

## **$P_m$ -SATURATED BIPARTITE GRAPHS WITH MINIMUM SIZE**

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### **Abstract**

A graph  $G$  is said to be  $H$ -saturated if  $G$  is  $H$ -free i.e., ( $G$  has no subgraph isomorphic to  $H$ ) and adding any new edge to  $G$  creates a copy of  $H$  in  $G$ . In 1986 L. Kászonyi and Zs. Tuza considered the following problem: for given  $m$  and  $n$  find the minimum size  $sat(n; P_m)$  of  $P_m$ -saturated graph of order  $n$ . They gave the number  $sat(n; P_m)$  for  $n$  big enough. We deal with similar problem for bipartite graphs.

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## **1. Preliminaries**

We deal with simple graphs without loops and multiple edges. As usual  $V(G)$  and  $E(G)$  denote the vertex set and the edge set, respectively,  $|G|, e(G)$  the order and the size of  $G$ , and  $d_G(v)$  the degree of  $v \in V(G)$ . By  $P_m$  we denote the path of order  $m$ , and by  $K_m$  the complete graph on  $m$  vertices. We define  $G_{a,b}$  to be a bipartite graph where  $a, b$  are the numbers of vertices in bipartition sets. Let us consider two graphs  $G$  and  $H$ . We say that  $G$  is  $H$ -free if it contains no copy of  $H$ , that is, no subgraph of  $G$  is isomorphic to  $H$ . A graph  $G$  is  $H$ -saturated if  $G$  is  $H$ -free and adding any new edge  $e$  to  $G$  creates a copy of  $H$ . In particular complete  $H$ -free graphs trivially satisfy this condition and therefore are  $H$ -saturated. We define also:

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$$ex(n; F) = \max\{e(G) : |G| = n, G \text{ is } F\text{-saturated}\},$$

$$Ex(n; F) = \{G : |G| = n, e(G) = ex(n; F), G \text{ is } F\text{-saturated}\},$$

$$sat(n; F) = \min\{e(G) : |G| = n, G \text{ is } F\text{-saturated}\},$$

$$Sat(n; F) = \{G : |G| = n, e(G) = sat(n; F), G \text{ is } F\text{-saturated}\}.$$

Observe that in the definitions of  $Ex(n; F)$  and  $ex(n; F)$  the word *saturated* may be replaced with *free*. The first results concerning saturated graphs were given by Turán [6] in 1941 who asked for  $ex(n; K_p)$  and  $Ex(n; K_p)$ . Later results were given by P. Erdős, A. Hajnal and J.W. Moon [3] (see also [2]) in 1964 who proved

$$sat(n; K_p) = \binom{p-2}{2} + (p-2)(n-p+2), \quad (n \geq p \geq 2)$$

$$Sat(n; K_p) = \{K_{p-2} * \bar{K}_{n-p+2}\}.$$

A corresponding theorem for bipartite graphs was given by N. Alon in 1983 (see [1]). The extremal problem for  $P_m$ -saturated bipartite graphs was solved by A. Gyárfás, C.C. Rousseau and R.H. Schelp [4]. We are interested in finding  $P_m$ -saturated bipartite graphs with minimum size. In Section 2 we present some results concerning  $P_m$ -saturated bipartite graphs. The proofs are given in Section 3.

In [5] L. Kászonyi and Zs. Tuza, gave the following results on  $Sat(n; P_m)$  and  $sat(n; P_m)$ .

**Theorem 1** ([5]).

$$sat(n; P_3) = \left\lfloor \frac{n}{2} \right\rfloor,$$

$$Sat(n; P_3) = \begin{cases} kK_2 & \text{if } n = 2k, \\ kK_2 \cup K_1 & \text{if } n = 2k + 1, \end{cases}$$

$$sat(n; P_4) = \begin{cases} k & \text{if } n = 2k, \\ k + 2 & \text{if } n = 2k + 1, \end{cases}$$

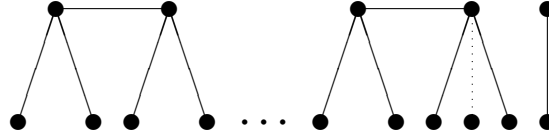
$$Sat(n; P_4) = \begin{cases} kK_2 & \text{if } n = 2k, \\ (k-1)K_2 \cup K_3 & \text{if } n = 2k + 1, \end{cases}$$

$$sat(n; P_5) = n - \left\lfloor \frac{n-2}{6} \right\rfloor - 1 \quad \text{for } n \geq 6.$$

Let

$$a_m = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if } m = 2k, k > 2, \\ 2^{k+1} - 2 & \text{if } m = 2k + 1, k \geq 2. \end{cases}$$

Then  $\text{sat}(n; P_m) = n - \lfloor \frac{n}{a_m} \rfloor$  for  $n \geq a_m$ .



$\text{Sat}(n; P_5)$  for  $n \geq 6$ .

Figure 1

## 2. $P_m$ -Saturated Bipartite Graphs with Minimum Size

Let  $G = (B, W; E)$  be a bipartite graph with vertex set  $V = B \cup W$ ,  $B \cap W = \emptyset$ . For convenience of the reader we call the set  $B$  the set of *black* vertices and the set  $W$  the set of *white* vertices. For bipartite graphs  $G = (B, W; E)$  and  $F = (B', W'; E')$  such that the sets  $B, W, B'$  and  $W'$  are mutually disjoint we define:  $G \cup F = (B \cup B', W \cup W'; E \cup E')$ .

**Definition 1.** Let  $G = (B, W; E)$  be a bipartite graph. Then  $G$  is called *F-saturated* if

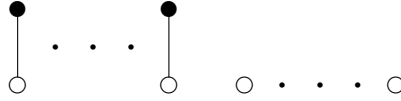
1.  $G$  is *F-free*,
2.  $(x \in B, y \in W, xy \notin E) \Rightarrow G \cup xy \supseteq F$ .

We denote also

$$\text{sat}_{bip}(p, q; F) = \min\{e(G) : |B| = p, |W| = q, G \text{ is } F\text{-saturated}\},$$

$$\text{Sat}_{bip}(p, q; F) = \{G = (B, W; E) : |B| = p, |W| = q, e(G) = \text{sat}_{bip}(p, q; F), G \text{ is } F\text{-saturated}\}.$$

**Proposition 2.**  $\text{sat}_{bip}(p, q; P_3) = p, p \leq q$ . ■



$Sat_{bip}(p, q; P_3)$ .

Figure 2

**Proposition 3.**  $sat_{bip}(p, q; P_4) = p, \quad 2 \leq p \leq q.$  ■



$Sat_{bip}(p, q; P_4)$ .

Figure 3

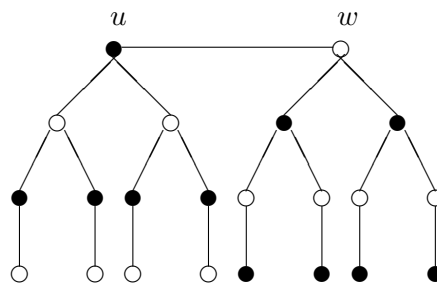
**Proposition 4.** *Let  $p \geq 2, q \geq 3, p \leq q.$  Then*

$$sat_{bip}(p, q; P_5) = \begin{cases} 2p & \text{if } 2p \leq q, p \text{ is even or } q = 2p - 2, \\ q & \text{if } 2p \leq q, p \text{ is odd,} \\ p + \left\lceil \frac{q}{2} \right\rceil & \text{if } 3 < q < 2p, q \neq 2p - 2, \\ 5 & \text{if } p = q = 3, \\ 4 & \text{if } p = 2. \end{cases}$$

**Proposition 5.** *Let  $3 \leq p \leq q.$  Then*

$$sat_{bip}(p, q; P_6) = \begin{cases} p + q - \left\lfloor \frac{p}{3} \right\rfloor - 1 & \text{if } p \equiv 2(\text{mod } 3) \text{ and } 3q \leq 4p - 2 \text{ or} \\ & p \equiv 1(\text{mod } 3) \text{ and } 3q \leq 4p - 1, \\ p + q - \left\lfloor \frac{p}{3} \right\rfloor & \text{if } p = q \equiv 1(\text{mod } 3) \text{ or} \\ & p \equiv 0(\text{mod } 3) \text{ and } 3q \leq 4p - 1, \\ 2p & \text{otherwise.} \end{cases}$$

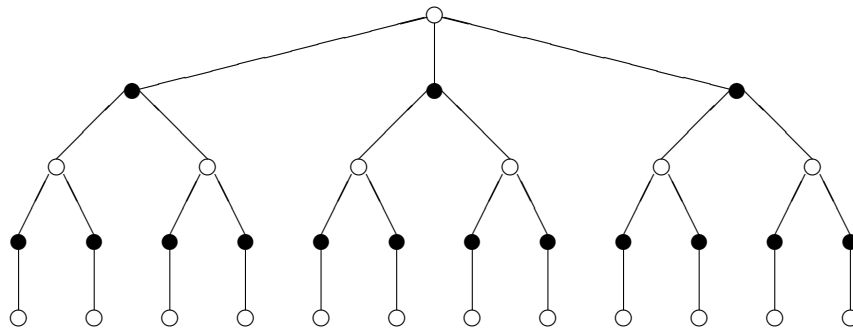
**Definition 2.** Let us suppose that  $m \geq 7$  is an integer. Then  $A_m$  is the following tree. All penultimate vertices of  $A_m$  have degree two and all vertices of  $A_m$  which are neither penultimate nor pendant have their degree equal to three. If  $m = 2k, k \geq 4$  then  $A_m$  has two centers  $u \in B$  and  $w \in W$  and each component of  $G - uw$  has  $k - 1$  levels (see Figure 4). If  $m = 2k + 1, k \geq 3$  then  $A_m$  has one center and  $k$  levels. The center is black when  $k$  is even (see Figure 5).



$A_{10}$

Figure 4

When  $m = 2k, k \geq 4$  we observe that  $|B| = |W| = 3 \cdot 2^{k-3} - 1$ .



$A_{11}$

Figure 5

If  $m = 2k + 1, k \geq 3$  and the center is white then in  $A_m$  we have  $|B| = 4 \cdot 2^{k-3} - 1$  and  $|W| = 5 \cdot 2^{k-3} - 1$  when  $k$  is odd or  $|W| = 4 \cdot 2^{k-3} - 1$  and  $|B| = 5 \cdot 2^{k-3} - 1$  when  $k$  is even. Denote by  $v$  the center of  $A_{2k+1}, k \geq 3$ .

Observe that if  $|B| \leq |W|$  then for  $m = 2k + 1, k \geq 3$  we obtain  $|B| = 4 \cdot 2^{k-3} - 1, |W| = 5 \cdot 2^{k-3} - 1$  and  $v \in B$  if  $k$  is even or  $v \in W$  if  $k$  is odd.

**Remark 1.** Observe that  $A_m$  is  $P_m$ -saturated and  $P_{m-1}$ -saturated for every  $m \geq 7$ .

**Remark 2.**  $lA_{2k}$  is  $P_{2k}$ -saturated for  $k \geq 4, l = 1, 2, \dots, n$ .

**Remark 3.** The union of two copies of  $A_{2k+1}$  is  $P_{2k+1}$ -saturated,  $k \geq 3$ , if and only if their centers have the same colour (see Figure 6).

**Theorem 6.** Let  $k \geq 4$  and let  $G = (B, W; E)$  be a  $P_{2k}$ -saturated bipartite graph without isolated vertices and with the minimum size,  $|B| = p, |W| = q, 3 \cdot 2^{k-3} - 1 \leq p \leq q$ . Then

$$e(G) = p + q - \left\lfloor \frac{p}{3 \cdot 2^{k-3} - 1} \right\rfloor.$$

**Theorem 7.** Let  $k \geq 3$  and let  $G = (B, W; E)$  be a  $P_{2k+1}$ -saturated bipartite graph without isolated vertices and with the minimum size,  $|B| = p \leq |W| = q, 4 \cdot 2^{k-3} - 1 \leq p, 5 \cdot 2^{k-3} \leq q$ . Then

$$e(G) = \begin{cases} p + q - \frac{q}{5 \cdot 2^{k-3} - 1} + 1 & \text{if } \frac{q}{5 \cdot 2^{k-3} - 1} = \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor < \frac{p}{4 \cdot 2^{k-3} - 1}, \\ p + q - \min \left\{ \left\lfloor \frac{p}{4 \cdot 2^{k-3} - 1} \right\rfloor, \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor \right\} & \text{otherwise.} \end{cases}$$

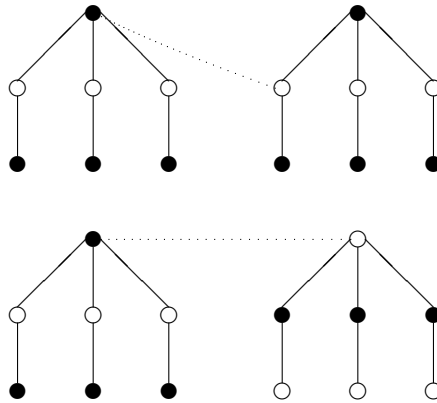


Figure 6

Theorem 6 and 7 imply the following corollary.

**Corollary 8.** *If  $|B| = p, |W| = q, 3 \cdot 2^{k-3} - 1 \leq p \leq q, k \geq 4$  then*

$$sat_{bip}(p, q; P_{2k}) \leq p + q - \left\lfloor \frac{p}{3 \cdot 2^{k-3} - 1} \right\rfloor.$$

*If  $|B| = p, |W| = q, p \leq q, 4 \cdot 2^{k-3} - 1 \leq p, 5 \cdot 2^{k-3} \leq q$ , then*

$$sat_{bip}(p, q; P_{2k+1}) = \begin{cases} p + q - \frac{q}{5 \cdot 2^{k-3} - 1} + 1 & \text{if } \frac{q}{5 \cdot 2^{k-3} - 1} = \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor < \frac{p}{4 \cdot 2^{k-3} - 1}, \\ p + q - \min \left\{ \left\lfloor \frac{p}{4 \cdot 2^{k-3} - 1} \right\rfloor, \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor \right\} & \text{otherwise.} \end{cases} \quad \blacksquare$$

### 3. Proofs

We first give some definitions. The graphs  $K_{1,n}$  and  $K_{n,1}$  are called *stars* when  $n \geq 1$  and *non-trivial stars* if  $n \geq 2$ . Let  $K_{1,b}$  and  $K_{a,1}$  be two vertex disjoint stars. Then the tree obtained by join of their centers is called *double star*  $S_{a,b}^2$  (see Figure 7). A double star  $S_{a,b}^2$  is said to be *non-trivial* if  $a > 0, b > 0$  and  $a + b \geq 3$ . Propositions 2 and 3 are evident. To prove Proposition 4 we give Lemmas 9–13 and Proposition 14 below.

**Lemma 9.** *Let  $G = (B, W; E)$  be a connected bipartite  $P_5$ -saturated graph  $|B| = p, |W| = q$ . Then either*

1.  $G$  is a star or
2.  $G$  is non-trivial double star  $S_{a,b}^2$  or else
3.  $G = K_{2,2}$ .

**Lemma 10.** *Let  $G = (B, W; E)$  be a bipartite  $P_5$ -saturated graph,  $|B| = p, |W| = q, p \leq q, p \geq 2, q \geq 3$ , such that there is at least one isolated vertex in  $W$ . Then  $p = 2k, k \geq 1$  and  $G = kK_{2,2} \cup K_{0,q-p}$ . In particular we have*

1.  $p < q$  and
2.  $e(G) = 2p$ .

Lemma 9 is evidently true. Lemma 10 follows from Lemma 9 easily. ■

**Lemma 11.** *Let  $G = (B, W; E)$  be a bipartite  $P_5$ -saturated graph without isolated vertices  $|B| = p, |W| = q, p \leq q, p \geq 2, q \geq 3$ . Then  $G$  is vertex disjoint union of*

1. complete graphs  $K_{2,2}$ ,
2. non-trivial double stars,
3. non-trivial stars,
4. at most one trivial star  $K_{1,1}$ .

*If  $K_{1,1}$  is a component of  $G$  then no other star is a component of  $G$ .*

Lemma 11 follows from Lemma 9. ■

If  $G = (B, W; E)$  is a bipartite  $P_5$ -saturated graph with  $|B| = p, |W| = q$  then we have either

$$(1) \quad G = nK_{2,2} \cup \bigcup_{i=1}^k S_{a_i, b_i}^2 \cup \bigcup_{j=1}^l K_{1, c_j} \cup \delta K_{1,1}$$

$$(2) \quad G = nK_{2,2} \cup \bigcup_{i=1}^k S_{a_i, b_i}^2 \cup \bigcup_{j=1}^l K_{d_j, 1} \cup \delta K_{1,1}$$

where  $S_{a_i, b_i}^2$  are non-trivial double stars,  $K_{1, c_j}$  and  $K_{d_j, 1}$  are non-trivial stars, and  $\delta \in \{0, 1\}$ . We have  $p = 2n + \left(\sum_{i=1}^k a_i + k\right) + l + \delta$ ,  $q = 2n + \left(\sum_{i=1}^k b_i + k\right) + \sum_{j=1}^l c_j + \delta$  if  $G$  is given by (1), and  $p = 2n + \left(\sum_{i=1}^k a_i + k\right) + \sum_{j=1}^l d_j + \delta, q = 2n + \left(\sum_{i=1}^k b_i + k\right) + l + \delta$  if  $G$  is given by (2), and  $\delta = 0$  if  $l > 0$ .

**Lemma 12.** *Let  $G = (B, W; E)$  be a union of non-trivial double stars, such that  $|B| = p, |W| = q, p \leq q, p \geq 2, q \geq 3$ . Then  $G$  has the minimum size if*

$$e(G) = \begin{cases} p + q - \lfloor \frac{p}{2} \rfloor & \text{if } 3p \leq 2q, \\ p + q - \lfloor \frac{1}{2} \left( p - \left\lceil \frac{3p-2q}{5} \right\rceil \right) \rfloor & \text{if } 2q \leq 3p. \end{cases}$$



**Proof.** Let  $G$  be a union of non-trivial double stars,  $G = \bigcup_{i=1}^l S_{a_i, b_i}^2$  where

$$S_{a_i, b_i}^2 = (B_i, W_i; E_i), |B_i| = a_i + 1, |W_i| = b_i + 1, i = 1, \dots, l,$$

$$\bigcup_{i=1}^l B_i = B, \bigcup_{i=1}^l W_i = W,$$

such that for fixed  $p$  and  $q$ ,  $G$  has the minimum size  $e(G)$ . We observe that  $e(G) = p + q - c$  where  $c = c(G)$  is the number of components of  $G$ . So,  $e(G)$  is the minimum whenever  $c(G)$  is the maximum. Since every component of  $G$  has at least two vertices in  $B$  then  $c(G) \leq \lfloor \frac{p}{2} \rfloor$ . If  $3p \leq 2q$ , then  $c = \lfloor \frac{p}{2} \rfloor$  and  $c$  components of  $G$  are  $S_{1, b_i}^2$  stars with  $b_i \geq 2, i = 1, 2, \dots, c - 1$  and  $a_c = 1, b_c \geq 2$  if  $p$  is even, and  $a_c = 2, b_c \geq 1$  when  $p$  is odd.

Therefore  $e(G) = p + q - \lfloor \frac{p}{2} \rfloor$  when  $3p \leq 2q$ . So we may assume from now that  $3p > 2q$ . Since the lemma is easy to verify for  $p \leq 4$  we shall assume  $p \geq 5$ . Observe that there are two different components  $C_1$  and  $C_2$  of  $G$  such that  $C_1 = S_{a_1, b_1}^2, C_2 = S_{a_2, b_2}^2, b_1 \geq 2$  and  $a_2 \geq 2$ .

If  $p \leq 6$  or  $q \leq 7$  then  $c(G) = 2$  and the proof is finished. So we suppose  $p \geq 7$  and  $q \geq 8$ . Then there is at least one component  $C, C \neq C_1, C \neq C_2$ . Let  $x, y$  be the centers of  $C, x_1, y_1$  be the centers of  $C_1, x_2, y_2$  be the centers of  $C_2$ , such that  $x, x_1, x_2 \in B, y, y_1, y_2 \in W$ . It is clear that the number of components of  $G$  will not change if we proceed the following operation:

- delete from  $C_1$  all but one black pendant vertices and all but two white pendant vertices (we denote then by  $C'_1$  the obtained component),
- delete from  $C_2$  all but two black pendant vertices and all but one white pendant vertices (we denote then by  $C'_2$  the obtained component),
- join  $x$  with all white vertices deleted from  $C_1$  and  $C_2$  and join  $y$  with all black vertices deleted from  $C_1$  and  $C_2$  (we denote then by  $C'$  the obtained component).

The new graph  $G'$  has exactly the same number of components as  $G$  and all the components of  $G'$  are non-trivial double stars. The number of components of  $G$  is equal to  $c = 2t + \lfloor (\frac{p-5t}{2}) \rfloor = \lfloor (\frac{p-t}{2}) \rfloor$  where  $t$  is the minimum integer verifying  $3(p - 5t) \leq 2(q - 5t), 3p - 2q \leq 5t$  and by consequence  $t = \lceil (\frac{3p-2q}{5}) \rceil$  and Lemma 12 is proved. ■

**Lemma 13.** *Let  $p \geq 4$  and let  $G = (B, W; E)$  be a bipartite  $P_5$ -saturated graph such that  $|B| = p \leq q = |W|, K_{1,1}$  is a component of  $G$  and  $G$  has the*

minimum size. Then

$$e(G) = \begin{cases} p + q - \lfloor \frac{p-1}{2} \rfloor - 2 & \text{if } 3p - 1 \leq 2q, \\ p + q - \lfloor \frac{1}{2} (p - 1 - \lfloor \frac{3p-2q-1}{5} \rfloor) \rfloor - 2 & \text{if } 2q \leq 3p - 1. \end{cases}$$

**Proof.** By Lemma 11 each component of  $G$  is either complete graph  $K_{2,2}$  or non-trivial double star  $S_{a,b}^2$  and exactly one component is isomorphic to  $K_{1,1}$ . The size of  $G$  is equal to  $e(G) = p + q - c - 1$  where  $c$  is the number of double stars. We have  $e(2K_{2,2}) = 8 > e(S_{3,3}^2) = 7$  and  $e(K_{2,2} \cup S_{a,b}^2) = e(S_{a+2,b+2}^2)$ . So we may suppose that  $G$  has no components isomorphic to  $K_{2,2}$ . The lemma follows from Lemma 12. ■

**Proposition 14.** Let  $G = (B, W; E)$  be a bipartite  $P_5$ -saturated graph such that  $|B| = p \leq q = |W|, 3 \leq p \leq q$  without isolated vertices and with the minimum size. Then

$$e(G) = \begin{cases} q & \text{if } 2p \leq q, \\ p + \lfloor \frac{q}{2} \rfloor & \text{if } q < 2p, 2p - q \neq 2, \\ p + \lfloor \frac{q}{2} \rfloor + 1 & \text{if } 2p - q = 2. \end{cases}$$

**Proof.** The proof starts with the observation that by Lemma 11  $G$  is a union of  $nK_{2,2}$  and  $S_{a_i,b_i}^2, i = 1, \dots, k$  and some stars such that there is at most one  $K_{1,1}$  and the remaining stars have their centers in exactly one set of bipartition  $B$  or  $W$ . Now observe that if  $n \geq 2$  then  $S_{2n-1,2n-1}^2$  is non-trivial double star which has less edges than  $nK_{2,2}$  and the same number of vertices. Thus there is at most one  $K_{2,2}$ . But then there is at least one component  $C$  which is a star, or non-trivial double star. Then  $K_{2,2} \cup C$  may be replaced with a double star  $S_{a,b}^2$  with the same vertex set and with the size  $e(S_{a,b}^2) = e(K_{2,2} \cup C)$ . So we may suppose that no component of  $G$  is isomorphic to  $K_{2,2}$ . So  $G$  is a union of stars and double stars. We may check easily that if  $G$  has more then one double star then it is always possible to find a union of non-trivial stars and at most one double non-trivial star with the same size. Moreover all the stars may have their centers in a given set of bipartition. Hence we may suppose that either  $G = \bigcup_{i=1}^k K_{1,q_i} \cup S_{a,b}^2, k + a + 1 = p, \sum_{i=1}^k q_i + b + 1 = q$  or  $G = \bigcup_{i=1}^l K_{p_i,1} \cup S_{a,b}^2, l + b + 1 = q, \sum_{i=1}^l p_i + a + 1 = p$ . Similarly we may suppose that all non-trivial stars are isomorphic to  $K_{1,2}$  or  $K_{2,1}$  and we

have  $2k + b + 1 = q$ ,  $k + a + 1 = p$  and  $2l + a + 1 = p$ ,  $l + b + 1 = q$ . Now the proof follows easily. ■

Clearly, Lemma 10 and Proposition 14 imply Proposition 4.

Proposition 5 follows from Lemma 18 and Corollary 17 given below. Let  $T_i, i \in \{1, 2\}$  be the tree defined in Figure 7.

**Lemma 15.** *Let  $G = (B, W; E)$  be a connected bipartite  $P_6$ -saturated graph. Then either  $G$  contains one of graphs  $S_{2,2}^2$ ,  $T_i, i \in \{1, 2\}$  or  $G = K_{r,s}$  with  $\min\{r, s\} \leq 2$ .*

**Proof.** Let us denote  $|B| = p, |W| = q$  and let  $p \leq q$ . For  $\min\{p, q\} \leq 2$  the lemma is evident. So let us suppose that  $p, q \geq 3$ . It is easily seen that there exists at least one vertex  $x \in V(G)$  such that  $d_G(x) \geq 3$ . Let us suppose that  $x \in B$ . Denote by  $y_i, i = 1, 2, \dots, n$  the neighbours of  $x$ . If there is a neighbour  $y_i, i = 1, 2, \dots, n$  such that  $d_G(y_i) \geq 3$  then  $G$  contains  $S_{2,2}^2$ . So we may suppose that  $d_G(y_i) \leq 2, i = 1, 2, \dots, n$ . Since  $p \geq 3$  at least two of  $y_i, i = 1, 2, \dots, n$  have their degrees equal to 2 and therefore  $G$  contains  $T_1$ . ■

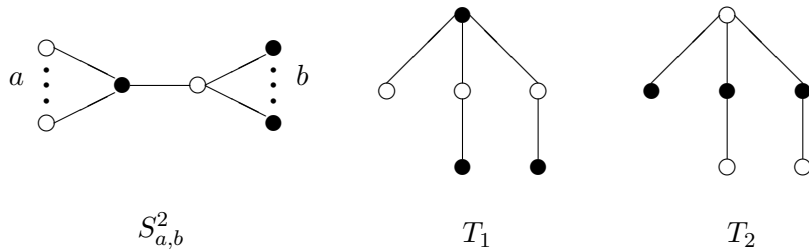


Figure 7

**Lemma 16.** *Let  $G = (B, W; E)$  be a bipartite  $P_6$ -saturated graph such that  $3 \leq |B| = p \leq q = |W|$ ,  $p \geq 3$  and there is a vertex  $w \in W$  which is isolated in  $G$ . Then all the isolated vertices of  $G$  are in  $W$  and  $G = \bigcup_{i=1}^k K_{a_i,2} \cup \bigcup_{j=1}^l K_{0,1}^j$  where  $a_i \geq 3, i = 1, 2, \dots, k, \sum_{i=1}^k a_i = p$  and  $q = 2k + l$ . ■*

**Proof.** The fact that all isolated vertices are in  $W$  is evident. Let  $B = \{b_1, b_2, \dots, b_p\}$ ,  $W = \{w_1, w_2, \dots, w_q\}$ . Denote by  $N_G(x)$  the set of the neighbours of the vertex  $x \in V(G)$ . It is clear that for every  $b \in B$  there is a path  $P_5$  starting from  $b$ . It is easy to check that every  $w \in N_G(b)$  belongs

to any path  $P_5$  starting from  $b$ . Thus  $d_G(b) \leq 2$ . Denote by  $b_1w_1b_2w_2b_3$  a path starting from  $b_1$ . It follows easily that for every  $x \in B$  such that  $w_i \in N_G(x), i \in \{1, 2\}$  we have  $N_G(x) \subseteq \{w_1, w_2\}$ . Therefore the component of  $G$  containing  $w_1$  and  $w_2$  is isomorphic to  $K_{a,2}, a \geq 3$ . ■

Corollary 17 follows immediately from Lemma 16.

**Corollary 17.** *Let  $G = (B, W; E)$  be a bipartite  $P_6$ -saturated graph such that  $|B| = p$  and there is an isolated vertex in  $W$ . Then  $e(G) = 2p$ . ■*

**Lemma 18.** *If  $G = (B, W; E)$  is a bipartite  $P_6$ -saturated graph without isolated vertices and with the minimum size and  $3 \leq |B| = p \leq q = |W|$ , then*

$$e(G) = \begin{cases} p + q - \left\lfloor \frac{p}{3} \right\rfloor & \text{if } p \equiv 0(\text{mod } 3) \text{ or } p = q \equiv 1(\text{mod } 3), \\ p + q - \left\lfloor \frac{p}{3} \right\rfloor - 1 & \text{if } p \equiv 1(\text{mod } 3) \text{ and } p < q \text{ or} \\ & p \equiv 2(\text{mod } 3). \end{cases}$$

**Proof.** For every graph  $G$  we have  $e(G) \geq |V(G)| - c$  where  $c$  is a number of components of  $G$  and equality holds if and only if  $G$  is a forest. The proof follows by Lemma 15. ■

Now, we turn to the case of  $m \geq 7$ .

**Lemma 19.** *Let  $T = (B, W; E)$  be a  $P_m$ -saturated tree,  $m \geq 7$ ,  $x \in B \cup W$ , with  $d_T(x) > 1$  and let  $x_1, x_2, \dots, x_k$  be the neighbors of  $x$ . For  $i = 1, 2, \dots, k$  denote by  $l_i$  the maximum number of vertices in a path starting from  $x$  and containing  $x_i, i = 1, 2, \dots, k, l_1 \geq l_2 \geq \dots \geq l_k$ . The following holds:*

- (i)  $m - 1 \leq l_1 + l_i \leq m, i = 2, 3,$
- (ii) *if  $d_T(v) = 2$  then  $v$  is the neighbour of a pendant vertex ( $v$  is penultimate).*

**Proof.** The inequality  $l_1 + l_i \leq m$  for  $i > 1$  is evident. Let  $x_1^i, x_2^i, \dots, x_{l_i}^i$  be a path of order  $l_i$  starting from  $x = x_1^i$  and containing  $x_i = x_2^i, i = 1, 2, \dots, k$  (see Figure 8).

Suppose first that  $k \geq 3$  and  $x$  is not a penultimate. Then adding to  $T$  the edge  $x_2^1x_3^2$  we create a path with  $m$  vertices. Thus  $l_1 - 1 + 2 + l_3 \geq m$  and therefore  $l_1 + l_3 \geq m - 1$ . So  $l_1 + l_2 \geq m - 1$  and (i) is proved.

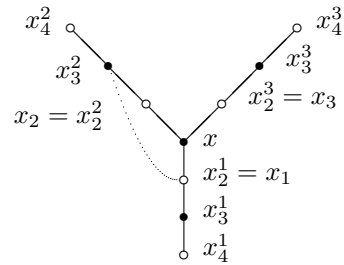
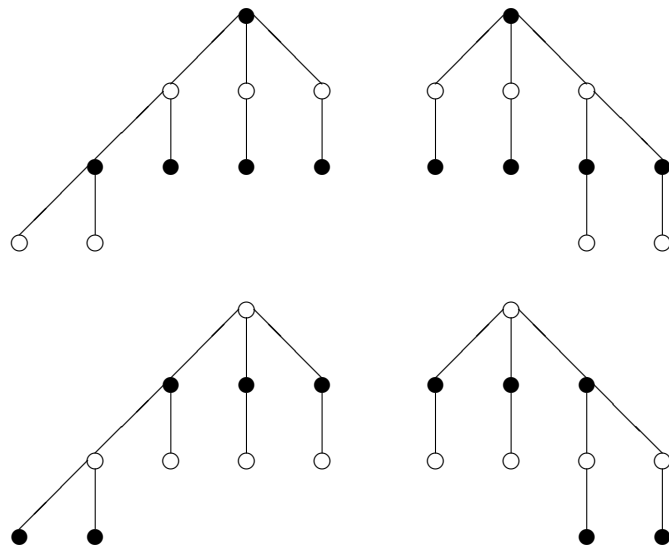


Figure 8

Suppose that  $v \in B \cup W$  and  $d_T(v) = 2$  and  $v$  is not penultimate vertex. Denote by  $u_1, v_1$  the neighbours of  $v$ ,  $P = v, u_1, \dots, u_s$  and  $P' = v, v_1, \dots, v_r$  the longest paths starting from  $v$  and passing by  $u_1, v_1$ , respectively. Then  $r, s \geq 2$ . The edge  $u_2 v_1$  create a  $P_m$  contradicting the maximality of  $P = v, u_1, \dots, u_s$ . ■



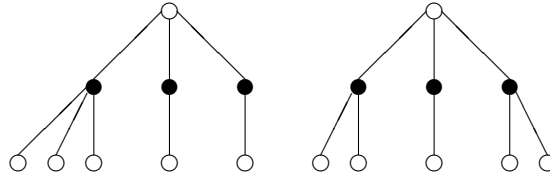
The  $P_7$ -saturated bipartite graphs with  $p = q = 5$ .

Figure 9

The next lemma follows from Lemma 19.

**Lemma 20.** *Let a tree  $T = (B, W; E)$  be a  $P_m$ -saturated bipartite graph  $m \geq 7$ . Then  $T$  contains  $A_m$ . ■*

**Proofs of Theorem 6 and 7.** Like in the proof of Lemma 18 we use the fact that for every graph  $G$  we have  $e(G) \geq |V(G)| - c$  and equality holds if and only if  $G$  is a forest with exactly  $c$  components. Hence for given  $p, q$  and  $m$ , if there is a  $P_m$ -saturated forest  $F = (B, W; E)$  with  $|B| = p, |W| = q$  and the maximum number of components then  $F$  is a  $P_m$ -saturated bipartite graph with the minimum size. On the other hand it is clear that if the assumptions of Theorem 6 or 7 are verified then there exists such a forest  $F$  that each component of  $F$  contains  $A_m, m \geq 7$  (see Figure 9 and Figure 10). ■



The  $P_7$ -saturated graphs with  $p = 3, q = 6$ .

Figure 10

Observe now that

- if  $m = 2k, k \geq 4$  and  $p = q = l(3 \cdot 2^{k-3} - 1)$ , or
- if  $m = 2k + 1, k \geq 3$  and  $p = l(4 \cdot 2^{k-3} - 1), q = l(5 \cdot 2^{k-3} - 1)$ ,

then the  $P_m$ -saturated bipartite graph  $F = (B, W; E)$  without isolated vertices and with the minimum size and with  $|W| = q, |B| = p$  is the forest containing  $l$  trees  $A_m$ .

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