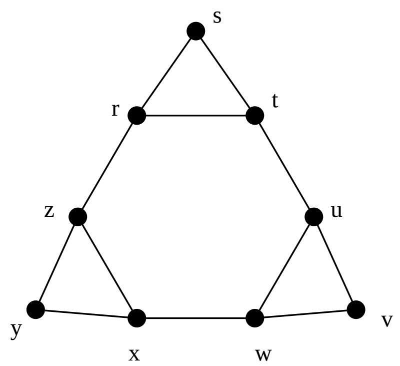

Figure 2: A second example

One property used repeatedly in the previous example is that the vertex set is partitioned into $\gamma(G)$ cliques, and each of these cliques contains a vertex that cannot be dominated from outside its own clique. This by itself is not enough to force $\gamma(G \Box K_2)$ to be $2\gamma(G)$. This can be seen by considering the graph H in Figure 2.

As with the graph in Figure 1, the vertex set of H partitions into $\gamma(H) = 3$ cliques, $C_1 = \{r,s,t\}$, $C_2 = \{u,v,w\}$ and $C_3 = \{x,y,z\}$. In addition, each of these cliques contains a vertex that cannot be dominated from outside its respective clique. However, the set $\{(r,1),(w,1),(y,2),(r,2),(w,2)\}$ dominates $H \Box K_2$, and thus $\gamma(H \Box K_2) < 2\gamma(H)$.

We now give a property of a graph G that is sufficient, although not necessary, to force $\gamma(G \Box K_2)$ to be $2\gamma(G)$. For an integer $k \geq 2$ we say a graph G satisfies *Property* $\mathcal{P}_k$ if $\gamma(G) = k$ and $V(G)$ can be partitioned into k cliques $C_1, C_2, \ldots, C_k$ in such a way that the following two conditions are satisfied:

- For each $1 \leq i \leq k$, the clique C_i contains a vertex h_i such that $N[h_i] \subseteq C_i$; and
- For every pair of disjoint subsets I and J of $\{1,2,\ldots,k\}$, if $S \subseteq \bigcup_{i \in I} C_i$ and S dominates $\bigcup_{j \in J}(C_j - \{h_j\})$, then $|S| \geq |I| + |J|$.

Theorem 2. *Let G be a graph that satisfies Property $\mathcal{P}_k$. If H is any spanning subgraph of G such that $\gamma(H) = \gamma(G)$, then $\gamma(H \Box K_2) = 2\gamma(H)$.*

Proof. Assume H and G are as stated in the theorem. We first show that any dominating set of $G\Box K_2$ contains at least $2k$ vertices. Let D be any dominating set of $G\Box K_2$, and for $i = 1, 2$, let $D_i = D \cap V(G_i)$. For ease of reference we think of the vertices of $G\Box K_2$ being laid out in two horizontal rows, corresponding to the vertices of K_2. We will say that a vertex (x, i) or a set $C_t \times \{i\}$ is dominated "horizontally" if it is contained in the closed neighborhood of D_i. So, for example, if a vertex $(x, 1)$ is not dominated horizontally by D_1, then it follows that $(x, 2) \in D$.

If $|(C_t \times \{i\}) \cap D| \geq 1$ for $1 \leq t \leq k$ and for $1 \leq i \leq 2$, then $|D| \geq 2k = 2\gamma(G)$. Hence, we assume this is not the case. For $i = 1, 2$ let A_i be the set of all cliques, C_t, such that $C_t \times \{i\}$ is not dominated horizontally. Note that for $i = 1$ (respectively, $i = 2$) if $C_t \in A_i$, then $(h_t, 2)$ (respectively, $(h_t, 1)$) is in D. From the set A_i we single out two subsets:

$A_i' = \{C_t \in A_i \mid (h_t, i)$ is the only vertex of $C_t \times \{i\}$ not dominated horizontally$\}$; and

$A_i'' = \{C_j \in A_i \mid$ for some $x \in C_j, x \neq h_j$, the vertex (x, i) is not dominated horizontally$\}$.

For $i = 1, 2$, we denote by R_i the set of cliques, C_r, such that for some vertex (y, i) in $(C_r \times \{i\}) \cap D$, there is a clique $C_t \in A_i'$ such that $y \in N_G(C_t)$. Let S_i be the set of all such vertices (y, i). Then by definition of A_i', the set S_i dominates $\bigcup_{C_t \in A_i'}((C_t \times \{i\}) - \{(h_t, i)\})$. Since G_i is isomorphic to G which satisfies Property $\mathcal{P}_k$, it follows that $|S_i| \geq |A_i'| + |R_i|$. Note that $(h_j, i) \notin S_i$ for any j. For each $C_m \in A_i''$ we see that $|(C_m \times \{j\}) \cap D| \geq 2$ for $j \in \{1, 2\} - \{i\}$. Hence we consider the vertex (h_m, j) as being "assigned" to $C_m \times \{i\}$ for counting purposes. Note that if $C_p \in A_2$, then the vertex $(h_p, 1)$ is in D, and so $C_p \notin A_1$ and $(h_p, 1) \notin S_1$. Also, since $(h_j, 2) \in D$ for every j such that $C_j \in A_1$, it follows that $(h_j, 2) \notin S_2$. This implies that $|D| \geq 2k = 2\gamma(G)$.

Since $H\Box K_2$ is a spanning subgraph of $G\Box K_2$, it follows that

$$\gamma(H\Box K_2) \geq \gamma(G\Box K_2) \geq 2k = 2\gamma(H). \qquad \blacksquare$$

We are now prepared to prove that the bounds of the following theorem are sharp. As we will show, those graphs that achieve the lower bound of the theorem possess minimum dominating sets that have special properties. We shall derive many of the structural properties of such graphs in Section 4.

Theorem 3 ([10], [11]). *Let G be any graph. Then $\gamma(G) \leq \gamma(G \square K_2) \leq 2\gamma(G)$, and these bounds are sharp.*

Proof. All that remains is to show that both bounds are achieved - infinitely often. We assume that $V(K_2) = \{1, 2\}$. For a positive integer $n \geq 3$, let G be the complete bipartite graph $K_{2,n}$, in which $D = \{u, v\}$ is the maximal independent set of order two. The set D is a minimum dominating set for G, and $G - N[u] = \{v\}$ while $G - N[v] = \{u\}$. Now it follows that $\{(u, 1), (v, 2)\}$ is a dominating set for $G \square K_2$, thus showing that the lower bound of the theorem is achieved infinitely often.

To show that the upper bound is assumed for an infinite class of graphs, we let H_k denote the graph obtained from the path P_k in the following way. Replace each of the $k - 2$ vertices of degree two by a clique of order five and each of the leaves by a clique of order three. Join each of two vertices of each clique of order five to a vertex of the clique preceeding it in "the path" and each of two other vertices to a clique following it in the path. One vertex from each of the k cliques thus has all of its neighbors entirely within its own clique. The graph in Figure 1 is H_3. The domination number of H_k is clearly k, and H_k satisfies Property $\mathcal{P}_k$. By Theorem 2 we see that $\gamma(H_k \square K_2) = 2\gamma(H_k)$. ■

The following theorem gives a characterization of the graphs that assume the lower bound of Theorem 3.

Theorem 4. *For a connected graph G, $\gamma(G \square K_2) = \gamma(G)$ if and only if G has a γ-set D that partitions into two nonempty subsets D_1 and D_2 such that $G - N[D_1] = D_2$ and $G - N[D_2] = D_1$.*

Proof. Assume that $\gamma(G \square K_2) = \gamma(G)$, and let A be a minimum dominating set for $G \square K_2$. Let $A_1 = A \cap V(G_1)$ and let $B = \{(u, 1) \in V(G_1) \mid (u, 1) \notin N[A_1]\}$. Then for every $(u, 1) \in B$ it must be the case that $(u, 2) \in A$, for otherwise the vertex $(u, 1)$ would not be dominated by A. Let $D_1 = \{v \in V(G) \mid (v, 1) \in A_1\}$ and let $D_2 = \{v \in V(G) \mid (v, 1) \in B\}$. By the way D_1 and D_2 are defined it follows that $G - N[D_1] = D_2$. Also, since the only vertices in G_2 that are dominated by A_1 are those whose first coordinate is in D_1, it is clear that D_2 dominates $V(G) - D_1$. Hence, $G - N[D_2] = D_1$. The set $D = D_1 \cup D_2$ is a minimum dominating set of G having the required properties.

Conversely, assume that G has a minimum dominating set D that partitions into two nonempty subsets D_1 and D_2 such that $G - N[D_1] = D_2$

and $G - N[D_2] = D_1$. It is straightforward to verify that the set $(D_1 \times \{1\}) \cup (D_2 \times \{2\})$ dominates $G \Box K_2$ and has cardinality $\gamma(G)$. ■

The results of Theorem 3 can be generalized to the case of a complete graph of order larger than two. For a general graph G the lower bound is somewhat more cumbersome because if n is large enough (specifically, if $n > |G| - \gamma(G) + 2$), then $V(G) \times \{1\}$ is a $\gamma(G \Box K_n)$-set. The lower bound is proved in Corollary 2.6 of [4], and the upper bound again follows from that of Vizing in [11].

Theorem 5 ([4], [11])**.** *Let G be any graph and let $n \geq 2$ be any positive integer. Then*

$$\min\{|G|, \gamma(G) + n - 2\} \leq \gamma(G \Box K_n) \leq n\gamma(G),$$

and these bounds are sharp.

Proof. As indicated above, the two inequalities follow from [4] and [11], respectively. Let t be an integer such that $t > (n-2)/2$. Let G_t be the graph with vertex set $V(G_t) = \{r, a_1, \ldots, a_t, b_1, \ldots, b_t, c_1, \ldots, c_t\}$ and edge set determined by the 4-cycles, r, a_i, b_i, c_i, r, $1 \leq i \leq t$, sharing a common vertex r. It is clear that $\gamma(G_t) = t + 1$ and that $\{(r,1), (r,2), \ldots, (r, n-1), (b_1, n), (b_2, n), \ldots, (b_t, n)\}$ dominates $G_t \Box K_n$. Hence $\gamma(G_t \Box K_n) = \gamma(G_t) + n - 2$. For graphs that assume the upper bound we will modify the definition of Property $\mathcal{P}_k$. Specifically, in the first condition of that definition we require that each clique C_i have $n-1$ distinct vertices, each having the property that all of its neighbors are inside C_i. Following a similar argument as that in the proof of Theorem 2, it can be seen that any graph that is a spanning subgraph of, and has the same domination number as, a graph satisfying this generalized Property $\mathcal{P}_k$ achieves the upper bound of this theorem. ■

4. Graphs with Two-Colored γ-Sets

Let $G = (V, E)$ be a connected graph. We say that G has a *two-colored* γ-set if some minimum dominating set D of G partitions into disjoint subsets R and B such that $G - N[R] = B$ and $G - N[B] = R$. For convenience we refer to R and B as the *parts* of D and to vertices in R as *red* vertices and those in B as *blue* vertices. In addition, we let $X = V - (R \cup B)$.

As an example of such a graph consider the one in Figure 3 which is obtained by deleting the edges of three vertex-disjoint 4-cycles from the complete bipartite graph $K_{6,6}$. The domination number of this graph is four and we may let $R = \{1, 7\}$ and $B = \{2, 8\}$. The set $R \cup B$ is a minimum dominating set for this graph. For this particular graph the vertex set can be partitioned into two-colored γ-sets. Using the obvious suggestive notation, if $R_2 = \{3, 9\}$, $B_2 = \{4, 10\}$, $R_3 = \{5, 11\}$ and $B_3 = \{6, 12\}$, then $\{R \cup B, R_2 \cup B_2, R_3 \cup B_3\}$ is such a partition.

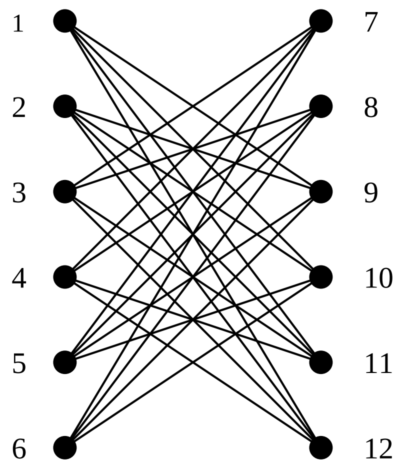

Figure 3: Bipartite graph with 2-colored γ-set

The example in Figure 4 shows that a graph can have a 2-colored γ-set whereas it may not be possible to partition its entire vertex set into 2-colored γ-sets. The set $D = \{u, v, w, x\}$ is a minimum dominating set for this graph and $\{u, v, w\}, \{x\}$ is the required partition of D.

Graphs with two-colored γ-sets were introduced in [3], where Hartnell and Rall gave a number of infinite classes of graphs that showed Vizing's conjecture, if true, is sharp. Many of these cases require one of the classes to contain graphs with vertex sets that can be partitioned into two-colored γ-sets. We are attempting to find a structural characterization of this class of graphs. The following propositions give some of the properties of any G with a two-colored γ-set.

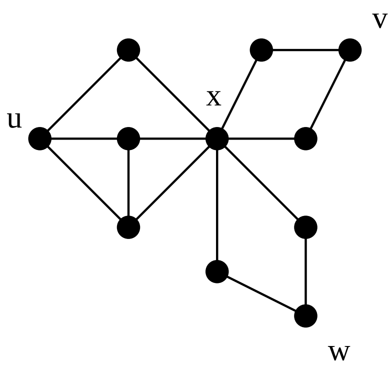

Figure 4: A non-partitionable example

Proposition 6. *Let G be a connected graph with a two-colored γ-set D having parts R and B. Then*

1. *Each of R and B is a 2-packing in G.*
2. *The set $R \cup B$ is independent in G.*
3. *The minimum degree of G is at least two.*
4. *Every vertex of G belongs to at least one minimum dominating set.*

Proof. Let $X = V(G) - (R \cup B)$ and assume that some vertex x of X is adjacent to two red vertices, say r_1 and r_2. Then $B \cup (R - \{r_1, r_2\}) \cup \{x\}$ dominates G and has cardinality $\gamma(G) - 1$. It follows from this contradiction that each vertex in X has at most one neighbor in R and similarly at most one in B. If two blue vertices, b_1 and b_2 were adjacent, then $R \cup (B - \{b_2\})$ would dominate G. Hence, B and R are both 2-packings. If there exist $r \in R$ and $b \in B$ such that $rb \in E(G)$, then $b \notin G - N[R]$, which contradicts our assumption about R and B. Hence $R \cup B$ is independent in G.

Note that by the proof of the first claim above, each vertex of X has exactly one red neighbor and exactly one blue neighbor. Suppose there is a red vertex r that has degree one, say $N(r) \cap X = \{u\}$. Since B dominates X, there is a blue vertex b that is adjacent to u. For any vertex $v \neq u$ in $N(b) \cap X$ it follows that $N(v) \cap (R - \{r\}) \neq \emptyset$ since R dominates X and $\deg(r) = 1$. Hence $((R \cup B) - \{r, b\}) \cup \{u\}$ dominates G, a contradiction. Hence $\deg(r) \geq 2$. Similarly, every vertex of B has degree at least two, and $\delta(G) \geq 2$. To prove the last claim let v be any vertex of X. Since $R \cup B$ is a two-colored γ-set, v has a red neighbor r and a blue neighbor b. It follows that $R \cup \{v\} \cup (B - \{b\})$ is a $\gamma(G)$-set containing v. ∎

Note in particular that every vertex in X has exactly one neighbor in each of R and B and every vertex of $R \cup B$ has at least two neighbors in X.

Proposition 7. *Let G be a connected graph with a two-colored γ-set D having parts R and B. If $r \in R$ and $b \in B$ have a common neighbor x, then r, b and x belong to a chordless 4-cycle. Hence, every vertex of G belongs to a chordless 4-cycle.*

Proof. If $N(r) \cap N(b) \cap X = \{x\}$, then $(R \cup B \cup \{x\}) - \{r, b\}$ dominates G. This contradicts our assumption that $\gamma(G) = |R \cup B|$. Therefore, if $A = N(r) \cap N(b)$, then $|A| \geq 2$. If x dominates $\langle A \rangle$, then again $(R \cup B \cup \{x\}) - \{r, b\}$ dominates G. Hence, there is a vertex $y \in A$ such that $xy \notin E(G)$, and so $\langle \{r, x, b, y\} \rangle$ is a chordless 4-cycle. ■

Proposition 8. *Let G be a connected graph with a two-colored γ-set D having parts R and B. If G has a path of order four consisting of vertices of degree two, then $G \simeq C_4$.*

Proof. Assume v, w, x and y are all of degree two such that $N(w) = \{v, x\}$, $N(x) = \{w, y\}$, $N(v) = \{u, w\}$ and $N(y) = \{x, z\}$. Since D dominates G, some vertex of $N[w]$ and some vertex of $N[x]$ belong to one of the parts. Without loss of generality, assume $v \in R$ and $x \in B$. By Proposition 7, v, w and x belong to a chordless 4-cycle, and since v, w, x and y all have degree two and G is connected, it follows that $G \simeq C_4$. ■

Proposition 9. *If G is not a 4-cycle but G has three vertices v, w and x of degree two that induce a subgraph isomorphic to P_3, then there is a vertex y such that y is a cut-vertex of G and $\langle \{v, w, x, y\} \rangle \simeq C_4$.*

Proof. Let $N(v) = \{u, w\}$, $N(w) = \{v, x\}$ and $N(x) = \{w, y\}$. If $v \in R \cup B$, then $x \in R \cup B$, and so by Proposition 7, v, w and x belong to a chordless 4-cycle, and it follows that $u = y$. Otherwise, $w \in R \cup B$, and since $\deg(v) = 2 = \deg(x)$ it follows that u and y also both belong to $R \cup B$. Again by Proposition 7 each of the sets $\{u, v, w\}$ and $\{w, x, y\}$ is part of a chordless 4-cycle. This implies that $u = y$, and the conclusion follows. ■

It is interesting at this point to note that no connected graph G has what might be called a three-colored γ-set. For suppose G is a graph that has a minimum dominating set D which is the disjoint union of sets R, B and W such that $G - N[R] = B \cup W$, $G - N[B] = R \cup W$ and $G - N[W] = R \cup B$.

Let $X = V(G) - (R \cup B \cup W)$. Let $x \in X$, $r \in N(x) \cap R$ and $b \in N(x) \cap B$. It can easily be checked that $(R \cup B \cup W \cup \{x\}) - \{r, b\}$ is a dominating set for G, and this contradicts the assumption that $D = R \cup B \cup W$ is a γ-set of G. Similarly, no graph can have a k-colored γ-set for $k > 3$.

5. Structure

Throughout this section we assume G is a graph such that $V(G)$ can be partitioned into two-colored γ-sets $D_1, D_2, \ldots, D_t$, where the parts of D_i are R_i and B_i. For ease of reference we say that such a graph is *partitionable.* It is easy to show that the 4-cycle is the smallest such graph. Recall from Section 4 that for each i, every vertex in $V(G) - (R_i \cup B_i)$ has exactly one neighbor in each of R_i and B_i. For each i, let $m_i = |R_i|$ and let $n_i = |B_i|$. Relabel them if necessary so that $m_i \leq n_i$ for every i, and such that $m_1 \leq m_2 \leq \cdots \leq m_t$. Assume $R_1 = \{u_1, u_2, \ldots, u_{m_1}\}$ and let $X_j = N(u_j) \cap (V(G) - D_1)$, for $1 \leq j \leq m_1$. Since B_2 is a 2-packing $|B_2 \cap X_j| \leq 1$ for every $1 \leq j \leq m_1$, and hence $m_1 \geq n_2 \geq n_3 \geq \cdots \geq n_t \geq m_1$. It follows immediately that $|R_1| = |B_1| = |R_i| = |B_i|$ for every i and that G is regular of degree $2(t-1)$. This establishes the following result.

Proposition 10. *Let G be a graph such that $V(G)$ can be partitioned into t two-colored γ-sets. Then G is regular of degree $2(t-1)$ and both parts of all the minimum dominating sets in the partition have the same cardinality.*

The vertex set of the graph in Figure 5 can be partitioned into two-colored γ-sets. Each of the parts of these minimum dominating sets has cardinality five and the graph is 6-regular. The vertices labelled "1" form R_1; those labelled "2" form B_1; those labelled "3" form R_2, etc. The labelling scheme is clear but not complete so as not to clutter the figure.

If G is bipartite satisfying the hypotheses of Proposition 10, we can say more about its structure.

Proposition 11. *Let G be a bipartite graph such that $V(G)$ can be partitioned into t two-colored γ-sets. Then G is regular of degree $2(t-1)$ and either G is a cycle of order four or both parts of all the sets in the partition have the same even cardinality.*

Proof. We use the notation set up before Proposition 10. All that remains to be proved is that if G is not C_4, then each part of a two-colored γ-set

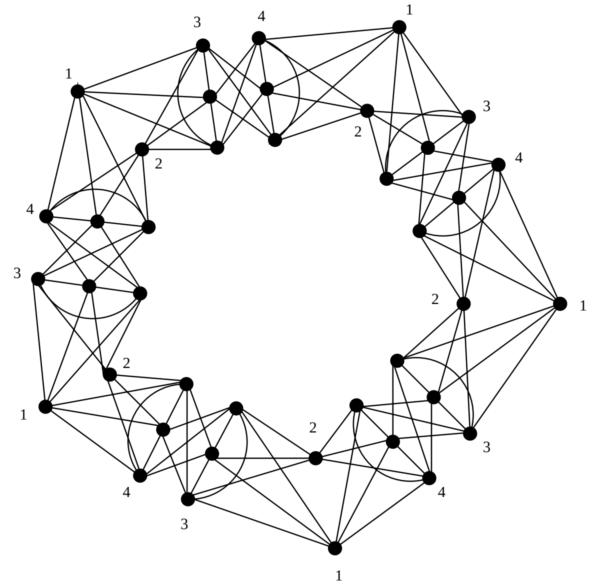

Figure 5: A partitionable example

in the partition contains an even number of vertices. Let C_1 and C_2 be the partite sets of G. Since G is regular by Proposition 10, $|C_1| = |C_2|$. Suppose $R_1 = \{u_1, u_2, \ldots, u_{m_1}\}$ and let $B_1 = \{v_1, v_2, \ldots, v_{m_1}\}$. Let $X = V(G) - D_1$ and for each i, let $X_i = N(u_i) \cap X$. Since G is bipartite, we conclude that for every i, either $X_i \subseteq C_1$ or $X_i \subseteq C_2$. Let x denote the number of values of i such that $X_i \subseteq C_1$ and let y denote the number of values of i such that $X_i \subseteq C_2$. If $x = 0$ or $y = 0$, then it follows that X is independent and $2(t-1) = 2$. In this case $G = C_4$. Otherwise, $|\bigcup_{X_i \subseteq C_1} X_i| = x2(t-1)$. By counting the edges between $\bigcup_{X_i \subseteq C_1} X_i$ and $\{v_j \mid v_j \in C_2\}$ in two different ways and using the fact that $|C_1| = |C_2|$ we conclude that $m_1 = 2x$. ■

We close with a conjecture concerning the structure of the partitionable, bipartite graphs.

Conjecture 2. *If G is a connected, bipartite graph such that $V(G)$ can be partitioned into two-colored γ-sets, then G is the 4-cycle or G can be obtained from $K_{2t,2t}$ by removing the edges of t vertex-disjoint 4-cycles.*

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