

## CYCLE-PANCYCLISM IN BIPARTITE TOURNAMENTS I

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### Abstract

Let  $T$  be a hamiltonian bipartite tournament with  $n$  vertices,  $\gamma$  a hamiltonian directed cycle of  $T$ , and  $k$  an even number. In this paper, the following question is studied: What is the maximum intersection with  $\gamma$  of a directed cycle of length  $k$ ? It is proved that for an even  $k$  in the range  $4 \leq k \leq \frac{n+4}{2}$ , there exists a directed cycle  $\mathcal{C}_{h(k)}$  of length  $h(k)$ ,  $h(k) \in \{k, k-2\}$  with  $|A(\mathcal{C}_{h(k)}) \cap A(\gamma)| \geq h(k) - 3$  and the result is best possible.

In a forthcoming paper the case of directed cycles of length  $k$ ,  $k$  even and  $k > \frac{n+4}{2}$  will be studied.

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## 1. Introduction

The subject of pancyclism has been studied by several authors (e.g. [1, 2, 5, 10, 12, 14]). Three types of pancyclism have been considered. A digraph  $D$  is pancyclic if it has directed cycles of all the possible lengths;  $D$  is vertex-pancyclic if given any vertex  $v$  there are directed cycles of every length containing  $v$ ; and  $D$  is arc-pancyclic if given any arc  $e$  there are directed cycles of every length containing  $e$ .

It is well known that a hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (with only very few exceptions) but not necessarily

arc-pancyclic (see e.g. [3, 11, 13]). Within the concept of cycle-pancyclicism the following question is studied: Given a directed cycle  $\gamma$  of a digraph  $D$ , find the maximum number of arcs which a directed cycle of length  $k$  (if such a directed cycle exists) contained in  $D[V(\gamma)]$  (the subdigraph of  $D$  induced by  $V(\gamma)$ ) has in common with  $\gamma$ . Cycle-pancyclicism in tournaments has been studied in [6, 7, 8] and [9]. In this paper, the cycle-pancyclicism in bipartite tournaments is investigated. In order to do so, it is sufficient to consider a hamiltonian bipartite tournament  $T$  where  $\gamma$  is a hamiltonian directed cycle (because we are looking for directed cycles of length  $k$  contained in  $D[V(\gamma)]$  whose arcs intersect the arcs of  $\gamma$  the most possible). We will assume (without saying it explicitly in Lemmas, Theorems or Corollaries) that we are working in a hamiltonian bipartite tournament with a vertex set  $V = \{0, 1, \dots, n-1\}$  and an arc set  $A$ . Also we assume without loss of generality that  $\gamma = (0, 1, \dots, n-1, 0)$  is a hamiltonian directed cycle of  $T$ ;  $k$  will be an even number;  $\mathcal{C}_{h(k)}$  will denote a directed cycle of length  $h(k)$  with  $h(k) \in \{k, k-2\}$  and  $\mathcal{J}(\mathcal{C}_{h(k)}) = |A(\mathcal{C}_{h(k)}) \cap A(\gamma)|$ . This paper is the first part of the study of the existence of a directed cycle  $\mathcal{C}_{h(k)}$  where  $\mathcal{J}(\mathcal{C}_{h(k)})$  is the maximum. For general concepts we refer the reader to [4].

## 2. Preliminaries

A chord of a cycle  $\mathcal{C}$  is an arc not in  $\mathcal{C}$  but with both terminal vertices in  $\mathcal{C}$ . The length of a chord  $f = (u, v)$  of  $\mathcal{C}$  denoted  $\ell(f)$  is equal to the length of  $\langle u, \mathcal{C}, v \rangle$ , where  $\langle u, \mathcal{C}, v \rangle$  denotes the  $uv$ -directed path contained in  $\mathcal{C}$ . We say that  $f$  is a  $c$ -chord if  $\ell(f) = c$  and  $f = (u, v)$  is an  $a$ - $c$ -chord if  $\ell(\langle v, \mathcal{C}, u \rangle) = c$ . Observe that if  $f$  is a  $c$ -chord, then it is also an  $a$ - $(n-c)$ -chord. All the chords considered in this paper are chords of  $\gamma$ . We will denote by  $\mathcal{C}_k$  a directed cycle of length  $k$ . In what follows all notation is taken modulo  $n$ .

For any  $a$ ,  $2 \leq a \leq n-2$ , denote by  $t_a$  the largest integer such that  $a + t_a(k-2) < n-1$ . The important case of  $t_{k-1}$  is denoted by  $t$  in the rest of the paper. Let  $r$  be defined as follows:  $r = n - [k-1 + t(k-2)]$ . Notice that: If  $a \leq b$  then  $t_a \geq t_b$ ;  $t \geq 0$  and  $3 \leq r \leq k-1$ ,  $r$  is odd.

**Lemma 2.1.** *If the  $a$ -chord with an initial vertex 0 (0 being an arbitrary vertex of  $T$ ) is in  $A$ , then at least one of the two following properties holds.*

- (i) *There exists a directed cycle  $\mathcal{C}_k$  with  $\mathcal{J}(\mathcal{C}_k) = k-2$ .*

- (ii) For every  $0 \leq i \leq t_a$ , the  $a + i(k - 2)$ -chord with an initial vertex 0 is in  $A$ .

**Proof.** Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1, 2, \dots, t_a\} \mid (a + i(k - 2), 0) \in A\},$$

then

$$\mathcal{C}_k = (0, a + (j - 1)(k - 2)) \cup \langle a + (j - 1)(k - 2), \gamma, a + j(k - 2) \rangle \cup (a + j(k - 2), 0)$$

is a directed cycle with  $J(\mathcal{C}_k) = k - 2$ . ■

**Corollary 2.2.** *At least one of the two following properties holds*

- (i) *There exists a directed cycle  $\mathcal{C}_k$  with  $J(\mathcal{C}_k) \geq k - 2$ .*
- (ii) *For every  $0 \leq i \leq t$ , every  $(k - 1 + i(k - 2))$ -chord is in  $A$ .*

**Proof.** Clearly, for any vertex 0,  $(0, k - 1) \in A$  since otherwise  $(k - 1, 0) \in A$  and  $\mathcal{C}_k = \langle 0, \gamma, k - 1 \rangle \cup (k - 1, 0)$  is a directed cycle with  $J(\mathcal{C}_k) = k - 1$  and thus (i) holds. Now applying Lemma 2.1 with  $a = k - 1$  we have that (i) or (ii) holds. ■

### 3. The Cases $k = 4, 6, 8$

**Theorem 3.1.** *There exists a directed cycle  $\mathcal{C}_4$  with  $J(\mathcal{C}_4) \geq 2$ .*

**Proof.** It follows from Corollary 2.2 for  $k = 4$  that we may assume that for every  $0 \leq i \leq t_3$ , every  $(3 + 2i)$ -chord is in  $A$ ; now recall  $3 \leq r \leq k - 1$ ,  $r$  is odd and  $r = n - [k - 1 + t(k - 2)]$ . Hence  $r = 3$  and we conclude that  $\mathcal{C}_4 = (0, 3 + 2t_3, 3 + 2t_3 + 1, 3 + 2t_3 + 2, 0)$  has  $J(\mathcal{C}_4) = 3$ . ■

**Theorem 3.2.** *There exists a directed cycle  $\mathcal{C}_{h(6)}$  with  $J(\mathcal{C}_{h(6)}) \geq h(6) - 2$ .*

**Proof.** It follows from Corollary 2.2 for  $k = 6$  that we can assume that for every  $i$ ,  $0 \leq i \leq t_5$ , every  $(5 + 4i)$ -chord is in  $A$ ; recall  $3 \leq r \leq 5$ , and  $r$  is odd; so  $r \in \{3, 5\}$ . When  $r = 3$ ,  $\mathcal{C}_4 = (0, 5 + 4t_5) \cup \langle 5 + 4t_5, \gamma, 0 \rangle$  satisfies  $J(\mathcal{C}_4) = 3$ ; and when  $r = 5$ ,  $\mathcal{C}_6 = (0, 5 + 4t_5) \cup \langle 5 + 4t_5, \gamma, 0 \rangle$  satisfies  $J(\mathcal{C}_6) = 5$ . ■

**Theorem 3.3.** *There exists a directed cycle  $\mathcal{C}_{h(8)}$  with  $\mathcal{J}(\mathcal{C}_{h(8)}) \geq h(8) - 3$ .*

**Proof.** It follows from Corollary 2.2 for  $k = 8$  that we may assume that for every  $i$ ,  $0 \leq i \leq t_7$  every  $(7 + 6i)$ -chord is in  $A$ ; recall  $3 \leq r \leq k - 1 = 7$  and  $r$  is odd, so  $r \in \{3, 5, 7\}$ . When  $r = 7$  we obtain  $\mathcal{C}_8 = (0, 7 + 6t_7) \cup \langle 7 + 6t_7, \gamma, 0 \rangle$  a directed cycle with  $\mathcal{J}(\mathcal{C}_8) = 7$ . When  $r = 5$  we have  $\mathcal{C}_6 = (0, 7 + 6t_7) \cup \langle 7 + 6t_7, \gamma, 0 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_6) = 5$ . When  $r = 3$ , since  $n \geq 2k - 4 = 12$  and  $r = 3$  we have  $t_7 \geq 1$  and every  $-9$ -chord is in  $A$  (notice that  $(0, n - 9) \in A$  because  $n - 9 = 7 + (t_7 - 1)6$  and  $0$  is an arbitrary vertex) in particular  $(n - 2, n - 11) \in A$ ; now observe that we can assume  $(n - 7, 0) \in A$  (otherwise  $(0, n - 7) \in A$  and  $\mathcal{C}_8 = (0, n - 7) \cup \langle n - 7, \gamma, 0 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_8) = 7$ ); also observe that  $r = 3$  implies  $(0, n - 3) \in A$ . We conclude that  $\mathcal{C}_8 = (n - 2, n - 11) \cup \langle n - 11, \gamma, n - 7 \rangle \cup (n - 7, 0, n - 3, n - 2)$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_8) = 5$ . ■

#### 4. The Case $n = 2k - 4$

**Theorem 4.1.** *If  $n = 2k - 4$ , then there exists a directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) = h(k) - 1$ .*

**Proof.** Consider the arc between  $0$  and  $k - 3$ ; when  $(0, k - 3) \in A$  we have  $\mathcal{C}_k = (0, k - 3) \cup \langle k - 3, \gamma, 0 \rangle$  a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 1$  and when  $(k - 3, 0) \in A$  we obtain  $\mathcal{C}_{k-2} = (k - 3, 0) \cup \langle 0, \gamma, k - 3 \rangle$  a directed cycle with  $\mathcal{J}(\mathcal{C}_{k-2}) = k - 3$ . ■

#### 5. The Cases $r = k - 1$ and $r = k - 3$

**Theorem 5.1.** *If  $r = k - 1$  or  $r = k - 3$ , then there exists a directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) = h(k) - 1$ .*

**Proof.** If  $r = k - 1$ , then  $(0, n - (k - 1)) \in A$  and  $\mathcal{C}_k = (0, n - (k - 1)) \cup \langle n - (k - 1), \gamma, 0 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 1$ . If  $r = k - 3$ , then  $(0, n - (k - 3)) \in A$  and  $\mathcal{C}_{k-2} = (0, n - (k - 3)) \cup \langle n - (k - 3), \gamma, 0 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_{k-2}) = k - 3$ . ■

**Corollary 5.2.** *If  $t = 0$ , then there exists a directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) = h(k) - 1$ .*

**Proof.** If  $t = 0$ , then  $n = k - 1 + r$ , where  $k - 3 \leq r \leq k - 1$ ,  $r$  is odd (because  $n \geq 2k - 4$ ), so  $r \in \{k - 3, k - 1\}$ . If  $r = k - 3$ , then the assertion follows by Theorem 4.1. If  $r = k - 1$  and thus  $n = 2k - 2$ , then we can distinguish the cases  $(0, k - 1) \in A$  and  $(0, k - 1) \notin A$ . If  $(0, k - 1) \in A$ , then  $\mathcal{C}_k = (0, k - 1) \cup \langle k - 1, \gamma, n \rangle$  is a cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 1$ . The other case follows analogously. ■

### 6. The General Case

In this section, we assume  $r \leq k - 5$ ,  $t \geq 1$ , and  $k \geq 10$  (so  $n \geq 16$ ) in view of the results in previous sections.

**Lemma 6.1.** *If the  $(k - 1 + \alpha)$ -chord,  $\alpha \leq r + 1$ , with an initial vertex 0 is in  $A$ , then at least one of the two following properties holds.*

- (i) *There exists a directed cycle  $\mathcal{C}_k$  with  $\mathcal{J}(\mathcal{C}_k) = k - 2$ .*
- (ii) *For every  $0 \leq i \leq t - 1$ , the  $k - 1 + \alpha + i(k - 2)$ -chord with an initial vertex 0 is in  $A$ .*

It follows directly from Lemma 2.1; only observe that since  $3 \leq r \leq k - 5$  we have  $k - 1 + r + 1 + (t - 1)(k - 2) \leq k + k - 5 + (t - 1)(k - 2) = k - 1 + t(k - 2) - 2 \leq k - 1 + t(k - 2) + r - 5 = n - 5$ .

**Lemma 6.2.** *At least one of the two following properties holds*

- (i) *There exists a directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3$ .*
- (ii) *All the following chords are in  $A$ : (a) Every  $(k - 1)$ -chord. (b) Every  $(-r)$ -chord. (c) Every  $(k - 3)$ -chord and (d) Every  $-(r + 2)$ -chord.*

**Proof.** Assume that (i) is false. Let us prove that (ii) holds. The proof of (a) follows directly from Corollary 2.2. The proof of (b) follows from Corollary 2.2, observing that  $n - r = k - 1 + t(k - 2)$ . To prove (c) assume that there is a  $-(k - 3)$ -chord, say  $f = (y, x)$ . It follows from (a) that  $(x - 2, y) \in A$ , and it follows from (b) that  $(x - 2 + r, x - 2) \in A$ . Hence, there exists a vertex  $z$  in  $\langle x - 2 + r, \gamma, y - 1 \rangle$  such that  $(z, x - 2) \in A$  and  $(x - 2, z + 2) \in A$ . We conclude

$$\mathcal{C}_{k-2} = (y, x) \cup \langle x, \gamma, z \rangle \cup (z, x - 2) \cup (x - 2, z + 2) \cup \langle z + 2, \gamma, y \rangle$$

is a directed cycle with  $\mathcal{J}(\mathcal{C}_{k-2}) = k - 5 = (k - 2) - 3$ .

Finally to prove (d), let  $(y, x)$  be a  $(r + 2)$ -chord. It follows from (c) and Lemma 2.1 that every  $k - 3 + i(k - 2)$ -chord is in  $A$  for  $0 \leq i \leq t$  in particular  $(x + k - 2, x + k - 2 + k - 3 + (t - 1)(k - 2)) \in A$ , now observe that  $x + k - 2 + k - 3 + (t - 1)(k - 2) = y + r + 2 + k - 2 + k - 3 + (t - 1)(k - 2) = y + r + k - 1 + t(k - 2) = y + n = y$ , so  $(x + k - 2, y) \in A$ ; we conclude that  $\mathcal{C}_k = (y, x) \cup \langle x, \gamma, x + k - 2 \rangle \cup (x + k - 2, y)$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 2$ . ■

**Lemma 6.3.** *Let  $0 \leq i \leq r + 1$ ,  $i$  being even. If all the  $-r$ -chords,  $-(r + 2)$ -chords,  $(k - 3 + i)$ -chords and  $(k - 1 + i)$ -chords are in  $T$ , then at least one of the following properties holds.*

- (i) *There exists a directed cycle  $\mathcal{C}_k$  with  $\mathcal{J}(\mathcal{C}_k) \geq k - 3$ .*
- (ii) *All the  $-(2r - i + 1)$ -chords,  $-(2r - i + 3)$ -chords and  $-(2r - i + 5)$ -chords are in  $T$ .*

**Proof.** Assume that the hypothesis of the Lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the  $[(k - 3) + i]$ -chords and all the  $[(k - 1) + i]$ -chords are in  $T$ , it follows from Lemma 6.1 (taking  $\alpha = i - 2$ ) that every  $[k - 3 + i + (t - 1)(k - 2)]$ -chord is in  $T$ , and that (taking  $\alpha = i$ ) every  $[k - 1 + i + (t - 1)(k - 2)]$ -chord is in  $T$ . Thus the following arcs are in  $T$ :  $(r, 0)$ ,  $(r + 2, 0)$ ,  $(0, k - 1 + (t - 1)(k - 2) + i)$  and  $(0, k - 1 + (t - 1)(k - 2) + i - 2)$ .

Let  $x_1 = r - 1$ ,  $x_2 = r + 1$ ,  $x_3 = k - 1 + (t - 1)(k - 2) + i - 2$ ,  $x_4 = x_3 + 2$ ,  $x_5 = x_4 + k - 2$ ,  $x_6 = x_5 + 2$ ,  $x_7 = x_5 - 2$ ,  $x_8 = x_7 - 2$ . Therefore  $(0, x_3) \in A$  and  $(0, x_4) \in A$ .

Observe that:  $\ell\langle x_5, \gamma, 0 \rangle = n - x_5 = r - i$  (because  $n - x_5 = k - 1 + t(k - 2) + r - (k - 1) - i - t(k - 2) = r - i$ ),  $\ell\langle x_6, \gamma, 0 \rangle = r - i - 2$ ,  $\ell\langle x_6, \gamma, x_1 \rangle = 2r - i - 3$  (because  $\ell\langle x_6, \gamma, x_1 \rangle = \ell\langle x_6, \gamma, 0 \rangle + r - 1 = r - i - 2 + r - 1$ ),  $\ell\langle x_7, \gamma, x_1 \rangle = \ell\langle x_6, \gamma, x_1 \rangle + 4 = 2r - i + 1$ ,  $\ell\langle x_7, \gamma, x_2 \rangle = \ell\langle x_7, \gamma, x_1 \rangle + 2 = 2r - i + 3$ ,  $\ell\langle x_8, \gamma, x_2 \rangle = \ell\langle x_7, \gamma, x_2 \rangle + 2 = 2r - i + 5$ ,  $\ell\langle x_4, \gamma, x_7 \rangle = \ell\langle x_4, \gamma, x_5 \rangle - 2 = k - 4$  and  $\ell\langle x_3, \gamma, x_8 \rangle = \ell\langle x_4, \gamma, x_7 \rangle = k - 4$ .

First, we prove that every  $-(2r - i + 1)$ -chord is in  $A$ . Suppose that there exists a  $(2r - i + 1)$ -chord. We can assume without loss of generality that  $(x_7, x_1)$  is that chord. Hence  $\mathcal{C}_k = (x_7, x_1, x_1 + 1 = r, 0, x_4) \cup \langle x_4, \gamma, x_7 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 3$ .

Now we prove that every  $-(2r - i + 3)$ -chord is in  $A$ . Assume the contrary; we may assume that  $(x_7, x_2)$  is a  $(2r - i + 3)$ -chord. Then  $\mathcal{C}_k = (x_7, x_2 = r + 1, r + 2, 0, x_4) \cup \langle x_4, \gamma, x_7 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 3$ .

Finally, we prove that every  $-(2r - i + 5)$ -chord is in  $T$ . Assuming the opposite, we may consider that  $(x_3, x_2)$  is a  $(2r - i + 5)$ -chord. Then  $\mathcal{C}_k = (x_3, x_2 = r + 1, r + 2, 0, x_3) \cup \langle x_3, \gamma, x_3 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 3$ . ■

**Lemma 6.4.** *At least one of the following properties holds.*

- (i) *There exists a directed cycle  $\mathcal{C}_k$  with  $\mathcal{J}(\mathcal{C}_k) \geq k - 3$ .*
- (ii) *For any even vertex  $x$  (resp. odd) there exist at most  $\frac{k-4}{2}$  consecutive odd (resp. even) vertices in  $\gamma$  which are in-neighbors of  $x$ .*

**Proof.** Assume that (i) does not hold. Assume without loss of generality that  $x = 0$ . It follows from Corollary 2.2 that the vertices  $k - 1 + i(k - 2)$ , for  $0 \leq i \leq t$  are not in-neighbors of 0.

So, there are at most  $\frac{k-4}{2}$  odd vertices consecutive in  $\langle k - 1, \gamma, 0 \rangle$  which are in-neighbors of 0. Since  $(0, 1) \in A$ , also in  $\langle 0, \gamma, k - 1 \rangle$  there are at most  $\frac{k-4}{2}$  odd vertices consecutive in-neighbors of 0. ■

The following corollary is a directed consequence of this Lemma (only observe that the hypothesis  $n \geq 2k - 4$  is not needed in the Lemma).

**Corollary 6.5.** *Let  $T$  be a bipartite tournament with  $n$  vertices and  $\gamma$  a hamiltonian cycle of  $T$ . For each even (resp. odd) vertex  $x$  of  $T$  such that the number of consecutive odd (resp. even) in-neighbors of  $x$  in  $\gamma$  is at least  $\frac{k-2}{2}$ ,  $3 \leq k \leq n$ ,  $k$  even, there exists a directed cycle  $\mathcal{C}_k$  containing  $x$  with  $\mathcal{J}(\mathcal{C}_k) \geq k - 2$ .*

**Lemma 6.6.** *If every  $(k + 1)$ -chord is in  $A$  then at least one of the two following properties holds.*

- (i) *There exists a directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3$ .*
- (ii) *For every odd  $\alpha$ ,  $0 < \alpha r < k$ ; every  $-(\alpha + 1)r + 1$ -chord is in  $A$ . And for every even  $\alpha$ ,  $0 \leq \alpha r < k$ ; every  $-(\alpha + 1)r$ -chord is in  $A$ .*

**Proof.** For  $\alpha = 0$ , we can assume that every  $-r$ -chord is in  $A$  (otherwise it follows from Lemma 6.2 that (i) holds and we are done). For  $\alpha = 1$ , suppose that  $(x_1, x_0)$  is a  $(2r + 1)$ -chord, let  $(x_0, x_2)$  the  $-r$ -chord with an initial vertex  $x_0$  and  $(x_2, x_3)$  the  $[(k - 1) + (t - 1)(k - 2)]$ -chord with an initial vertex  $x_2$  (It follows from Corollary 2.2 that we can assume such a chord exists); clearly,  $\ell \langle x_1, \gamma, x_2 \rangle = r + 1$  and  $\ell \langle x_3, \gamma, x_1 \rangle = n - (r + 1) - [k - 1 + (t - 1)(k - 2)] = k - 3$ . Now notice that  $x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}$

(because  $r < k - 3$  and  $k \geq 10$ ), thus  $\mathcal{C}_k = \langle x_3, \gamma, x_1 \rangle \cup (x_1, x_0, x_2, x_3)$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) \geq k - 3$ .

We have proved the assertion of Lemma 6.6 for  $\alpha = 0$  and  $\alpha = 1$ . To complete the proof, assume (ii) does not hold for some  $\alpha' \geq 2$  and we show that (i) holds. Let  $\alpha$  be the least integer  $\alpha \geq 2$  for which (ii) does not hold. We analyze two possible cases.

*Case 1.*  $\alpha$  is odd.

We have  $\alpha \geq 3$ ,  $0 < \alpha r < k$  and there exists an  $[(\alpha + 1)r + 1]$ -chord in  $A$ . Since  $\alpha - 1$  is even, the choice of  $\alpha$  implies that every  $-((\alpha - 1) + 1)r$ -chord is in  $A$ .

Let  $(x_1, x_0)$  be an  $[(\alpha + 1)r + 1]$ -chord,  $(x_0, x_2)$  the  $-\alpha r$ -chord with an initial vertex  $x_0$  and  $(x_2, x_3)$  the  $(k + 1) + (t - 1)(k - 2)$ -chord with an initial vertex  $x_2$  (it follows from the hypothesis and Lemma 2.1 that these chords are in  $A$ ). Clearly,  $\ell\langle x_1, \gamma, x_2 \rangle = (\alpha + 1)r + 1 - \alpha r = r + 1$  and  $\ell\langle x_3, \gamma, x_1 \rangle = n - (r + 1) - [(k + 1) + (t - 1)(k - 2)] = k - 5$ . Notice  $x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}$  because  $\alpha r < k$ , and  $\ell\langle x_2, \gamma, x_3 \rangle \geq k + 1$ . We conclude that  $\mathcal{C}_{k-2} = \langle x_3, \gamma, x_1 \rangle \cup (x_1, x_0, x_2, x_3)$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_{k-2}) \geq k - 5$ .

*Case 2.*  $\alpha$  is even.

We have  $\alpha \geq 2$ ,  $0 < \alpha r < k$ , and there exists an  $(\alpha + 1)r$ -chord in  $A$ . Since  $\alpha - 1$  is odd, the choice of  $\alpha$  implies that every  $-[(\alpha - 1 + 1)r + 1]$ -chord is in  $A$ . Let  $(x_1, x_0)$  be an  $(\alpha + 1)r$ -chord,  $(x_0, x_2)$  the  $-(\alpha r + 1)$ -chord with an initial vertex  $x_0$ , and  $(x_2, x_3)$  the  $(k + 1) + (t - 1)(k - 2)$ -chord with an initial vertex  $x_2$ .

Clearly,  $\ell\langle x_1, \gamma, x_2 \rangle = (\alpha + 1)r - \alpha r - 1 = r - 1$  and  $\ell\langle x_3, \gamma, x_1 \rangle = n - (r - 1) - [k + 1 + (t - 1)(k - 2)] = k - 3$ . Moreover,  $x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}$  because  $\ell\langle x_2, \gamma, x_0 \rangle = \alpha r + 1$ ,  $0 < \alpha r + 1 \leq k$  and  $\ell\langle x_2, \gamma, x_3 \rangle \geq k + 1$ . We obtain  $\mathcal{C}_k = \langle x_3, \gamma, x_1 \rangle \cup (x_1, x_0, x_2, x_3)$  a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 3$ . ■

**Lemma 6.7.** *At least one of the following properties holds.*

- (i) *There exists a directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3$ .*
- (ii) *For  $i$  even  $-2 \leq i \leq r + 1$ ; every  $-(2r + 1 - i)$ -chord and every  $(k - 1 + i)$ -chord is in  $A$ .*

**Proof.** Suppose (i) does not hold, we shall prove that property (ii) holds by induction on  $i$ . We start with  $i = -2$  and  $i = 0$ ; namely, we prove that the



following chords are in  $A$ : (a) every  $(k - 3)$ -chord, (b) every  $(k - 1)$ -chord, (c) every  $-(2r + 3)$ -chord and (d) every  $-(2r + 1)$ -chord.

The proof of (a) and (b) follows directly from Lemma 6.2. Let  $0$  be any vertex of  $T$ . It follows from Lemma 6.1 (with  $\alpha = 0$ ) and from Lemma 6.2 (part (b) and (d)) that the following chords are in  $A$ :  $(0, k - 1 + (t - 1)(k - 2))$ ,  $(r + 2, 0)$  and  $(r, 0)$ . Part (c): every  $-(2r + 3)$ -chord is in  $A$ . If  $(n - r - 1, r + 2) \in A$ , then  $\mathcal{C}_k = (n - r - 1, r + 2, 0, k - 1 + (t - 1)(k - 2)) \cup \langle k - 1 + (t - 1)(k - 2), \gamma, n - r - 1 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 3$ , a contradiction. (Notice that  $k - 1 + (t - 1)(k - 2) \in \langle r + 2, \gamma, n - r - 1 \rangle - \{r + 2, n - r - 1\}$  since  $r \leq k - 5$ ; moreover  $\ell\langle k - 1 + (t - 1)(k - 2), \gamma, n - r - 1 \rangle = n - r - 1 - (k - 1) - (t - 1)(k - 2) = k - 3$ ).

Part (d): every  $-(2r + 1)$ -chord is in  $A$ . If  $(n - r - 2, r - 1) \in A$ , then  $\mathcal{C}_k = (n - r - 2, r - 1, r, 0, k - 1 + (t - 1)(k - 2)) \cup \langle k - 1 + (t - 1)(k - 2), \gamma, n - r - 2 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 3$ , a contradiction. (Notice that since  $r \leq k - 5 < k - 1$  and  $k \geq 10$  we have  $r < k - 1 + (t - 1)(k - 2) < n - (r + 2)$ , and  $\ell\langle k - 1 + (t - 1)(k - 2), \gamma, n - r - 2 \rangle = n - (r + 2) - [k - 1 + (t - 1)(k - 2)] = k - 4$ ). Assume that (ii) in Lemma 6.7 holds for each  $i'$  even,  $0 \leq i' \leq i$  and let us prove it for  $i + 2$ ; namely, we prove

- ( $\alpha$ ) Every  $(k + 1 + i)$ -chord is in  $A$ ,  $0 \leq i \leq r - 1$ .
- ( $\beta$ ) Every  $-(2r - 1 - i)$ -chord is in  $A$ ,  $0 \leq i \leq r - 1$ .

**Proof of ( $\alpha$ ).** It follows from the inductive hypothesis that for each  $j$  even,  $0 \leq j \leq i$ , every  $[(k - 1) + j]$ -chord and every  $[(k - 3) + j]$ -chord is in  $A$  (because for  $j = 0$  we have proved that every  $(k - 3)$ -chord is in  $A$ ). It follows from Lemma 6.2 that every  $(-r)$ -chord and every  $-(r + 2)$ -chord is in  $A$ . Therefore it follows from Lemma 6.3 that for even  $j$ ,  $0 \leq j \leq i$  every  $-(2r - j + 1)$ -chord,  $-(2r - j + 3)$ -chord and  $-(2r - j + 5)$ -chord is in  $A$ . That means that for each even  $j$ ,  $-4 \leq j \leq i \leq r - 1$  every  $-(2r - j + 1)$ -chord is in  $A$ . These are  $\frac{i}{2} + 3$  chords with initial odd (resp. even) vertices consecutive in  $\gamma$ .

Assume by contradiction that  $(x_3, 0)$  is a  $-(k + 1 + i)$ -chord,  $i$  being even  $0 \leq i \leq r - 1$ . Let  $x_0 = n - (2r - i - 1)$ . Hence letting  $x_2 = 2$  we have that  $(x_2, x_0)$  is a  $-(2r - i + 1)$ -chord (we have observed that every  $-(2r - i + 1)$ -chord is in  $A$ ).

First, we prove that  $x_0 \in \langle x_3 + 1, \gamma, n - 1 \rangle$ :  $\ell\langle x_0, \gamma, 0 \rangle = 2r - i - 1 \geq r \geq 3$ ,  $\ell\langle x_3, \gamma, x_0 \rangle = n - (k + 1 + i + 2r - i - 1) = k - 1 + t(k - 2) + r - k - 2r \geq k - 1 + k - 2 - r - k = k - 3 - r \geq 2$  (remember  $3 \leq r \leq k - 5$ ).

Now, there exists an out-neighbor of  $x_0$ , say  $x$ , such that  $x$  is in  $\langle x_2, \gamma, x_3 - 1 \rangle$  this is a direct consequence of Lemma 6.4 and the fact that the number even vertices in  $\langle x_2, \gamma, x_3 - 1 \rangle$  is at least  $\frac{k-2}{2}$  (Notice that  $x_0$  is odd,  $\ell\langle x_2, \gamma, x_3 - 1 \rangle = k + 1 + i - 3 = k + i - 2 \geq k - 2$ ). Let  $x_4$  be the smallest (the nearest to 0 in  $\gamma$ ) such vertex.

Let  $x_1 = 0$ , we will prove that  $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3 \rangle$ . Since for each  $j$ ,  $j$  even  $-4 \leq j \leq i \leq r - 1$ , every  $-(2r - j + 1)$ -chord is in  $A$ , it follows that

$$\{(2, x_0), (4, x_0), (6, x_0), \dots, (i + 4, x_0), (i + 6, x_0)\} \subseteq A.$$

Hence the selection of  $x_4$  implies  $x_4 \geq i + 8$ , so  $x_4 - i - 4 > 3$ .

Finally, since  $\ell\langle x_4, \gamma, x_3 \rangle + \ell\langle x_1, \gamma, x_4 - i - 4 \rangle = k + 1 + i - (i + 4) = k - 3$  it follows that  $\mathcal{C}_k = (x_4 - i - 4, x_0, x_4) \cup \langle x_4, \gamma, x_3 \rangle \cup (x_3, x_1) \cup \langle x_1, \gamma, x_4 - i - 4 \rangle$  is a directed cycle with  $\mathcal{J}(\mathcal{C}_k) = k - 3$  (Notice  $(x_4 - i - 4, x_0) \in A$  by the choice of  $x_4$  and the fact  $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3 \rangle$ ).

**Proof of  $(\beta)$ .** Part  $(\beta)$  follows from Lemma 6.3 (taking  $i + 2$  instead of  $i$ ) and the following facts:

Every  $(k - 1 + i)$ -chord is in  $A$  for even  $i$ ,  $-2 \leq i \leq r + 1$  (it follows from part  $(\alpha)$ ).

Every  $(k - 3 + i)$ -chord is in  $A$  for even  $i$ ,  $0 \leq i \leq r + 1$  (it follows from the assertion of above).

Every  $(-r)$ -chord and every  $-(r + 2)$ -chord is in  $A$  (it follows from Lemma 6.2). ■

**Theorem 6.8.** *If  $n \geq 2k - 4$ , then there exists a directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3$ .*

**Proof.** The case  $n = 2k - 4$  is considered in Section 4. Assume  $n > 2k - 4$  and suppose by contradiction that there is no directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3$ .

It follows from Lemma 6.7 that for each even  $i$ ,  $-2 \leq i \leq r + 1$  every  $(k - 1 + i)$ -chord is in  $A$ , in particular

$$(1) \quad \{(0, k-3), (0, k-1), (0, k+1), (0, k+3), \dots, (0, k+r-2), (0, k+r)\} \subset A.$$

(Notice that  $k + r < n - 1$  because  $t \geq 1$  and  $k \geq 10$ ).

It follows from Lemma 6.2 that every  $(-r)$ -chord is in  $A$ , and by Lemma 6.7 that every  $(k+1)$ -chord is in  $A$ . Therefore, by Lemma 6.6 we have: For every odd  $\alpha$ ,  $0 < \alpha r < k$ , every  $-[(\alpha+1)r+1]$ -chord is in  $A$ . And for every even  $\alpha$ ,  $0 \leq \alpha r < k$ , every  $-(\alpha+1)r$ -chord is in  $A$ . Let  $\alpha_0 = \max\{\alpha \in \mathbb{N} \mid \alpha r < k\}$ . Clearly,  $\alpha_0 r < k$ . We will analyze the two possible cases:

*Case 1.*  $\alpha_0$  is even.

It follows from Lemma 6.6 that every  $-(\alpha_0+1)r$ -chord is in  $A$ , in particular  $((\alpha_0+1)r, 0) \in A$ . On the other hand,  $\alpha_0 r < k$  implies  $(\alpha_0+1)r < k+r$  and the selection of  $\alpha_0$  implies  $k < (\alpha_0+1)r < k+r$ . Thus  $y = (\alpha_0+1)r \in \{k+1, k+3, k+5, \dots, k+r\}$ ; thus we have  $(y, 0) \in A$  and (1) implies  $(0, y) \in A$ . A contradiction.

*Case 2.*  $\alpha_0$  is odd.

It follows from Lemma 6.6 that every  $-[(\alpha_0+1)r+1]$ -chord is in  $A$ , in particular  $((\alpha_0+1)r+1, 0) \in A$ . On the other hand,  $\alpha_0 r < k$  and the choice of  $\alpha_0$  implies  $k+1 \leq (\alpha_0+1)r+1 \leq k+r$ ,  $y = (\alpha_0+1)r+1$  is odd and  $y \in \{k+1, k+3, k+5, \dots, k+r\}$ . So it follows from (1) that  $(0, y) \in A$  and we have proved  $(y, 0) \in A$ . A contradiction. ■

## 7. Remarks

In this section, it is proved that the hypothesis of Theorem 6.8 is tight.

**Definition 7.1.** A digraph  $D$  with vertex set  $V$  is called *cyclically  $p$ -partite complete* ( $p \geq 3$ ) provided one can partition  $V = V_0 + V_1 + \dots + V_{p-1}$  so that  $(u, v)$  is an arc of  $D$  if and only if  $u \in V_i, v \in V_{i+1}$  (notation modulo  $p$ ).

**Remark 7.2.** The cyclically 4-partite complete digraph  $T_4$  is a bipartite tournament and clearly every directed cycle of  $T_4$  has length  $\equiv 0 \pmod{4}$ . So for  $k = 4m + 2$ ,  $T_4$  has no directed cycles of length  $k$  and for  $k = 4m$ ,  $T_4$  has no directed cycles of length  $k - 2$ .

Now we consider the following simple lemma.

**Lemma 7.3.** Let  $\mathcal{C}_{h(k)}$  be a directed cycle with  $\mathcal{J}(\mathcal{C}_{h(k)}) = h(k) - 2$ . If  $f_1 = (0, x_1), f_2 = (y_1, y_2)$  are the arcs of  $\mathcal{C}_{h(k)}$  not in  $\gamma$ , then  $y_2 = y_1 + n - (h(k) - 2 + x_1)$ . Namely,  $f_2$  is a  $-(x_1 + (h(k) - 2))$ -chord of  $\gamma$ .

**Remark 7.4.** For  $n \geq 5, k \geq 5$ , such that  $n \neq k + s(k - 2) + m(k - 4)$  and  $n \neq s(k - 2) + m(k - 4)$  with  $s, m \in \mathbb{N}$ , there exists a bipartite hamiltonian tournament  $T_n$  with no directed cycles  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) = h(k) - 2$ .

**Proof.** Define  $T_n$  as follows:

Let

$$C = \{(i, i + k - 1 + s(k - 2) + m(k - 4)) \mid i \in \{0, 1, \dots, n - 1\}, s, m \in \mathbb{N} \text{ with}$$

$$(k - 1) + s(k - 2) + m(k - 4) < n - 1\}$$

and

$$F = \{(i, i + k - 3 + s(k - 2) + m(k - 4)) \mid i \in \{0, 1, \dots, n - 1\}, s, m \in \mathbb{N} \text{ with}$$

$$(k - 3) + s(k - 2) + m(k - 4) < n - 1\},$$

$$A(T_n) = C \cup F \cup \left( \left\{ (i + j, i) \mid j \in \left\{ 2, 3, \dots, \left\lfloor \frac{n - 1}{2} \right\rfloor \right\} \right\} - (C \cup F) \right)$$

$$\cup \{(i, i + 1) \mid i \in \{0, 1, \dots, n - 1\}\} \cup \left\{ \left( i + \frac{n}{2}, i \right) \mid i \in \left\{ 0, 1, \dots, \frac{n}{2} - 1 \right\} \right\}.$$

Clearly, there is no directed cycle  $\mathcal{C}_{h(k)}$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) = h(k) - 1$  (Notice that  $T_n$  has every  $(k - 1)$ -chord and every  $(k - 3)$ -chord). Now assume for contradiction that  $\mathcal{C}_{h(k)}$  is a directed cycle of  $T_n$  with  $\mathcal{J}(\mathcal{C}_{h(k)}) = k - 2$ , and let  $f_1 = (0, x_1), f_2 = (y_1, y_2)$  the only arcs of  $\mathcal{C}_k$  not in  $\gamma$ . Without loss of generality we can assume  $\ell(f_1) < \frac{n}{2}$ . The definition of  $T_n$  implies that  $x_1 = k - 1 + s(k - 2) + m(k - 4)$  or  $x_1 = k - 3 + s(k - 2) + m(k - 4)$ . It follows from Lemma 7.3 that  $y_2$  has one of the following forms:

(a)  $y_2 = y_1 + n - [k - 1 + (s + 1)(k - 2) + m(k - 4)].$

When  $k - 1 + (s + 1)(k - 2) + m(k - 4) < n - 1$  we obtain that  $f_2$  is a  $-(k - 1 + (s + 1)(k - 2) + m(k - 4))$ -chord, contradicting the definition of  $T_n$ .

When  $k - 1 + (s + 1)(k - 2) + m(k - 4) \geq n - 1$  we have that  $\ell\langle x_1, \gamma, 0 \rangle \leq k - 1$  and the fact  $\mathcal{C}_k - \{(0, x_1), (y_1, y_2)\} \subseteq \langle x_1, \gamma, 0 \rangle$  implies  $\ell\langle x_1, \gamma, 0 \rangle \geq k - 3$ ; and since  $\ell\langle x_1, \gamma, 0 \rangle$  is odd we have  $\ell\langle x_1, \gamma, 0 \rangle \in \{k - 1, k - 3\}$ . Now if  $\ell\langle x_1, \gamma, 0 \rangle = k - 1$ , then  $n = x_1 + k - 1 = k - 1 + s(k - 2) + m(k - 4) + k - 1 =$

$k + (s + 1)(k - 2) + m(k - 4)$ , a contradiction. If  $\ell\langle x_1, \gamma, 0 \rangle = k - 3$ , then  $n = x_1 + k - 3 = k - 1 + s(k - 2) + m(k - 1) + k - 3 = k + s(k - 2) + (m + 1)(k - 4)$ , a contradiction.

$$(b) \quad y_2 = y_1 + n - [(k - 1) + s(k - 2) + (m + 1)(k - 4)].$$

$$(c) \quad y_2 = y_1 + n - [(k - 3) + (s + 1)(k - 2) + m(k - 4)].$$

$$(d) \quad y_2 = y_1 + n - [(k - 3) + s(k - 2) + (m + 1)(k - 4)].$$

Cases (b), (c) and (d) can be analyzed in a completely analogous form as the case (a) to get a contradiction.

It is easy to verify that if  $n = k + s(k - 2) + m(k - 4)$  or  $n = s(k - 2) + m(k - 4)$  with  $s, m \in \mathbb{N}$ , then  $T_n$  (any bipartite hamiltonian tournament with  $n$  vertices) has a directed cycle  $\mathcal{C}_{h(k)}$  with  $J(\mathcal{C}_{h(k)}) \geq h(k) - 2$ . ■

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