

## ANALOGUES OF CLIQUES FOR ORIENTED COLORING

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### Abstract

We examine subgraphs of oriented graphs in the context of oriented coloring that are analogous to cliques in traditional vertex coloring. Bounds on the sizes of these subgraphs are given for planar, outerplanar, and series-parallel graphs. In particular, the main result of the paper is that a planar graph cannot contain an induced subgraph  $D$  with more than 36 vertices such that each pair of vertices in  $D$  are joined by a directed path of length at most two.

**Keywords:** graph coloring, oriented coloring, clique, planar graph.

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## 1. Introduction

A *homomorphism* of a directed graph (digraph)  $D_1$  to a digraph  $D_2$  is a function  $f$  that maps the vertices of  $D_1$  to the vertices of  $D_2$  such that if  $xy$  is an arc of  $D_1$ , then  $f(x)f(y)$  is an arc of  $D_2$ . The existence of a homomorphism of  $D_1$  to  $D_2$  is denoted by  $D_1 \rightarrow D_2$ . If there exists a homomorphism  $D_1 \rightarrow D_2$ , then we say that  $D_1$  is *homomorphic* to  $D_2$ .

Let  $D$  be an oriented graph, i.e., a digraph in which at most one of the arcs  $uv, vu$  exists for each pair of vertices  $u, v$ . An *oriented  $k$ -coloring* of  $D$  is a homomorphism of  $D$  to some oriented graph  $T$  on  $k$  vertices. Since arcs joining non-adjacent vertices of  $T$  can be added without destroying the existence of a homomorphism  $D \rightarrow T$ , the oriented graph  $T$  can be taken to be a tournament. The *oriented chromatic number* of  $D$ , denoted  $\chi_o(D)$ , or simply  $\chi_o$ , is the smallest integer  $k$  such that there is an oriented  $k$ -coloring of  $D$ . From the definition of an oriented coloring of digraph  $D$ , if  $u \rightarrow v$  and  $w \rightarrow u$ , then each of  $u, v, w$  must receive distinct colors. In other words, adjacent vertices receive different colors and all arcs between vertices of color  $i$  and color  $j$ ,  $i \neq j$ , must be oriented in the same direction.

Recent results concerning oriented chromatic number of digraphs can be found in [2, 3, 4, 6]. Raspaud and Sopena proved that all oriented planar graphs have oriented chromatic number at most eighty [4]. It was shown in [8] that there exists an orientation of a planar graph with oriented chromatic number at least sixteen. Tightening this bound is a significant problem in the domain of oriented coloring.

Complete subgraphs (cliques) on  $k$  vertices obviously prevent an undirected graph from being  $(k - 1)$ -colored. Similarly, a subgraph that is a tournament on  $k$  vertices is a similar obstruction for oriented coloring. The objective of this paper is to study another fundamental obstruction for oriented coloring, which we shall term *ocliques*. Ocliques will be defined in Section 2. In Sections 3 through 5, we bound the maximum cardinalities of ocliques in outerplanar, planar, and series-parallel graphs.

## 2. Preliminaries

Recall that the famous Kuratowski/Wagner theorem states that a graph  $G$  is planar if and only if it contains neither a  $K_5$  minor nor a  $K_{3,3}$  minor [9]. Similarly, a graph is *outerplanar*, i.e., it can be embedded in the plane with all vertices bordering the outside face, if and only if it contains neither a  $K_4$  minor nor a  $K_{2,3}$  minor. A graph is *series-parallel* if and only if it contains no  $K_4$  minor.

Let  $P_n$  denote a path with  $n$  vertices. In a digraph, vertex  $x$  *dominates* vertex  $y$  if  $x \rightarrow y$  is an arc. We say that vertices  $x$  and  $y$  are *adjacent* if  $x$  dominates  $y$  or  $y$  dominates  $x$  and in either event we also say that  $x$  and  $y$  are neighbors. We say that vertex set  $A$  dominates vertex set  $B$  if, for each pair of vertices  $x \in A$ ,  $y \in B$ ,  $x$  dominates  $y$ .

Define an *oclique* to be an oriented graph  $D$  such that for any two distinct vertices  $x, y \in V(D)$  there exists a directed  $xy$  path of length at most two or a directed  $yx$  path of length at most two. Let  $\omega_o(D)$  denote the cardinality of the largest induced subgraph of  $D$  that is an oclique (i.e., the number of vertices it contains) in oriented graph  $D$ .

The next fact is implicit in Sopena [7].

**Fact 1.** *Let  $D$  be an oriented graph. Then  $\chi_o(D) \geq \omega_o(D)$ .*

The following results states that, in terms of the number of colors used, ocliques play the same role in oriented colorings as cliques do in colorings.

**Corollary 2.** *Let  $D$  be an oriented graph with  $n$  vertices. Then  $\chi_o(D) = n$  if and only if  $D$  is an oclique.*

**Proof.** Suppose  $D$  is an oclique. By Fact 1,  $\chi_o(D) \geq n$ . Since the identity mapping is a homomorphism  $D \rightarrow D$ , this implies  $\chi_o(D) = n$ .

Now suppose  $D$  is not an oclique. Then there exist nonadjacent vertices  $x$  and  $y$  that are not joined by a directed path of length two. Let  $D'$  be the digraph that results from identifying  $x$  and  $y$ , and creating the new vertex  $z$ . Since  $x$  and  $y$  are not joined by a directed path of length two,  $D'$  is an oriented graph. The mapping that sends  $x$  and  $y$  to  $z$  and every other vertex of  $D$  to its clone in  $D'$  is a homomorphism  $D \rightarrow D'$ . Thus,  $\chi_o(D) \leq n - 1$ . ■

The next proposition is straightforward.

**Proposition 3.** *If  $D$  is an oclique, then  $G[D]$  has diameter at most two.*

The following lemma is used several times. The proof is trivial and is omitted.

**Lemma 4.** *Let  $D$  be an oclique whose underlying graph has a cut vertex,  $v$ , whose deletion partitions  $G[D]$  into components  $C_1, C_2, \dots, C_k, k \geq 2$ . Then either  $V(C_i)$  dominates  $v$ , or  $v$  dominates  $V(C_i)$ , for all  $i, 1 \leq i \leq k$ .*

We use the term *2-path* to mean a directed path of length two. If  $D$  is a directed graph, then  $G = G[D]$  denotes the underlying undirected graph of  $D$ . Also, let  $dist(u, v)$  denote the length of the shortest directed  $uv$  path in digraph  $D$ .

### 3. Outerplanar Ocliques

Sopena [7] proved that every oriented outerplanar graph admits a homomorphism to the quadratic residue tournament on seven vertices ( $V = \{0, 1, \dots, 6\}$  and  $uv \in E$  if and only if  $u - v = 1, 2,$  or  $4 \pmod{7}$ ). Thus we have the following.

**Fact 5.** *Every outerplanar oclique has at most seven vertices.*

**Proposition 6.** *Let  $D$  be an outerplanar oclique with  $|V| \geq 6$ . Then  $G[D]$  has minimum degree two.*

**Proof.** Since  $G[D]$  is outerplanar, it is well-known that  $\delta(G[D]) \leq 2$ . Suppose  $x$  has degree one in  $G = G[D]$ . Without loss of generality,  $x \rightarrow y$  in  $D$ . Since  $D$  is an oclique, the vertex  $y$  dominates all vertices in  $V - \{x, y\}$ . Fix an outerplanar embedding of  $D$ . Let  $z_1, z_2, z_3, z_4$  be vertices in  $V - \{x, y\}$  such that the arcs  $xy, yz_1, yz_2, yz_3, yz_4$  occur consecutively in clockwise order about  $y$  in the embedding. But, any directed path of length at most two joining  $z_1$  and  $z_4$  creates a cycle that separates  $x$  from  $z_2$  or  $z_3$ . ■

An outerplanar oclique on seven vertices, called  $O_7$ , is shown in Figure 1. Two additional outerplanar ocliques on seven vertices can be constructed from  $O_7$  by adding an arc (oriented in either direction) between the leftmost vertex and the rightmost vertex. Another pair of ocliques on seven vertices can be formed from  $O_7$  by adding an arc (oriented in either direction) between the third and fifth vertices from the left.

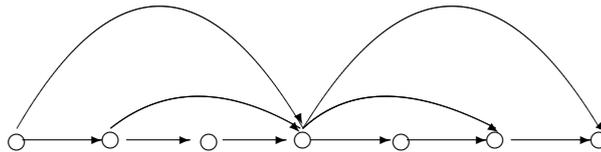


Figure 1. Outerplanar oclique  $O_7$

**Proposition 7.** *Every outerplanar oclique on seven vertices contains  $O_7$  as a spanning subgraph.*

**Proof.** Let  $D$  be an outerplanar oclique with seven vertices and fix an outerplanar embedding of  $G[D]$ . We consider two cases.

*Case 1.* Suppose  $G[D]$  has a cut-vertex  $v$ . Clearly, each component of  $D - v$  must be an outerplanar oclique and each vertex in each component of  $G[D] - v$  must be adjacent to  $v$ . It is not hard to see that there can be at most two components in  $G[D - v]$ . Observe that each of these components can have at most three vertices, else  $G[D]$  cannot be outerplanar. It follows that  $D$  is the digraph  $O_7$ .

*Case 2.* Suppose  $G[D]$  is 2-connected. Since  $G[D]$  is outerplanar, it has a Hamiltonian cycle  $C$ . Let  $C = u_0u_1u_2u_3u_4u_5u_6$ . Since  $G[D]$  has diameter two, there is at least one edge  $u_i, u_{i+2}$  (addition modulo 7). Assume without loss of generality that  $u_0u_2$  is an edge in  $G[D]$ . For  $u_1$  being distance at most two from any other vertex,  $u_4$  and  $u_5$  have to be adjacent to either  $u_0$  or  $u_2$ . If we have edges  $u_4u_2$  and  $u_5u_0$ , then the distance between  $u_3$  and  $u_6$  is greater than two (in  $G[D]$ ). Therefore  $u_4$  and  $u_5$  are either both adjacent to  $u_0$  or both adjacent to  $u_2$ . Assume without loss of generality that  $u_4$  and  $u_5$  are both adjacent to  $u_0$ . In order for the distance between  $u_3$  and  $u_6$  to be less than three, we need edge  $u_0u_3$ . Thus the degree of  $u_0$  is six.

Consider now  $D$ . Without loss of generality, assume  $u_0u_1$  is an edge in  $D$ . For  $u_4, u_5$  and  $u_6$  being (oriented) distance at most two from  $u_1$ , we necessarily have edges  $u_4u_0, u_5u_0$  and  $u_6u_0$ . By symmetry, because of edge  $u_6u_0$ , we necessarily have edges  $u_0u_2$  and  $u_0u_3$ . Finally, for  $u_1$  and  $u_3$  (respectively,  $u_4$  and  $u_6$ ) being oriented distance at most two from each other,  $u_1u_2u_3$  (respectively  $u_4u_5u_6$ ) is necessarily a directed 2-path (in any direction) so that  $O_7$  is a spanning subgraph of  $D$ . ■

## 4. Planar Graphs

As a consequence of Raspaud and Sopena's theorem, we know that if  $D$  is an orientation of a planar graph, then  $\omega_o(D) \leq 80$ .

The following is quite simple.

**Fact 8.** *If  $D$  is a bipartite planar oclique, then  $D$  has at most six vertices.*

**Proof.** Let  $V = A \cup B$  denote the bipartition of  $D$ . It is clear that each vertex in  $A$  must be adjacent to each vertex in  $B$ . Hence one part must contain fewer than three vertices. It is easy to verify that neither  $K_{2,p}, p > 4$  nor  $K_{1,q}, q > 4$  can be oriented to be an oclique. ■

The following implication of Fact 8 is used several times, so we state it as a corollary.

**Corollary 9.** *The complete bipartite graph  $K_{2,5}$  cannot be oriented to be an oclique.*

**Lemma 10.** *Let  $D$  be a planar oclique with  $|V| \geq 10$ . Then  $G[D]$  has minimum degree at least two.*

**Proof.** Suppose vertex  $x$  has degree one in  $G[D]$ . Without loss of generality  $x \rightarrow y$  in  $D$ . Since  $D$  is an oclique, the vertex  $y$  dominates all vertices in  $V - \{x, y\}$ . Then  $G[V - \{x, y\}]$  is an outerplanar oclique and by Fact 5 has at most seven vertices, thus  $G[D]$  can have at most nine vertices. ■

The vertex  $v$  on 2-path  $u \rightarrow v \rightarrow w$  is designated as the *intermediate vertex*.

**Theorem 11.** *Let  $D$  be an orientation of a triangle-free planar graph such that  $D$  is an oclique. Then  $D$  has at most fourteen vertices.*

**Proof.** Suppose the theorem is false and let  $G[D]$  be planar and triangle-free, where  $D$  is an oclique with at least fifteen vertices. It is known that since the diameter of  $G[D]$  is two,  $G[D]$  has a vertex,  $v$ , of degree at least ten [1, 5]; see also

<http://www.unc.edu/~rpratt/degdiam.html>

Then in  $D$ , at least five of  $v$ 's neighbors are, without loss of generality, dominated by  $v$ . These neighbors induce an independent set, since  $G[D]$  is triangle-free. We claim that an independent set with five vertices, all dominated by a sixth vertex is a *forbidden configuration* for a planar oclique, meaning that such a subgraph cannot exist in  $D$  if  $D$  is a planar oclique.

We prove this is a forbidden configuration. Fix a planar embedding of  $D$ , and let  $v$  dominate  $v_1, v_2, \dots, v_5$  where the  $v_i$  vertices are arranged around  $v$  in clockwise order. Each pair  $v_i, v_j$  must be connected by a 2-path using an intermediate vertex  $w_k \neq v$ . One can easily verify that at least three such  $w_k$  vertices are needed to connect five independent vertices via 2-paths. Suppose that  $v_1, w_1, v_5$  is a 2-path, assuming without loss of generality that  $v_1 \rightarrow w_1$ . This forms a cycle in the underlying graph,  $v, v_1, w_1, v_5, v$ .

We consider two cases.

*Case 1.* Suppose  $v_2, w_2, v_4$  is a 2-path where  $w_2 \neq w_1$ . Then  $v_3, w_2, v_5$  and  $v_3, w_2, v_1$  are 2-paths (note that if  $v_3 \rightarrow w_2$ , then  $w_2 \rightarrow v_5, w_2 \rightarrow v_1$  and vice versa). But now there can be no 2-path between  $v_1$  and  $v_4$ .

*Case 2.* Suppose  $v_2, w_1, v_4$  is a 2-path. Then  $v_3, w_1, v_5$  is a 2-path, where  $v_3 \rightarrow w_1$ . But now there can be no 2-path between  $v_1$  and  $v_3$ . ■

**Theorem 12.** *If  $D$  is a planar oclique, then  $D$  has at most 36 vertices.*

**Proof.** Suppose the theorem were false and let  $D$  be an oclique with at least 37 vertices. Fix a planar embedding of  $G[D]$ . Let  $v$  be a vertex of degree at most five, which exists because  $G[D]$  is planar [9]. By Lemma 10,  $v$  has degree at least two. Let  $v_1, v_2, \dots, v_r$ , where  $2 \leq r \leq 5$ , be the neighbors of  $v$ . There are at least 31 additional vertices that  $v$  is connected to via a 2-path. Since  $G[D]$  has diameter two, the subgraph induced by  $V(G[D]) - \{v, v_1, \dots, v_r\}$  is outerplanar and, as such, 3-colorable. Hence there exists an independent set,  $I$ , of at least eleven vertices among these 31 or more vertices. Partition  $I = \{u_1, u_2, \dots, u_k\}$  into sets,  $S_1, S_2, \dots, S_r$ , according to which  $v_i$  vertex is the intermediate vertex on a  $v \rightarrow v_i \rightarrow u_j$  2-path or a  $u_j \rightarrow v_i \rightarrow v$  2-path. That is,  $v_i$  dominates each vertex in  $S_i$  (or vice versa). Of course, a vertex  $u_j$  may have several such 2-paths to/from vertex  $v$ . We deal with this by re-numbering the  $v_i$  vertices as necessary so that we can fix a partition  $S_1, S_2, \dots, S_r$  of  $I$  so that the set  $S_i$  of vertices with intermediate vertex  $v_i$  is at least as large as the set of vertices with intermediate vertex  $v_{i+q}$ , for all  $q > 0$ . Furthermore, we require that if  $u_j \in S_i$ , then there is no 2-path connecting  $u_j$  with  $v$  via any  $v_p$  where  $p < i$ .

Since there are at least eleven vertices in the union of the  $S_i$  sets, the pigeonhole principle assures us that either (i)  $|S_1| \geq 5$  or (ii)  $|S_1| = 4$  and  $|S_2| = 4$  or (iii)  $|S_1| \geq 3, |S_2| = 3, |S_3| = 3$  or (iv)  $|S_1| = 4, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2, |S_5| \geq 1$  or (v)  $|S_1| = 3, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2, |S_5| \geq 1$  or (vi)  $|S_1| = 4, |S_2| = 3, |S_3| = 2, |S_4| = 1, |S_5| = 1$  or (vii)  $|S_1| = 4, |S_2| = 3, |S_3| = |S_4| = 2$ .

(Note that (iv), (v), and (vi) are similar, we separate them to clarify the details as we proceed below). A basic diagram to aid the reader's understanding is given in Figure 2.

Denote the vertices in  $S_1$  as  $a_1, a_2, \dots$  in clockwise order around  $v_1$  and the vertices  $S_2$  as  $b_1, b_2, \dots$  in clockwise order around  $v_2$ . Assume, without

loss of generality, that  $v$  dominates  $v_1$  and thus,  $v_1$  dominates  $a_1, a_2, \dots$ . A basic illustration is given in Figure 2.

In case (i), we have the forbidden configuration described in the proof of Theorem 11.

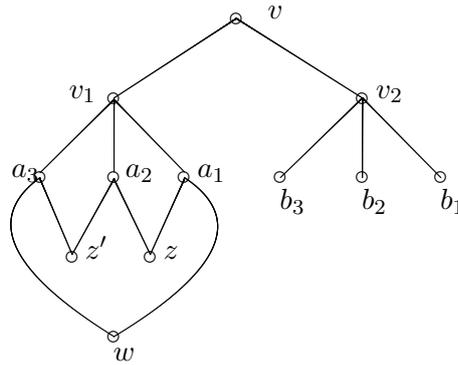


Figure 2. Possible structure of  $S_1, S_2$

In case (ii),  $|S_1| = 4, |S_2| = 4$ . Vertex  $v_1$  cannot be an intermediate vertex between any two of the  $b_i$  vertices in  $S_2$ , because in that case, one of the  $b_i$  vertices would belong in  $S_1$ . Then the 2-path between  $b_1$  and  $b_3$  uses intermediate vertex  $z \neq v_1$ . Consider the 2-path between  $a_1$  and  $a_4$ . First suppose this 2-path uses intermediate vertex  $w \neq v_2, w \neq z$ . Then there can be no 2-path between  $a_2$  and  $b_2$ , as the cycles  $v_1, a_1, w, a_4, v_1$  and  $v_2, b_1, z, b_3, v_2$  are disjoint separating cycles, separating  $a_2$  from  $b_2$ . If however,  $z = w$ , we have the following. A 2-path between  $a_2$  and  $b_2$  must use intermediate vertex  $z$ . And though  $b_1$  and  $b_3$  may have 2-paths to the  $S_1$  vertices via intermediate vertex  $v_1$ ,  $b_2$  cannot, nor can  $b_2$  use  $z$  as an intermediate vertex to both  $a_1$  and  $a_4$ .

On the other hand,  $v_2$  may be an intermediate vertex between  $a_1$  and  $a_4$ . But then case (i) applies as one of  $a_1$  or  $a_4$  could be moved to set  $S_2$ .

In case (iii),  $|S_1| \geq 3, |S_2| = 3, |S_3| = 3$ . Denote the vertices in  $S_3$  as  $c_1, c_2, c_3$ . Vertex  $v_1$  cannot be an intermediate vertex between any two of the  $b_i$  vertices in  $S_2$  because in that case, one of the  $b_i$  vertices would belong in  $S_1$ . Likewise  $v_3$  cannot be an intermediate vertex between two vertices in  $S_2$ , else a  $c_i$  vertex would belong to  $S_2$ . Similarly, neither  $v_1$  nor  $v_2$  can serve as intermediate vertices for any pair of vertices in  $S_3$ .

Consider the 2-paths between  $b_1$  and  $b_3$ , as well as between  $c_1$  and  $c_3$  via intermediate vertices,  $w, z$ , respectively. First suppose  $w \neq z$ . Then it is not possible to form a 2-path between  $b_2$  and  $c_2$ . On the other hand, suppose  $w = z$ . Then  $b_2$  cannot have a 2-path to both  $c_1$  and  $c_3$ .

Now examine case (iv), with  $|S_1| = 4, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2, |S_5| \geq 1$ . Denote the vertices in  $S_3$  as  $c_1, c_2$  in clockwise order around  $v_3$ .

To connect the four vertices in  $S_1$  via 2-paths, at least three intermediate vertices are needed. If only two are used, then a  $K_{3,3}$  subgraph results, because  $v_1$  dominates all the vertices in  $S_1$  and one can easily verify that an orientation of  $K_{2,4}$  is the only way to connect four independent vertices by 2-paths using two intermediate vertices. For the purposes of the argument in case (iv) (and later in case (vi)), only two such intermediate vertices need to be specified.

Consider the 2-path between  $a_1$  and  $a_4$ , using intermediate vertex  $w$ . Any two vertices in  $S_1$  may have at most one common neighbor in the set  $\{v_2, v_3, v_4, v_5\}$ , else  $G[D]$  has a  $K_{3,3}$  subgraph. We may assume, without loss of generality, that  $w \neq v_2, w \neq v_3$  (else we can replace one of  $S_2, S_3$  with  $S_4$ ). The 2-path between  $a_1$  and  $a_4$  creates a cycle  $C$  containing the vertices  $v_1, a_1, w, a_4$ . There also exists an intermediate vertex  $z \neq w, z \neq v_i, 1 \leq i \leq 5$ , such that  $z$  is used to form another 2-path between two vertices in  $S_1$  besides  $a_1$  and  $a_4$ .

First suppose that  $z$  is not adjacent to  $w$ . Regardless of the routing of  $C$  within the given embedding of  $G[D]$ ,  $C$  separates each of  $b_1, b_2, c_1, c_2$  from  $z$ . This forces that  $v_1$  is the intermediate vertex on 2-paths between  $b_1, b_2, c_1, c_2$  and  $z$ . Then 2-paths cannot be formed between all the vertices in  $S_2$  and  $S_3$ .

On the other hand, suppose that each intermediate vertex, other than  $w$  itself, between vertices in  $S_1$  is adjacent to  $w$ . Denote these intermediate vertices as  $z_1, z_2, \dots, z_q, q \geq 2$ . Then in  $D$ ,  $w$  cannot dominate (or be dominated by) the set  $\{a_2, a_3, z_1, z_2, \dots, z_q\}$ , else there can be either no 2-path between  $a_2$  and  $a_4$  or no 2-path between  $a_1$  and  $a_3$ . Now we have that each vertex in  $S_2$  and  $S_3$  must be adjacent to  $v_1$ , which precludes the formation of 2-paths between the vertices in  $S_2$  and  $S_3$  (again, recalling that  $v_1$  cannot be an intermediate vertex between vertices in  $S_2$  and  $S_3$ ).

Now consider case (v). Extending the argument of case (iv), we show that it is forbidden to have  $|S_1| = 3, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2, |S_5| \geq 1$ . We use the same terminology as in case (iv) when referring to vertices. Let  $w$  be the intermediate vertex on a 2-path between  $a_1$  and  $a_3$ . As in case (iv),

we may assume, without loss of generality, that  $w \neq v_2, w \neq v_3$  (else we can replace one of  $S_2, S_3$  with  $S_4$ ).

The case when  $z$  is not adjacent to  $w$  follows identically (with  $a_3$  in the place of  $a_4$ ) as in case (iv). However, some differences arise when  $z$  is adjacent to  $w$ . Assume without loss of generality that  $a_3$  dominates  $w$  and  $w$  dominates  $a_1$  (the argument is symmetric in the other event). In this case, we suppose that  $w$  dominates both  $z$  and  $a_2$  in  $D$ , otherwise the argument proceeds as in case (iv), as each vertex in  $S_2$  and  $S_3$  would have to be adjacent to  $v_1$ . Each vertex in  $S_2$  and  $S_3$  must be adjacent to at least one of  $v_1$  or  $w$ , in order to have a 2-path to  $a_2$  and  $z$ .

If the vertices in  $S_2$  and  $S_3$  use but two intermediate vertices to form 2-paths between them, then each vertex in  $S_2$  and  $S_3$  must be adjacent to both (as an orientation of  $K_{2,4}$  is the only way to connect four independent vertices by 2-paths using two intermediate vertices). The resulting graph has a  $K_{3,3}$  minor.

So we assume the vertices in  $S_2 \cup S_3$  use at least three intermediate vertices to form the 2-paths between them. Of course, none of these intermediate vertices can be  $v_1$ . And at most one can be  $w$ , so let us use  $x, x'$  to denote two intermediate vertices not equal to  $w$ . Note that  $x$  and  $x'$  must each be adjacent to at least one of  $v_1$  or  $w$ .

Suppose  $w$  is the intermediate vertex on the  $b_1b_2$  2-path and the  $c_1c_2$  2-path. But then the planarity of  $G[D]$  makes it impossible to form at least one of the remaining two 2-paths between pairs of the form  $b_ic_j$ .

On the other hand, suppose  $x$  is the intermediate vertex on the  $b_1b_2$  2-path. Then the cycle (in  $G[D]$ )  $v_2, b_1, w, a_3, v_1, v, v_2$  (or  $v_2, b_1, v_1, v, v_2$ ) along with the corresponding cycle containing either edge  $b_2w$  or  $b_2v_1$  (rather than  $b_1w$  or  $b_1v_1$ ) force  $c_1$  and  $c_2$  to be adjacent to  $w$ , in order to have a 2-path to/from  $x$ . Which then makes it impossible for both  $c_1$  and  $c_2$  to each have a 2-path to each of  $a_1$  and  $a_3$  (since  $w$  dominates only one of  $a_1, a_3$ ).

The argument for case (vi) is again nearly identical to case (iv). The only difference being the following. Suppose that the intermediate vertex  $w$  between  $a_1$  and  $a_4$  is  $v_2$ . Then we must replace  $S_2$  as we did in case (iv), but we replace it with both  $S_4$  and  $S_5$  (since we know that  $w$  cannot be  $v_4$  or  $v_5$ , else  $G[D]$  has a  $K_{3,3}$  subgraph). In the language of case (iv), we then let  $c_1$  be the lone vertex in  $S_4$  and  $c_2$  be the lone vertex in  $S_5$  and the argument proceeds as in case (iv) (even though, for example, there may be a 2-path between  $c_1$  and  $c_2$  via  $v_5$ ).

Case (vii) follows exactly as case (iv). ■

## 5. Series-Parallel Graphs

It is proved in [7] that orientations of graphs having treewidth at most two have oriented chromatic number at most seven. Since series-parallel graphs have treewidth at most two,  $\omega_o(D) \leq 7$  for any digraph  $D$  whose underlying graph is series-parallel. The proof in [7] is based on graph homomorphisms. We present an alternate, direct proof of this result based on graph minors/subgraphs. Our method of proof yields that Proposition 7 also holds for series-parallel ocliques.

We first state a lemma given in [9], page 251.

**Lemma 13** [9]. *Let  $H$  be a graph with  $\Delta(H) \leq 3$ . Then  $H$  is a minor of  $G$  if and only if  $G$  contains a subdivision of  $H$ .*

The following is easily verified by inspection.

**Fact 14.** *No subdivision of  $K_{2,3}$ , other than  $K_{2,3}$  itself, can be oriented to be an oclique.*

One can easily confirm that there exist ocliques whose underlying graph is  $K_{2,3}$ .

**Theorem 15.** *Any oclique whose underlying graph is series-parallel has at most seven vertices. Further, there are seven such ocliques on seven vertices, and each contains  $O_7$  as a spanning subgraph.*

**Proof.** Suppose by way of contradiction that  $D$  is an oclique with at least eight vertices; so  $G[D]$  is a series-parallel graph. If  $G[D]$  contains no  $K_{2,3}$  minor, then  $G[D]$  is outerplanar, and the first part of the theorem follows from Fact 5. So suppose  $G[D]$  contains a  $K_{2,3}$  minor. By Lemma 13, we may assume that  $G[D]$  contains a  $K_{2,3}$  subdivision. Because of Fact 14, we shall focus attention on  $K_{2,3}$  itself and not worry about its subdivisions. Let  $H = (A, B, E)$  be a  $K_{2,3}$  subgraph of  $G[D]$ . Let  $A = \{a_1, a_2\}$  and  $B = \{b_1, b_2, b_3\}$ . Let  $V(G[D]) - V(H) = \{g_1, g_2, \dots\}$

Some key facts are now stated, which form the heart of the argument.

(1)  $B$  induces an independent set in  $G[D]$ .

**Proof of (1).** Otherwise,  $G[D]$  contains a  $K_4$  minor.

Likewise we have:

(1a) Every path in  $G[D]$  joining two distinct vertices of  $B$  contains a vertex in  $A$ .

**Proof.** Otherwise there is a  $K_4$  minor.

Note that (1a) implies that if  $g_i$  is adjacent to  $g_j$ , then  $N(g_i) \cap B = N(g_j) \cap B$ . It also implies that each  $g_i$  is adjacent to at most one vertex of  $B$ .

(2) Each  $g_i$  vertex must be adjacent to at least one of  $a_1, a_2$ .

**Proof of (2).** This is in order that the diameter of  $G$  be at most two and follows from (1a), since no  $g_i$  can be adjacent to two vertices from  $B$ .

(3) Each  $g_i$  vertex must have degree at least two.

**Proof of (3).** Otherwise, say  $g_i$  is adjacent to only  $a_1$ , since we know from (2) that  $g_i$  must be adjacent to one of  $a_1, a_2$ . The logic from Lemma 4 then tells us that  $a_1$  must dominate (or be dominated by)  $B$  in  $D$ . As  $B$  is an independent set and vertices in  $B$  have no common neighbors other than vertices in  $A$ , it is then not possible to orient the edges in  $G[D]$  to form 2-paths between the three vertices in  $B$ .

(4)  $I = \{g_1, g_2, \dots\}$  is an independent set.

**Proof of (4).** As stated above, each vertex in  $I$  must be adjacent to at least one of  $a_1, a_2$  in order for  $D$  to be an oclique. Let us assume that  $g_1$  is adjacent to  $g_2$ . By the note following (1a), if both  $g_1$  and  $g_2$  have a neighbor in  $B$  (and each can have at most one), then they are adjacent to the same vertex of  $B$ . Thus, there are two possibilities to consider, as (2) assures us that one of the following must occur.

(a)  $g_1$  and  $g_2$  have a common neighbor in  $A$ . Then using the logic from (3), each of  $g_1, g_2$  must be adjacent to another vertex from  $H$  (and it may be that each is adjacent to a different vertex). Then  $G[D]$  contains a  $K_4$  minor.

(b)  $g_1$  is adjacent to  $a_1$  and  $g_2$  is adjacent to  $a_2$ . Then either  $g_1$  and  $g_2$  have a neighbor in  $B$ , in which case  $G[D]$  has a  $K_4$  minor, which violates (1a), or (a) applies.

(5)  $g_i$  cannot be adjacent to all of  $a_1, a_2$  and  $b_1, b_2, b_3$ .

**Proof of (5).** Otherwise,  $G[D]$  has a  $K_4$  minor.

Combining the above, we deduce that each  $g_i$  has degree two. If two  $g_i$ 's are both adjacent to  $a_1$ , and  $a_2$ , then  $G[D]$  contains a  $K_{2,5}$  subgraph. But, since no vertex in  $I = \{g_1, g_2, \dots\}$  can be an intermediate vertex on a 2-path

joining two vertices of  $I$ , in this case it follows from the structure of  $G[D]$  that  $D$  cannot be oriented to be an oclique. Hence at most one  $g_i$  is adjacent to both  $a_1$  and  $a_2$ .

If three  $g_i$ 's are each adjacent to  $a_1$  and  $b_1$ , say, then without loss of generality each must dominate  $a_1$ , in order that the  $g_i$ 's can have 2-paths to  $b_q, q \neq 1$ . But then, by the structure of  $G[D]$ , we cannot form 2-paths amongst the three  $g_i$ 's.

Suppose for the moment that  $a_1$  is not adjacent to  $a_2$  in  $G[D]$ , and consider the possible neighbors of  $g_1$  and  $g_2$ . It follows from our work above that there are three possibilities (up to relabelling the elements of  $A, B$  and  $I$ ):

- (i) Both  $g_1$  and  $g_2$  are adjacent to  $a_1$  and  $b_1$ ;
- (ii)  $g_1$  is adjacent to  $a_1$  and  $b_1$ , and  $g_2$  is adjacent to  $a_1$  and  $b_2$ ;
- (iii)  $g_1$  is adjacent to  $a_1$  and  $a_2$ , and  $g_2$  is adjacent to  $a_1$  and  $b_1$ . We show that, in each case,  $G[D]$  cannot be oriented as an oclique.

In case (i), without loss of generality, we must have that  $g_1$  and  $g_2$  both dominate  $a_1$ , in order for there to be 2-paths to  $b_2$  and  $b_3$ . In order for there to be 2-paths from the  $g_i$ 's to  $a_2$ , both  $g_1$  and  $g_2$  must dominate  $b_1$ . But now there cannot be a 2-path joining  $g_1$  and  $g_2$ . In case (ii), without loss of generality, we must have that  $g_1$  and  $g_2$  both dominate  $a_1$ , in order for there to be 2-paths  $b_3$ . Again, we have that there cannot be a 2-path joining  $g_1$  and  $g_2$ . In case (iii), without loss of generality,  $g_2$  must dominate  $a_1$ . In turn,  $a_1$  must dominate  $b_2, b_3$  and  $g_1$  so that 2-paths between  $g_1$  and these vertices are formed. Similarly,  $g_1$  must dominate  $a_2$ , which must dominate  $b_1$  and  $b_3$ . But now there cannot be a 2-path joining  $b_2$  and  $b_3$ .

It follows from the above argument that  $a_1$  and  $a_2$  must be adjacent. We assume from now on that this is the case. As above, one of cases (i), (ii) and (iii) must arise. Cases (ii) and (iii) can be eliminated similarly to the arguments above.

Suppose  $g_1$  and  $g_2$  are both adjacent to  $a_1$  and  $b_1$ . In order for  $g_1$  and  $g_2$  to have 2-paths to  $b_2$  and  $b_3$ , without loss of generality, both of  $g_1$  and  $g_2$  must dominate  $a_1$ , which in turn must dominate  $b_2$  and  $b_3$ . There must be a 2-path joining  $g_1$  and  $g_2$ . Without loss of generality, again, assume it is  $g_1 \rightarrow b_1 \rightarrow g_2$ . Then,  $a_1$  must dominate  $a_2$  so that  $g_2$  can have a 2-path to  $a_2$ . In order for there to be a 2-path joining  $b_2$  and  $b_3$ , without loss of generality (still),  $b_2$  dominates  $a_2$  and  $a_2$  dominates  $b_3$ . This implies that  $b_1$  dominates  $a_1$ , so that it can be joined to  $b_2$  by a 2-path. The subgraph induced by the edges whose orientation has been determined so far - all

edges in the subgraph induced by  $A \cup B \cup \{g_1, g_2\}$  except the edge joining  $b_1$  and  $a_2$  is  $O_7$ . Two seven vertex ocliques whose underlying graph is series-parallel and contains  $K_{2,3}$  arise from the two choices for the orientation of this edge.

We claim that we must have  $|I| = 2$ . Suppose  $I$  contains a third vertex,  $g_3$ . From our work above,  $g_3$  cannot be adjacent to  $a_1$  and  $b_1$ . The other possibilities for neighbors of  $g_3$  give rise to cases (ii) or (iii), using  $g_3$  in place of  $g_2$ .

It now follows that there are exactly seven ocliques whose underlying graph is series-parallel: the five mentioned in Section 3, and the two arising in the proof above. All of these contain  $O_7$  as a spanning subgraph. This completes the proof. ■

## 6. Conclusions

We state our main conjecture.

**Conjecture 1.** *Let  $D$  be an orientation of a planar graph such that for any two vertices  $x, y \in V(G)$  there exists an  $xy$  directed path or a  $yx$  directed path of length at most two. Then  $D$  has at most fifteen vertices.*

It is interesting to note that Sopena's planar graph with  $\chi_o = 16$  has  $\omega_o = 15$  [7].

It is obvious that  $k$ -cliques are not the only "obstruction" to  $k$ -coloring an undirected graph. Imperfect graphs illustrate this concept. Hadwiger's conjecture perhaps lies at the heart of this issue: it claims that each  $k$ -chromatic graph contains a subgraph that "becomes  $K_k$  via edge contractions" [9]. It would be interesting to formulate an analogous conjecture for oriented coloring, as there are infinitely many digraphs having  $\chi_o > \omega_o$ , such as the directed cycle on  $n$  vertices, where  $n > 5$  and  $n$  is not divisible by 3. Such graphs can have  $\omega_o = 3$  and  $\chi_o = 4$ .

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