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SOME APPLICATIONS OF *pq*-GROUPS IN GRAPH THEORY

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Abstract

We describe some new applications of nonabelian pq-groups to construction problems in Graph Theory. The constructions include the smallest known trivalent graph of girth 17, the smallest known regular graphs of girth five for several degrees, along with four edge colorings of complete graphs that improve lower bounds on classical Ramsey numbers.

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For primes p and q such that $p < q$ and $q \equiv 1 \pmod{p}$, there exists a nonabelian group of order pq, which is a semidirect product of Z_q by Z_p. Details on the properties of these groups can be found in [3]. In this note, we use nonabelian pq-groups to construct several graphs and graph colorings.

Let $\mathcal{G}$ be a nonabelian pq-group and let a and b be elements of orders p and q, respectively. The subgroup generated by b is normal in $\mathcal{G}$ and $a^{-1}ba = b^r$ where $r^p \equiv 1 \pmod{q}$. So the elements of $\mathcal{G}$ can be represented in the form a^ib^j, for $0 \le i < p$ and $0 \le j < q$. Group multiplication can be defined by

$$a^ib^j * a^sb^t = a^{i+s}b^{jr^s+t}.$$

In what follows, it will be convenient to refer to elements of a pq-group using the notation (i, j) instead of a^ib^j.

1. Ramsey Graphs

Recall that an (s, t)-coloring of the complete graph K_n is a 2-coloring of its edges such that there is neither a complete subgraph of order s, all of whose edges are color 1, nor a complete subgraph of order t, all of whose edges are color 2. The Ramsey number $R(s, t)$ is the minimum n such that no (s, t)-coloring on K_n exists.

Our first application of pq-groups gives us a new lower bound for $R(4, 8)$. For this we use a Cayley coloring on the nonabelian group $\mathcal{G}$ of order 55, where we take $p = 5$, $q = 11$, and $r = 3$. Recall that in a Cayley coloring, the vertices of the graph are identified with group elements, and we assign a color to each element $g \in \mathcal{G}$, and then assign that color to all edges of the form (x, xg), for all $x \in \mathcal{G}$. In the array below, the entry in row i and column j gives the color assigned to each element (i, j), i.e., $a^i b^j$, of $\mathcal{G}$. It can be easily verified by computer that this indeed determines a $(4, 8)$-coloring of K_{55}, thereby improving the lower bound on $R(4, 8)$ to 56 [4].

1	2	2	1	1	1	1	1	1	2	2
2	2	2	1	1	2	1	1	2	2	2
2	1	2	2	2	2	1	2	2	2	1
2	2	2	2	1	2	2	1	2	1	2
2	1	2	1	2	2	1	2	1	2	2

To obtain more new Ramsey colorings we use a well known property of pq-groups: they have a permutation representation of degree q. One can use **GAP** to obtain such a representation [6]. The representation is applied to the construction of Ramsey graphs on q vertices by considering the induced action on the edges of K_q. Each edge orbit contains pq edges, and so we only need to make $\frac{q-1}{2p}$ color choices to determine a coloring. Thus dealing with searches for relatively large Ramsey graphs is much easier than when dealing with circle colorings, for example.

Following up on this idea, and using the combinatorial search technique outlined in [1], we are able to improve three classical Ramsey bounds [4]. These colorings may be of particular interest, since they are not circle colorings.

$$R(3, 27) > 157,$$

$$R(3, 31) > 197,$$

$$R(5, 17) > 283.$$

Since these graphs are so large, rather than attempt to present them here, we note that they can be obtained in several electronic formats from the author's web site at `http://ginger.indstate.edu/ge/RAMSEY`.

2. A TRIVALENT GRAPH OF GIRTH 17

In this section, we use a construction method, first discussed in [2], that can be viewed as a generalization of Cayley graphs. Instead of using one copy of a group as the vertex set, as with Cayley graphs, we use multiple copies. The groups that seem to be most useful for the cage problems below are the nonabelian pq-groups.* A property of pq-groups that makes them useful in this regard is the absence of elements of order less than p (save the identity).

Our next construction is made by using the nonabelian group $\mathcal{G}$ of order $301 = 7 \times 43$. To complete the definition of the group, we take $r = 4$, then start with a set V of 2408 vertices, viewed as 8 copies of $\mathcal{G}$. Define a group action on V by fixing an ordering of the group elements, $g_0, \ldots, g_{300}$, and identify the vertices with the integers $0 \ldots 2407$. The ordering of the group elements is given by having $(i_1, j_1) < (i_2, j_2)$ if and only if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$. We associate vertex $301i + j$ with group element g_j for $1 \leq i \leq 8$. Then the action of $\mathcal{G}$ of V is defined as follows. For $g \in \mathcal{G}$, let $g(301i + j) = 301i + k$ if and only if $g_j g = g_k$ in $\mathcal{G}$. Then it is simply a matter of selecting a set of orbits from the induced action on the edges. The orbits used are indicated below, where we list one edge from each edge orbit chosen.

0	593
0	1185
0	1329
301	1343
301	2206
602	1158
602	1967
602	2278
903	1506
1204	2006
1505	1923
1505	2374

*Recall that a (k, g)-cage is a smallest k-regular graph of girth g.

Adjacency lists for all the cage constructions presented here can be obtained from `http://ginger.indstate.edu/ge/CAGES`.

Similarly, we were able to construct the smallest known four-regular graph of girth 9 [5] from five copies of the nonabelian group G of order 55, using $p = 5$, $q = 11$, $r = 3$. Edge orbit representatives are again listed below.

0	55
0	73
0	212
0	239
55	219
55	227
110	176
110	214
110	258
110	263

3. Small Regular Graphs of Girth 5

Finally, we undertook a search for Cayley graphs of girth 5 on nonabelian pq-groups. We were able to find the smallest known regular graphs of girth 5 for several even degrees in the range 12 to 32. The table below summarizes our results, listing the degree, girth, and order of the graph along with the values of p, q, and r for the group. The column labeled "ratio" gives the order of the graph divided by the Moore bound, which of course is $1 + k^2$ for graphs of degree k and girth 5.

degree	girth	order	p	q	r	ratio
12	5	203	7	29	7	1.400
14	5	355	5	71	5	1.802
16	5	497	7	71	20	1.934
18	5	655	5	131	53	2.015
20	5	889	7	127	2	2.217
22	5	1027	13	79	8	2.118
24	5	1255	5	251	20	2.175
26	5	1655	5	331	64	2.445
28	5	2005	5	401	39	2.554
30	5	2359	7	337	8	2.618
32	5	2947	7	421	33	2.875

Below we give the generators for each of the Cayley graphs. An entry of (i, j) means that the group element $a^i b^j$ is a generator. The inverses of these generators are not listed.

For $d = 12$, $n = 203$:

$(1, 0)(2, 2)(2, 27)(2, 3)(3, 11)(3, 28)$

For $d = 14$, $n = 355$:

$(0, 10)(0, 61)(1, 54)(1, 69)(2, 35)(2, 41)(2, 53)(2, 64)$

For $d = 16$, $n = 497$:

$(1, 24)(1, 41)(1, 51)(2, 20)(2, 22)(2, 45)(3, 4)(3, 56)$

For $d = 18$, $n = 655$:

$(1, 10)(1, 113)(1, 29)(1, 37)(1, 87)(2, 118)(2, 62)(2, 86)(2, 97)$

For $d = 20$, $n = 889$:

$(1, 112)(1, 122)(1, 21)(1, 75)(1, 90)(1, 98)(2, 50)(2, 56)$
$(2, 59)(3, 41)$

For $d = 22$, $n = 1027$:

$(1, 22)(1, 30)(1, 52)(2, 0)(2, 27)(2, 39)(3, 27)(3, 71)(4, 13)$
$(5, 14)(5, 16)$

For $d = 24$, $n = 1255$:

$(1, 140)(1, 159)(1, 198)(1, 246)(1, 33)(1, 63)(2, 110)(2, 126)$
$(2, 150)(2, 173)(2, 209)(2, 32)$

For $d = 26$, $n = 1655$:

$(1, 108)(1, 109)(1, 260)(1, 265)(1, 267)(1, 38)(2, 107)(2, 191)$
$(2, 215)(2, 237)(2, 25)(2, 312)(2, 54)$

For $d = 28$, $n = 2005$:

$(1, 109)(1, 140)(1, 166)(1, 376)(1, 390)(1, 92)(2, 133)(2, 161)$
$(2, 196)(2, 204)(2, 340)(2, 372)(2, 377)(2, 40)$

For $d = 30$, $n = 2359$:

$(1,183)(1,201)(1,302)(1,334)(1,75)(2,140)(2,181)(2,233)$
$(2,293)(2,306)(3,179)(3,238)(3,327)(3,4)(3,96)$

For $d = 32$, $n = 2947$:

$(1,190)(1,214)(1,328)(1,336)(1,83)(2,161)(2,220)(2,340)$
$(2,341)(2,360)(3,25)(3,253)(3,330)(3,334)(3,340)(3,392)$

References

[1] G. Exoo, *Some New Ramsey Colorings*, Electronic J. Combinatorics **5** (1998) #R29.

[2] G. Exoo, *A Small Trivalent Graph of Girth* 14, to appear.

[3] M. Hall, The Theory of Groups (The Macmillan Company, New York, 1959).

[4] S.P. Radziszowski, *Small Ramsey Numbers*, Dynamic Survey DS1, Electronic J. Combinatorics **1** (1994) pp. 28.

[5] G. Royle, Cubic Cages, ⟨`http://www.cs.uwa.edu.au/~gordon/cages/index.html`⟩, February, 2001 (Accessed: January 20, 2002).

[6] M. Schönert et al, Groups, Algorithms and Programming, version 4 release 2, Department of Mathematics, University of Western Australia.

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