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# LOWER BOUND ON THE DOMINATION NUMBER OF A TREE

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**Abstract**

We prove that the domination number $\gamma(T)$ of a tree $T$ on $n \geq 3$ vertices and with $n_1$ endvertices satisfies inequality $\gamma(T) \geq \frac{n+2-n_1}{3}$ and we characterize the extremal graphs.

**Keywords:** domination number, tree.

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## 1. Introduction

In a simple undirected graph $G = (V, E)$ a subset $D$ of $V$ is *dominating* if every vertex of $V - D$ has at least one neighbour in $D$ and $D$ is *independent* if no two vertices of $D$ are adjacent. A set is *independent dominating* if it is independent and dominating. Let $\gamma(G)$ be the minimum cardinality of a dominating set and let $i(G)$ denotes the minimum cardinality of an independent dominating set of $G$. The *neighbourhood* $N_G(v)$ of a vertex $v$ is the set of all vertices adjacent to $v$. For a set $X \subseteq V$, the *neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$. The degree of a vertex $v$ is $d_G(v) = |N_G(v)|$. For unexplained terms and symbols see [2].

Here we consider trees on at least three vertices. If $T$ is a tree, let $n = n(T)$ be the order of $T$ and let $n_1 = n_1(T)$ denote the number of endvertices of $T$. The set of endvertices of $T$ is denoted by $\Omega(T)$.

Let $D$ be a dominating set of a tree $T$. We say that $D$ has the property $\mathcal{F}$ if $D$ contains no endvertex of $T$. It is obvious that in every tree on at least 3 vertices exists a minimum dominating set having property $\mathcal{F}$.

Favaron [1] has proved that $i(T) \leq \frac{n+n_1}{3}$ for a tree $T$. The number $\frac{n+n_1}{3}$ is also an upper bound on the domination number, because $\gamma(T) \leq i(T)$. In this paper we give a lower bound on the domination number of a tree in terms of $n$ and $n_1$. Precisely, we prove that $\gamma(T) \geq \frac{n+2-n_1}{3}$ for a tree $T$ on $n \geq 3$ vertices and we characterise all trees $T$ for which $\gamma(T) = \frac{n+2-n_1}{3}$.

## 2. Results

**Theorem 1.** *If $T$ is a tree of order at least 3, then $n_1(T) \geq n(T)+2-3\gamma(T)$.*

***Proof.*** We use induction on $n$, the order of a tree. The result is trivial for a tree of order 3. Let $T$ be a tree of order $n > 3$ and assume that $n_1(T') \geq n(T') + 2 - 3\gamma(T')$ for each tree $T'$ with $3 < n(T') \leq n-1$. Let $D$ be a minimum dominating set of $T$ having property $\mathcal{F}$, let $P = (v_0, v_1, \ldots, v_l)$ be a longest path in $T$ and let $T' = T - \{v_0\}$ be the subtree of $T$. Without loss of generality we may assume that $P$ is chosen in such a way that $d_T(v_1)$ is as large as possible. We consider two cases: $d_T(v_1) > 2$ or $d_T(v_1) = 2$.

*Case* 1. $d_T(v_1) > 2$. In $T'$ we have $n_1(T') \geq n(T') + 2 - 3\gamma(T')$ (by induction), and therefore $n_1(T) \geq n(T) + 2 - 3\gamma(T)$ as $n_1(T') = n_1(T) - 1$, $\gamma(T') = \gamma(T)$ and $n(T') = n(T) - 1$.

*Case* 2. If $d_T(v_1) = 2$, we consider two subcases: $\gamma(T') < \gamma(T)$ or $\gamma(T') = \gamma(T)$.

*Subcase* 2.1. If $\gamma(T') < \gamma(T)$, then it is easy to observe, that $\gamma(T') = \gamma(T) - 1$. By induction, $n_1(T') \geq n(T') + 2 - 3\gamma(T')$ and consequently $n_1(T) \geq n(T) + 2 - 3\gamma(T)$ as $n_1(T') = n_1(T)$, $n(T') = n(T) - 1$.

*Subcase* 2.2. If $\gamma(T') = \gamma(T)$, then $v_2 \notin N_T(\Omega(T))$ (otherwise $D - \{v_1\}$ would be a dominating set of $T'$ and $\gamma(T') = \gamma(T - v_0) < \gamma(T)$) and therefore $l \geq 4$. By $T_1$ and $T_2$ we denote the subtrees of $T - v_2v_3$ to which belong vertices $v_3$ and $v_2$, respectively. If $n(T_1) = 2$, then certainly $n_1(T_1) \geq n_1(T_1) - 2 + 3\gamma(T_1)$. Thus assume that $n(T_1) \geq 3$.

Let $\Omega_2$ denotes the set $\Omega(T_2) \cap \Omega(T)$ and let $D_2$ be a minimum dominating set of $T_2$ which does not contain $v_2$. Since $d_T(v_1) = 2$, from the choice of

$P$ it follows that all neighbours of $v_2$ in $T_2$ are of degree two and this implies $|\Omega_2| = |D_2|$.

It is no problem to observe, that $\gamma(T) = \gamma(T_1) + \gamma(T_2) = \gamma(T_1) + |D_2|$ and $n(T) = n(T_1)+|\Omega_2|+|D_2|+1$. If $v_3$ is an endvertex of $T_1$ we have $n_1(T) = n_1(T_1)+|\Omega_2|-1$, otherwise $n_1(T) = n_1(T_1)+|\Omega_2| \geq n_1(T_1)+|\Omega_2|-1$ as well. Now, since $n(T_1) \geq 3$, we have by induction $n_1(T_1) \geq n(T_1) + 2 - 3\gamma(T_1)$. In both cases, for $n(T_1) = 2$ and for $n(T_1) \geq 3$ we get $n(T_1) + 2 - 3\gamma(T_1) \leq n_1(T_1) \leq n_1(T) - |\Omega_2| + 1$. Thus $n(T) - |\Omega_2| - |D_2| - 1 + 2 - 3(\gamma(T) - |D_2|) \leq n_1(T) - |\Omega_2| + 1$ and $n_1(T) \geq n(T) + 2|D_2| - 3\gamma(T) \geq n(T) + 2 - 3\gamma(T)$. ■

By $\mathcal{R}$ we denote the family of all trees in which the distance between any two distinct endvertices is congruent to 2 modulo 3, i.e., a tree $T \in \mathcal{R}$ if $d(x, y) \equiv 2 \pmod 3$ for distinct $x, y \in \Omega(T)$. The next lemma describes main properties of trees belonging to $\mathcal{R}$.

**Lemma 2.** *Let $T$ be a tree belonging to $\mathcal{R}$ and let $D$ be a minimum dominating set having property $\mathcal{F}$ in $T$. Then $d(u, v) \equiv 0 \pmod 3$ for every two vertices $u, v \in D$. In addition, $n_1(T) = n(T) + 2 - 3\gamma(T)$.*

***Proof.*** We use induction on $n$, the order of a tree. The result is obvious for stars $K_{1,n-1}, n \geq 3$. Thus, let $T \in \mathcal{R}$ be a tree of order $n > 3$ which is not a star, and let $D$ be a minimum dominating set with property $\mathcal{F}$ in $T$. Let $P = (v_0, v_1, \ldots, v_l)$ be a longest path in $T$. Since $T$ is not a star and $T \in \mathcal{R}$ we certainly have $l \geq 5$ and $l \equiv 2 \pmod 3$. We consider two cases.

*Case* 1. At least one of the vertices $v_1, v_{l-1}$ is of degree at least three, say $d_T(v_1) \geq 3$. Then $T' = T - v_0$ belongs to $\mathcal{R}$, the set $D$ is a minimum dominating set with property $\mathcal{F}$ in $T'$ and by induction $d(u, v) \equiv 0 \pmod 3$ for every two vertices $u, v \in D$. Consequently, $D$ has the same property in $T$. By induction, $n_1(T') = n(T') + 2 - 3\gamma(T')$ and therefore $n_1(T) = n(T)+2-3\gamma(T)$ as $n_1(T') = n_1(T)-1$, $n(T') = n(T)-1$ and $\gamma(T') = \gamma(T)$.

*Case* 2. $d_T(v_1) = d_T(v_{l-1}) = 2$. Since $D$ is a minimum dominating set having property $\mathcal{F}$ in $T$, vertices $v_1$ and $v_{l-1}$ belong to $D$. Because $T \in \mathcal{R}$, $d_T(v_2) = d_T(v_3) = 2$ and it is possible to choose $D$ containing $v_4$ and not $v_3$. In this case, the subgraph $T' = T - v_0 - v_1 - v_2$ is a tree belonging to $\mathcal{R}$ and $v_3$ is an end vertex of $T'$. The set $D' = D - \{v_1\}$ is a minimum dominating set with property $\mathcal{F}$ in $T'$. Since $v_3 \notin D'$, it follows that $v_4 \in D'$. By induction, $d(u, v) \equiv 0 \pmod 3$ if $u, v \in D'$. From this property and from the fact that

$T^{'} \in \mathcal{R}$ it follows that all vertices belonging to $V(T^{'}) - (D^{'} \cup \Omega(T^{'}))$ are of degree two in $T^{'}$. Since $d(u,v) \equiv 0 \,(\mathrm{mod}\, 3)$ for every two vertices $u, v \in D^{'}$, $d(v_1, v) = d(v_1, v_4) + d(v_4, v)$ is a multiple of 3 for every $v \in D$ and therefore the distance between any two vertices from $D$ is a multiplicity of 3. This easily implies that each vertex belonging to $V(T) - (D \cup \Omega(T))$ is of degree two and this forces $|V(T) - (D \cup \Omega(T))| = 2(\gamma(T) - 1)$. Thus $n(T) = |V(T)| = |\Omega(T) \cup D \cup (V(T) - (D \cup \Omega(T)))| = n_1(T) + \gamma(T) + 2(\gamma(T) - 1)$ and so $n_1(T) = n(T) + 2 - 3\gamma(T)$. ■

Now we characterise trees $T$ for which the following equality $n_1(T) = n(T) + 2 - 3\gamma(T)$ holds.

**Theorem 3.** *If $T$ is a tree, then $n_1(T) = n(T) + 2 - 3\gamma(T)$ if and only if $T$ belongs to $\mathcal{R}$.*

***Proof.*** If the tree $T$ belongs to $\mathcal{R}$ then $n_1(T) = n(T) + 2 - 3\gamma(T)$ by Lemma 1. Now assume that $T$ does not belong to $\mathcal{R}$. Then $T$ has at least four vertices and it suffices to show that $n_1(T) > n(T) + 2 - 3\gamma(T)$.

If $T$ is of order four, then $T = P_4$ and certainly $n_1(P_4) > n(P_4) + 2 - 3\gamma(P_4)$. Assume that $T$ has at least five vertices and let $P = (v_0, v_1, v_2, \ldots, v_l)$ be a longest path in $T$ and let $D$ be a minimum dominating set satisfying property $\mathcal{F}$ in $T$. We consider three cases.

*Case* 1. If $d_T(v_1) > 2$, then the tree $T^{'} = T - v_0$ does not belong to $\mathcal{R}$ and $n_1(T^{'}) > n(T^{'}) + 2 - 3\gamma(T^{'})$ (by induction), which implies $n_1(T) > n(T) + 2 - 3\gamma(T)$ as $n_1(T^{'}) = n_1(T) - 1, n(T^{'}) = n(T) - 1, \gamma(T^{'}) = \gamma(T)$.

*Case* 2. If $d_T(v_1) = 2$ and $d_T(v_2) \geq 3$ then we consider $T^{'} = T - v_0 - v_1$. Notice, that $v_1 \in D$, since $D$ safisfies property $\mathcal{F}$, $D^{'} = D - \{v_1\}$ is a dominating set of $T^{'}$ and certainly it is the smallest. Thus $\gamma(T^{'}) = \gamma(T) - 1$. For a tree $T^{'}$ we have also $n_1(T^{'}) = n_1(T) - 1$ and $n(T^{'}) = n(T) - 2$. Then $n_1(T) - n(T) + 3\gamma(T) = n_1(T^{'}) + 1 - n(T^{'}) - 2 + 3(\gamma(T^{'}) + 1) = n_1(T^{'}) - n(T^{'}) + 3\gamma(T^{'}) + 2 \geq 2 + 2 > 2$ by Theorem 1 applied to $T^{'}$.

*Case* 3. If $d_T(v_1) = 2$ and $d_T(v_2) = 2$, then we consider $T^{'} = ((T - v_0) - v_1) - v_2$. Like in *Case* 2, $v_1 \in D$, since $D$ safisfies property $\mathcal{F}$, and $D^{'} = D - \{v_1\}$ is a minimum dominating set of $T^{'}$. Thus $\gamma(T^{'}) = \gamma(T) - 1$.
If $d_T(v_3) > 2$, then $n_1(T^{'}) = n_1(T) - 1, n(T^{'}) = n(T) - 3$ and $n_1(T) - n(T) + 3\gamma(T) = n_1(T^{'}) + 1 - n(T^{'}) - 3 + 3\gamma(T^{'}) + 3 = n_1(T^{'}) - n(T^{'}) + 3\gamma(T^{'}) + 1 \geq 2 + 1 > 2$ by Theorem 1 applied to $T^{'}$. If $d_T(v_3) = 2$ then notice, that $T^{'} \notin \mathcal{R}$

(since $T \notin \mathcal{R}$). Hence $n_1(T') > n(T') + 2 - 3\gamma(T')$ by induction and finally we have $n_1(T) > n(T) + 2 - 3\gamma(T)$ as $n_1(T') = n_1(T), n(T') = n(T) - 3$. ■

## 3. Concluding Remarks

From [1] and above results it follows that $\frac{n(T)+2-n_1(T)}{3} \leq \gamma(T) \leq \frac{n(T)+n_1(T)}{3}$ for every tree $T$ on at least 3 vertices. The example of caterpillar given in Figure 1 proves that the difference between $\gamma(T)$ and $\frac{n(T)+2-n_1(T)}{3}$ can be arbitrarily large. It is no problem to observe that $\gamma(T_l) - \frac{n(T_l)+2-n_1(T_l)}{3} = \frac{2l-2}{3}$ for any integer $l \geq 3$.

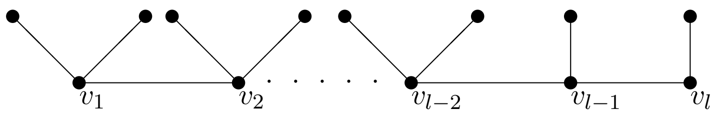


Figure 1. Caterpillar

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## References

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