Discussiones Mathematicae
Graph Theory 24 (2004) 125–135

A NOTE ON TOTAL COLORINGS OF PLANAR GRAPHS WITHOUT 4-CYCLES

Ping Wang*

Department of Mathematics, Statistics and Computer Science
St. Francis Xavier University, Antigonish, Nova Scotia, Canada

e-mail: pwang@stfx.ca

and

Jian-Liang Wu†

School of Mathematics, Shandong University
Jinan, Shandong, 250100, P.R. China

Abstract

Let G be a 2-connected planar graph with maximum degree Δ such that G has no cycle of length from 4 to k, where $k \geq 4$. Then the total chromatic number of G is $\Delta+1$ if $(\Delta, k) \in \{(7,4),(6,5),(5,7),(4,14)\}$.

Keywords: total coloring, planar graph, list coloring, girth.

2000 Mathematics Subject Classification: 05C15.

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G respectively. Let $d(v)$ denote the degree of vertex v. A *k-vertex* is a vertex of degree k.

A *total k-coloring* of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi_T(G)$ is the smallest integer k such that G has a total k-coloring. Behzad and Vizing (see page 86 in [8]) conjectured independently that any graph G is totally $(\Delta(G)+2)$-colorable in 1965.

*Research supported by The Natural Sciences and Engineering Council of Canada.
†Research supported by the James Chair from St. Francis Xavier University.

Various coloring techniques have been introduced in effort to prove this conjecture for some special graph classes (see survey papers [7] and [11]). In 1989, Sanchez-Arroyo [10] proved that for any graph G it is NP-complete to decide if $\chi_T(G) = \Delta(G)+1$. In 1997, Borodin et al [3] proved that a planar graph G with maximum degree $\Delta \geq 11$ has $\chi_T(G) = \Delta(G)+1$, and they also obtained several related results by adding girth restrictions [4]. Note that the added girth requirement in [4] prohibits the appearance of triangles. The forbidden cycle or the girth restriction plays an important role in considering list-coloring planar graphs. For example, Kratochvíl and Tuza showed that every triangle-free planar graph is 4-choosable and Thomassen observed that a planar graph is 3-choosable if the girth of the graph is at least 5 (both results can be found in Section 2.13 of [8]). Recently, Lam, Xu and Liu [9] proved that every C_4-free planar graph is 4-choosable. We shall adopt a similar approach and prove the following theorem. Note that triangles are allowed in the graph G in our theorem.

Let a planar graph G be charged by an initial charge $w(v) = d(v) - 4$ if $v \in V(G)$ and $w(f) = r(f) - 4$ if $f \in F$, where $r(f)$ is the degree of the face f. Euler's formula implies that $\sum_{x \in V \cup F} w(x) < 0$. The discharging method distributes the positive charge to neighbors so as to leave as little positive charge remaining as possible. This leads to $\sum_{x \in V \cup F} w(x) > 0$. A contradiction follows and this shows the unavoidability of a set of special elements in G (see Claims 2, 3 and 4).

Theorem. *Let G be a connected planar graph with maximum degree Δ such that G has no cycle of length from 4 to k, where $k \geq 4$. If*

(1) $\Delta \geq 7$ *and* $k \geq 4$ *or*

(2) $\Delta \geq 6$ *and* $k \geq 5$, *or*

(3) $\Delta \geq 5$ *and* $k \geq 7$, *or*

(4) $\Delta \geq 4$ *and* $k \geq 14$,

then $\chi_T(G) = \Delta(G) + 1$.

Lemma 1 [6]**.** *Every region of a planar imbedding of a graph has a simple cycle for its boundary if and only if G is 2-connected.*

This lemma is equivalent to the assertion that no three edges incident with any vertex v lie on the same face. It implies that each vertex v is incident with $d(v)$ faces. We shall use this fact often in the proof of the Theorem.

An *edge coloring* of a graph G is a coloring of $E(G)$ such that no two adjacent edges receive the same color. A graph G is said to be *edge-f-choosable* if, whenever we give lists A_e of $f(e)$ colors to each edge $e \in E(G)$, there exists an edge coloring of G where each edge is colored with a color from its own list.

Lemma 2 [5]**.** *A bipartite graph G is edge-f-choosable where $f(e) = \max\{d(u), d(v)\}$ for $e = uv \in E(G)$.*

Proof of Theorem. Let $G = (V, E, F)$ be a minimal counterexample to any of (1) – (4) in the Theorem. Then

(a) G is 2-connected and

(b) any vertex is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces, and

(c) G contains no even cycle $v_1v_2 \cdots v_{2t}v_1$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$, and

(d) G contains no edge uv with $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta(G)}{2} \rfloor$ and $d_G(u) + d_G(v) \leq \Delta(G) + 1$.

(a) and (b) are obvious. The proofs of (c) and (d) can be found in [2] and [5], respectively.

Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. Since $\Delta(G) \geq 4$ in all four cases in the Theorem, (d) implies that G does not contain two adjacent 2-vertices. Hence, G_2 does not contain any odd cycle. It follows from (c) that G_2 does not contain any even cycle. Therefore, any component of G_2 is a tree. For any component in G_2 that is a path of even length, one can easily find a set of edges saturating all 2-vertices. For any component that is not a path of even length, we can select a vertex t with $d_{G_2}(t) \geq 3$ as the root of the tree. We denote edges of distance i from the root to be at level $i + 1$ where $i = 0, 1, ..., d$ and d is the depth of the tree. Since G does not contain two adjacent 2-vertices, the distance from any leaf to the root is even. We can select all the edges at even level to form a matching saturating all 2-vertices in this component. Thus, there exists a matching M such that all 2-vertices in G_2 are saturated. If $uv \in M$ and $d(u) = 2$, v is called the 2-*master* of u and u is called the *dependent* of v. Each 2-vertex has a 2-master and each vertex of degree Δ can be the 2-master of at most one 2-vertex.

Since G is a planar graph, by Euler's formula, we have

$$(E) \qquad \sum_{v\in V}(d(v)-4)+\sum_{f\in F}(r(f)-4)=-4(|V|-|E|+|F|)=-8<0,$$

where $r(f)$ is the *degree* of the face f, that is, the number of edges around f. A k-*face* is a face of degree k. Now we define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(x) = d(x) - 4$ if $x \in V$ and $w(x) = r(x) - 4$ if $x \in F$. It follows from (E) that $\sum_{x\in V\cup F} w(x) < 0$.

We begin the proof of (1) in the Theorem. First we prove a claim establishing a relation between the set of vertices of degree 3 or less and the set of vertices of degree at least $\Delta - 1$. We adopt the classic technique used in proving Hall's Matching Theorem (see page 72 in [1]). Let X be the set of vertices of degree at most 3 and $Y = \cup_{x\in X} N(x)$. By (d), X is an independent set of G. Let K be the induced bipartite subgraph of G with partite sets X and Y.

Claim 1. If $X \neq \emptyset$, then G contains a bipartite subgraph $B = (X, Y)$ such that $d_B(x) = 1$ and $d_B(y) \leq 2$ whenever $x \in X$ and $y \in Y$.

***Proof of Claim* 1.** Let $H = (X', Y)$, where $X' \subseteq X$, be a maximum bipartite subgraph such that $d_H(x) = 1$ and $d_H(y) \leq 2$ whenever $x \in X'$ and $y \in Y$. Note that there may be some isolated vertices in Y. Clearly, H is not empty since there is at least one edge from X to Y. Suppose that $X \backslash X' \neq \emptyset$. Let $v \in X \backslash X'$. An *alternating path*, P_v, in G is a path whose origin is v and edges are alternating between $E(K) \setminus E(H)$ and $E(H)$. By the maximality of H, there exists no alternating path that will terminate at a vertex $v' \in Y$ with $d_H(v') \leq 1$. Let Z denote the set of of all vertices connected to v by alternating paths. Set $X'' = Z \cap X'$ and $Y'' = Z \cap Y$ (see Figure 1).

Clearly, $Y'' \subseteq \cup_{x\in X''} N(x)$. Suppose $\cup_{x\in X''} N(x) \nsubseteq Y''$. It follows that there exists a vertex $x \in X''$ such that $xy \in E(G)$ and $y \notin Y''$. This implies that an alternating path P_v terminates at a vertex $y \in Y$, a contradiction. Hence, $Y'' = \cup_{x\in X''} N(x)$.

Now we show that $d_H(y) \geq 2$ for any $y \in Y''$. Suppose, on the contrary, there exists a vertex $y_i \in Y''$ where $vy_1x_1...x_{i-1}y_i$ is an alternating path such that $d_H(y_i) = 1$. Let $H' = H - \{y_1x_1, ..., y_{i-1}x_{i-1}\} + \{vy_1, x_1y_2, ..., y_ix_{i-1}\}$ if $i \geq 2$ and let $H' = H + \{vy_1\}$ if $i = 1$. It follows that $|E(H')| > |E(H)|$, a contradiction to H being maximum.

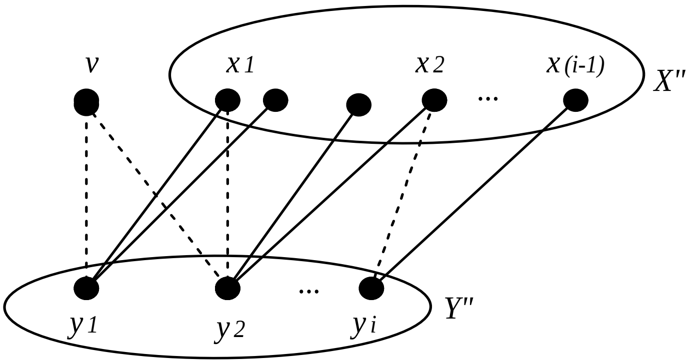

Figure 1. Subgraph F

Let $F = (X'', Y'')$. It follows that $d_F(y) \geq d_H(y) + 1 \geq 3$ for any $y \in Y''$. Note that $d_G(x) = d_F(x) \leq 3$ for any $x \in X''$.

Now $G - X''$ has a total $(\Delta + 1)$-coloring by the minimality of G. By Lemma 2, we can color all edges in F using the same set of colors by choosing the colors unused on $y \in Y''$. Since the maximum degree in X'' is 3, all vertices in X'' can be easily colored by $(\Delta+1)$ colors. Therefore, G has a total $(\Delta + 1)$-coloring, a contradiction with the fact that G is a counterexample. This implies $X = X'$, and which in turn, proves Claim 1. ■

We call y the 3-*master* of x if $xy \in B$ and $x \in X$. It follows from this claim that each vertex of degree at most 3 has a 3-master. Each vertex of degree at least $\Delta - 1$ can be a 3-master of at most two vertices.

Claim 2. If $\Delta \geq 7$, then G does not contain a 3-face uvw such that $d(u) = d(v) = 4$.

***Proof of Claim* 2.** Suppose it does contain such a 3-face. Let $G' = G - uv$. By the minimality of G, G' has a total $(\Delta + 1)$-coloring φ. Since $d_{G'}(u) = d_{G'}(v) = 3$ and $\Delta \geq 7$, we may assume that $\varphi(u) \neq \varphi(v)$. Let C be the set of colors used to color edges adjacent to uv. If $\varphi(w) \notin C$, then color uv with $\varphi(w)$. Otherwise, without loss of generality, we may assume that an edge e incident with u is colored with $\varphi(w)$. Then we erase the color on u. It follows that at least one color is available for uv, and then we re-color u. This is possible because $d(u) = 4$ and both e and w share the same color. Now, G has a total coloring with $(\Delta + 1)$ colors, a contradiction with the fact that G is a counterexample. ■

Claim 2 and (d) imply that every 3-face is incident with at least two vertices of degree at least 5. To prove (1), we are ready to construct a new charge $w^*(x)$ on G as follows:

R11: Each $r(\geq 5)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R12: Each 2-vertex receives $\frac{3}{5}$ from its 3-master, and receives $\frac{16}{15}$ from its 2-master if it is incident with a 3-face and receives 1 from its 2-master otherwise.

R13: Each 3-vertex receives $\frac{8}{15}$ from its 3-master. In addition, if v is incident with a 3-face f, then each 3-vertex v receives $\frac{1}{15}$ from u where u is a neighbor of v but not incident with f.

R14: Each 3-face receives $\frac{1}{2}$ from its incident vertices of degree at least 5.

By (d), $d(v) = \Delta \geq 7$ if a vertex v is the 2-master of some vertex, $d(v) \geq \Delta - 1 \geq 6$ if v is the 3-master of some vertices, and $d(u) \geq 6$ if a vertex u gives $\frac{1}{15}$ via R13. Note that a vertex can be the 3-master of most two vertices and, in turn, it may give at most $2 \times \max\{\frac{3}{5}, \frac{8}{15}\} = \frac{6}{5}$. Let f be a face of G. Clearly, $w^*(f) = 0$ if $r(f) \geq 5$. By Claim 2, each 3-face f is incident with at least two vertices of degree at least 5. Hence, $w^*(f) \geq w(f) + 1 = 0$. Let v be an arbitrary vertex of G. First, we consider the case of $d(v) = 2$. It will receive $\frac{3}{5}$ from its 3-master. By Lemma 1, v is incident with two faces. If v is incident with a 3-face, then the other incident face of v must have degree at least 6 since G is a C_4-free graph. This implies that v receives at least $\frac{1}{3}$ from the face of degree ≥ 6. If v is not incident with a 3-face, then v receives at least $2 \times \frac{1}{5}$ from its incident faces. So $w^*(v) \geq w(v) + \min\{\frac{3}{5} + \frac{16}{15} + \frac{1}{3}, \frac{3}{5} + 1 + \frac{2}{5}\} = 0$. Consider $d(v) = 3$. If it is incident with a 3-face, then the other two vertices on the same face must be of degree at least 5 and this implies that v receives at least $\frac{2}{5}$ from its incident faces. If v is not incident with a 3-face, then it must be incident with three r-faces where $r \geq 5$. It follows that it receives at least $\frac{3}{5}$ from its incident faces. Hence, $w^*(v) \geq w(v) + \min\{\frac{8}{15} + \frac{1}{15} + \frac{2}{5}, \frac{8}{15} + \frac{3}{5}\} = 0$. If $d(v) = 4$, then it is incident with at most two 3-faces and its other two incident faces must be of degree ≥ 5. Hence, $w^*(v) \geq w(v) + \frac{2}{5} > 0$. If $d(v) = 5$, then v is incident with at least three r-faces where $r \geq 5$ and at most two 3-faces. Hence, $w^*(v) \geq w(v) + \frac{3}{5} - 2 \times \frac{1}{2} > 0$. If $d(v) = 6$, it can be 3-master of at most two vertices. Consider any two neighbors of v, say u_1 and u_2. If they form a 3-face, then v gives $\frac{1}{2}$ to the 3-face. If each of them is a 3-vertex on some 3-face, then v gives $2 \times \frac{1}{15}$. However, these two cases can not happen simultaneously; that is, vu_1u_2 is a 3-face and u_1,

u_2 have another common neighbor $w \neq v$, such that either $d(u_1) = 3$ or $d(u_2) = 3$ since G is C_4-free graph. In the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case when v gives $3 \times \frac{1}{15}$ to its incident 3-faces. It follows that $w^*(v) \geq w(v) + \frac{3}{5} - 2 \times \frac{8}{15} - 3 \times \frac{1}{2} > 0$. Now consider $d(v) = 7$. Suppose v is a 2-master of a vertex u. If u and v are incident with the same 3-face, then v receives at least $3 \times \frac{1}{5} + (1 - \frac{4}{6})$ from its incident faces and gives $\frac{16}{15}$ to u. Otherwise v receives at least $4 \times \frac{1}{5}$ from its incident faces and gives 1 to u. Vertex v may be incident with at most three 3-faces and the remaining neighbor of v not incident with any three 3-faces may be a 3-vertex and in another 3-face, in turn, v may give $\frac{1}{15}$ to the 3-vertex. Vertex v may also be the 3-master of two other vertices. Hence, $w^*(v) \geq w(v) + \min\{3 \times \frac{1}{5} + \frac{1}{3} - \frac{16}{15}, \frac{4}{5} - 1\} - (3 \times \frac{1}{2} + \frac{1}{15} + 2 \times \frac{3}{5}) > 0$. In general, if $d(v) \geq 8$, then $w^*(v) \geq w(v) + \min\{\lfloor \frac{d(v)}{2} \rfloor \times \frac{1}{5} + \frac{1}{3} - \frac{16}{15}, \lceil \frac{d(v)}{2} \rceil \times \frac{1}{5} - 1\} - (\lfloor \frac{d(v)}{2} \rfloor \times \frac{1}{2} + \frac{1}{15} + 2 \times \frac{3}{5}) > 0$. It follows that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) \geq 0$, a contradiction with (E). This completes the proof of (1).

Note that (1) implies that (2) is true if $\Delta \geq 7$. Hence, it is sufficient to prove (2) by assuming $\Delta = 6$. Similarly, we may assume that $\Delta = 5$ in the proof of (3) and $\Delta = 4$ in the proof of (4).

Claim 3. If $\Delta \geq 5$, then G does not contain a 3-face uvw such that $d(u) = d(v) = 3$.

The proof of Claim 3, which we omit, is the same as Claim 2. Claim 3 implies that each 3-face is incident with at least two vertices of degree at least 4. To prove (2), we construct the new charge $w^*(x)$ on G as follows:

R21: Each $r(\geq 6)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R22: Each 2-vertex receives $\frac{11}{7}$ from its 2-master if it is incident with a 3-face and receives $\frac{4}{3}$ from its 2-master otherwise.

R23: Each 3-vertex v receives $1/3$ from u if v is incident with a 3-face f and u is a neighbor of v but not incident with f.

R24: Each 3-face receives $\frac{1}{2}$ from its incident vertex v if $d(v) \geq 5$ and receives $\frac{1}{3}$ if $d(v) = 4$.

Clearly, we have $w^*(f) \geq 0$ for any face f. Let v be an arbitrary vertex of G. Consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face must have degree at least 7 since G is a C_4-free and C_5-free graph. It follows that v receives at least $1 - \frac{4}{7} = \frac{3}{7}$ from the incident face and $\frac{11}{7}$ from its 2-master; that is, $w^*(v) \geq w(v) + \frac{3}{7} + \frac{11}{7} = 0$. Otherwise if

v is not incident with any 3-face, then it receives at least $2 \times (1 - \frac{4}{6}) = \frac{2}{3}$ from its two incident faces of degree at least 6 and $\frac{4}{3}$ from its 2-master. Hence, $w^*(v) \geq w(v) + \frac{2}{3} + \frac{4}{3} = 0$. Suppose $d(v) = 3$. If v is incident with a 3-face, then v receives at least $\frac{2}{3}$ from its two incident faces and $\frac{1}{3}$ from its 3-master not lying on the same 3-face. Otherwise if v is not incident with any 3-face, then v receives at least 1 from its three incident faces. Hence, $w^*(v) \geq w(v) + 1 = 0$. Note that v gives either $\frac{1}{3}$ if $d(v) = 4$ or $\frac{1}{2}$ if $d(v) \geq 5$ to an incident 3-face, say vuw where $u, w \in N(v)$, or gives $\frac{1}{3}$ to u and $\frac{1}{3}$ to w by R23 but v will then receive at least $1 - \frac{4}{6} = \frac{1}{3}$ from the face whose partial boundary contains u, v, w sequentially if $uw \notin E(G)$. In the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case when v gives $\frac{1}{3}$ or $\frac{1}{2}$ to its incident 3-faces. If $d(v) = 4$, then it receives at least $\frac{2}{3}$ from its two incident faces of degree ≥ 6 and gives at most $\frac{2}{3}$ to its incident 3-faces since any 4-vertex is incident with at most two 3-faces. It follows that $w^*(v) \geq w(v) + \frac{2}{3} - \frac{2}{3} = 0$. If $d(v) = 5$, then there are five faces incident with v by Lemma 1. It follows that v is incident with at most two 3-faces and at least three r-faces ($r \geq 6$). If four neighbors of v form two 3-faces and a 3-face is pending on the remaining neighbor of v, then v discharges at most $2 \times \frac{1}{2} + \frac{1}{3}$ via R23. This implies that $w^*(v) \geq w(v) + 3 \times \frac{1}{3} - (2 \times \frac{1}{2} + \frac{1}{3}) > 0$. Suppose $d(v) = 6$. It follows that v can be the 2-master of some vertex u. In this case, either u is on 3-face vuu' (and it follows that v gives $\frac{11}{7} + \frac{1}{2}$), or v is the 2-master of u and 3-master of u' where v and u are not on the same 3-face, and it follows that v gives $\frac{4}{3} + \frac{1}{3}$. To find a low bound for $w^*(v)$, it suffices to consider the first case when v is the 2-master of u and vuu' forms a 3-face. If v is the 3-master of some 3-vertex u_1, then v gives at most $2 \times \frac{1}{3}$ to its dependents and $\frac{1}{2}$ to another 3-face. In this case, v receives $\frac{3}{7} + 3 \times \frac{1}{3}$ from its incident faces. If v is not a 3-master of any 3-vertex, then v gives at most $2 \times \frac{1}{2}$ to its two incident 3-faces. In this case, v receives $\frac{3}{7} + 2 \times \frac{1}{3}$ from its incident faces. Hence, $w^*(v) \geq w(v) + \min\{\frac{3}{7} + 1 - (\frac{11}{7} + \frac{1}{2} + \frac{2}{3} + \frac{1}{2}), \frac{3}{7} + \frac{2}{3} - \frac{11}{7} - \frac{1}{2} - 1\} = 2 - \frac{83}{42} = \frac{1}{42} > 0$. It follows that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (2).

To prove (3), we construct a new charge $w^*(x)$ on G as follows:

R31: Each $r(\geq 8)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R32: Each 2-vertex receives $\frac{13}{9}$ from its 2-master if it is incident with a 3-face and receives 1 from its 2-master otherwise.

R33: Each 3-face receives $\frac{1}{2}$ from its incident vertices of degree at least 4.

We also have $w^*(f) \geq 0$ for any face f. Let v be an arbitrary vertex of G. If $d(v) = 2$, then $w^*(v) \geq w(v) + \min\{\frac{13}{9} + \frac{5}{9}, 2 \times \frac{1}{2} + 1\} = 0$. If $d(v) = 3$, then v is incident with at most one 3-face and at least two faces of degree ≥ 8. It follows that v receives at least $2 \times \frac{1}{2} = 1$ from its incident faces, and in turn, $w^*(v) \geq w(v) + 1 = 0$. If $d(v) = 4$, then it receives at least $2 \times \frac{1}{2} = 1$ from its incident faces and gives at most $2 \times \frac{1}{2} = 1$ to its incident 3-faces, that is, $w^*(v) \geq w(v) + 1 - 1 = 0$. Suppose $d(v) = 5$. If v is the 2-master of a 2-vertex u, and u is incident with a 3-face, then v receives at least $3 \times \frac{1}{2}$ from its incident faces and gives at most $\frac{13}{9} + 2 \times \frac{1}{2}$ to its dependent and 3-faces. Otherwise v receives at least $3 \times \frac{1}{2}$ from its incident faces and gives at most $1 + 2 \times \frac{1}{2}$ to its dependent and 3-faces. It follows that $w^*(v) \geq w(v) + \min\{\frac{3}{2} - \frac{13}{9} - 1, \frac{3}{2} - 2\} = \frac{1}{18} > 0$. This implies that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (3).

We will prove the following claim before we prove (4).

Claim 4. If $\Delta = 4$, then G contains no 4-vertex z where z is incident with two 3-faces zux, zvy and $d(x) = d(y) = 2$.

***Proof of Claim* 4.** Suppose, on the contrary, such vertex z does exist. We can totally color the edges and vertices of $G - \{xz, yz\}$ with a set of five colors, say C, by the minimality of G. First, we erase the colors assigned on x and y. Let c_1, c_2, c_3, c_4, c_5 be colors used on xu, zu, zv, yv, z, respectively.

We will show that $c_1 \neq c_4$. Otherwise if $c_1 = c_4$, we claim that $c_1 \neq c_5$. If $c_1 = c_5$, then we can color xz by $\alpha \in C\backslash\{c_1, c_2, c_3\}$ and yz by a color in $C\backslash\{c_1, c_2, c_3, \alpha\}$. It easy to see that x and y can be colored because they are only adjacent to two vertices and incident with two edges. This implies that G can be totally colored by five colors, a contradiction. Now we show it is impossible that $c_1 = c_4$ and $c_1 \neq c_5$. If $c_1 = c_4$, then we can interchange colors c_3 and c_1 at v and color zx by c_3. It follows that we can also color zy by a color in $C\backslash\{c_1, c_2, c_3, c_5\}$. Similarly we can color vertices x and y since they are both vertices of degree 2. This implies that G can be totally colored by five colors, a contradiction.

Similarly, we can show that $c_1 \neq c_3$ and $c_1 \neq c_5$. Since $c_1 \notin \{c_2, c_3, c_4, c_5\}$, c_1 can be assigned to zy and there is a color available for zx, x and y. This implies that G can be totally colored by five colors, a contradiction. ■

To prove (4), construct a new charge $w^*(x)$ on G as follows:

R41: Each $r(\geq 15)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R42: Each 2-vertex receives $\frac{19}{24}$ from its neighbors if it is incident with a 3-face and receives $\frac{8}{15}$ from its 2-master otherwise.

R43: Each 3-face receives $\frac{1}{3}$ from its incident vertices.

It is obvious that $w^*(f) = 0$ for any face f. Let v be an arbitrary vertex of G. First consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face f must have degree at least 16. From (d), any neighbor of v should be of degree at least $(\Delta + 2) - 2 = 4$. Hence, they can not be 2-vertices. It follows that v receives at least $1 - \frac{4}{16} = \frac{3}{4}$ from f and $2 \times \frac{19}{24} = \frac{19}{12}$ from its neighbors, and gives $\frac{1}{3}$ to its incident 3-face. Otherwise v receives at least $2 \times \frac{11}{15} = \frac{22}{15}$ from its incident faces and $\frac{8}{15}$ from its 2-master. Hence, $w^*(v) \geq w(v) + \min\{\frac{3}{4} + \frac{19}{12} - \frac{1}{3}, \frac{22}{15} + \frac{8}{15}\} = 0$. Now consider the case of $d(v) = 3$. v receives at least $2 \times \frac{11}{15} = \frac{22}{15}$ from its incident faces. Hence, $w^*(v) = w(v) + \frac{22}{15} - \frac{1}{3} = \frac{2}{15} > 0$. If $d(v) = 4$ and it is incident with two 3-faces, then v is adjacent to at most one 2-vertex by Claim 4. It follows that $w^*(v) \geq w(v) + \frac{22}{15} - (\frac{2}{3} + \frac{19}{24}) = \frac{1}{120} > 0$. Otherwise it receives at least $3 \times \frac{11}{15}$ from its incident faces, and gives at most $\frac{1}{3}$ to its incident 3-face and $\frac{19}{24} + \frac{8}{15}$ to its adjacent 2-vertices. It follows that $w^*(v) \geq w(v) + \frac{33}{15} - (\frac{1}{3} + \frac{19}{24} + \frac{8}{15}) = \frac{13}{24} > 0$. This implies that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (4). ∎

In the proof of the Theorem, we showed that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$. It implies the following corollary.

Corollary 1. *Let G be a graph with maximum degree Δ embedded in a surface of nonnegative characteristic, and G has no cycle of length from 4 to k, where $k \geq 4$. Then $\chi_T(G) = \Delta + 1$ if $(\Delta, k) \in \{(7,4), (6,5), (5,7), (4,14)\}$.*

Acknowledgements

We sincerely thank the anonymous reviewer whose useful and critical comments have significantly enhanced the content, organization and presentation of this paper.

References

[1] J.A. Bondy and U.S.R. Murty (Graph Theory with Applications, North-Holland, 1976).

[2] O.V. Borodin, *On the total coloring of planar graphs*, J. Reine Angew. Math. **394** (1989) 180–185.

[3] O.V. Borodin, A.V. Kostochka and D.R. Woodall, *Total colorings of planar graphs with large maximum degree*, J. Graph Theory **26** (1997) 53–59.

[4] O.V. Borodin, A.V. Kostochka and D.R. Woodall, *Total colourings of planar graphs with large girth*, European J. Combin. **19** (1998) 19–24.

[5] O.V. Borodin, A.V. Kostochka and D.R. Woodall, *List edge and list total colourings of multigarphs*, J. Combin. Theory (B) **71** (1997) 184–204.

[6] J.L. Gross and T.W. Tucker (Topological Graph Theory, John and Wiley & Sons, 1987).

[7] A.J.W. Hilton, *Recent results on the total chromatic number*, Discrete Math. **111** (1993) 323–331.

[8] T.R. Jensen and B. Toft (Graph Coloring Problems, John Wiley & Sons, 1995).

[9] Peter C.B. Lam, B.G. Xu, and J.Z. Liu, *The* 4*-choosability of plane graphs without* 4*-cycles*, J. Combin. Theory (B) **76** (1999) 117–126.

[10] A. Sanchez-Arroyo, *Determining the total coloring number is NP-hard*, Discrete Math. **78** (1989) 315–319.

[11] H.P. Yap, *Total colourings of graphs*, Lecture Notes in Mathematics **1623** (Springer, 1996).

Received 26 February 2002
Revised 21 October 2003