Discussiones Mathematicae
Graph Theory 24 (2004) 85–107

LIGHT CLASSES OF GENERALIZED STARS IN POLYHEDRAL MAPS ON SURFACES

Stanislav Jendrol'

Department of Geometry and Algebra
P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovakia

e-mail: jendrolkosice.upjs.sk

and

Heinz-Jürgen Voss †

Department of Algebra, Technical University Dresden
Mommsenstrasse 13, D–01062 Dresden, Germany

e-mail: vossmath.tu-dresden.de

Dedicated to Professor Hanjo Walther on the occasion of his 60th birthday

Abstract

A generalized s-star, $s \geq 1$, is a tree with a root Z of degree s; all other vertices have degree ≤ 2. S_i denotes a generalized 3-star, all three maximal paths starting in Z have exactly $i+1$ vertices (including Z). Let $\mathbb{M}$ be a surface of Euler characteristic $\chi(\mathbb{M}) \leq 0$, and $m(\mathbb{M}) := \lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \rfloor$. We prove:

(1) Let $k \geq 1, d \geq m(\mathbb{M})$ be integers. Each polyhedral map G on $\mathbb{M}$ with a k-path (on k vertices) contains a k-path of maximum degree $\leq d$ in G or a generalized s-star $T, s \leq m(\mathbb{M})$, on $d+2-m(\mathbb{M})$ vertices with root Z, where Z has degree $\leq k \cdot m(\mathbb{M})$ and the maximum degree of $T \setminus \{Z\}$ is $\leq d$ in G. Similar results are obtained for the plane and for large polyhedral maps on $\mathbb{M}$.

(2) Let k and i be integers with $k \geq 3, 1 \leq i \leq \frac{k}{2}$. If a polyhedral map G on $\mathbb{M}$ with a large enough number of vertices contains a k-path then G contains a k-path or a 3-star S_i of maximum degree $\leq 4(k+i)$ in G. This bound is tight. Similar results hold for plane graphs.

Keywords: polyhedral maps, embeddings, light subgraphs, path, star, 2-dimensional manifolds, surface.

2000 Mathematics Subject Classification: 05C10, 05C75, 52B10.

1. Introduction

In this paper all manifolds are compact 2-dimensional manifolds. We shall consider graphs without loops and multiple edges. Multigraphs can have multiple edges and loops. If a multigraph G is embedded in a manifold $\mathbb{M}$ then the connected components of $\mathbb{M} - G$ are called the *faces* of G. If each face is an open disc then the embedding is called a *2-cell embedding.* If each vertex of a 2-cell embedding has degree ≥ 3 and each vertex of degree h is incident with h different faces then G is called a *map* in $\mathbb{M}$. If, in addition, G is 3-connected and the embedding has representativity at least three, then G is called a *polyhedral map* in $\mathbb{M}$, see e.g. Robertson and Vitray [19] or Mohar [17]. Let us recall that the *representativity* rep$(G, \mathbb{M})$ (or the *face width*) of a (2-cell) embedded graph G into a compact 2-manifold $\mathbb{M}$ is equal to the smallest number k such that $\mathbb{M}$ contains a noncontractible closed curve that intersects the graph G in k points.

Let $\mathbb{S}_g$ ($\mathbb{N}_q$) be an orientable (a non-orientable) compact 2-dimensional manifold (called also a surface , see [18]) of genus g (q, respectively). Let us recall that the relationship between Euler characteristic and the genus of a surface is the following

$$\chi(\mathbb{S}_g) = 2 - 2g \quad \text{and} \quad \chi(\mathbb{N}_q) = 2 - q.$$

We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The boundary of a face α of an embedded graph consists of all vertices and edges incident with α. Note that the boundary of α can be disconnected. Let $D_1, D_2, \cdots, D_s$ be the components of the boundary of α. Let W_i be the shortest closed walk induced by all edges of D_i, and let $\partial(W_i)$ be its length, i.e., the number of edges met at the walk W_i (edges met twice are counted twice). The degree of a face α is

$$\deg_G(\alpha) = \sum_{i=1}^{s} \partial(W_i).$$

Hence the *degree* $\deg_G(\alpha)$ of a face α of a 2-cell embedding is the length of its facial walk. Vertices and faces of degree i are called i-vertices and i-faces, respectively. The number of i-vertices and j-faces in a map is denoted by v_i and f_j, respectively. For a map G let $V(G), E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. The degree of a vertex A in G is denoted by $\deg_G(A)$ or $\deg(A)$ if G is known from the context. A path and a cycle on k distinct vertices is defined to be the *k-path* and the *k-cycle*, respectively. P_k will denote a k-path. The *length* of a path or a cycle is the number of its edges.

A generalized s-star, $s \geq 1$, is a tree with a root Z of degree s; all other vertices have degree ≤ 2. The maximal paths starting in Z are called beams. The symbol $S_i, i \geq 0$, denotes a generalized 3-star, all three beams of it are paths with $i+1$ vertices (including the root). Obviously, $S_0 = K_1$, and $S_1 = K_{1,3}$.

It is a consequence of Euler's formula that each planar graph contains a vertex of degree at most 5. It is well known that any graph embedded in a surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ has minimum degree

(1) $\delta(G) \leq \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor =: m(\mathbb{M})$, if $\mathbb{M} \neq \mathbb{S}_0$, and
$\delta(G) \leq 5 =: m(\mathbb{S}_0)$, where $\mathbb{S}_0$ is the sphere.

(For a proof see e.g. Sachs [20], p. 227).

A further consequence of Euler's formula is

$$\sum_{A \in V(G)} (\deg(A) - 6) + 2 \sum_{\alpha \in F(G)} (\deg(\alpha) - 3) = 6(-\chi(\mathbb{M})).$$

For any graph G embedded in a surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ this implies

(2) if $\sum_{\deg(A)>6}(\deg(A) - 6) > 6|\chi(\mathbb{M})|$ then $\delta(G) \leq 5$, and

(3) if G has more than $6|\chi(\mathbb{M})|$ vertices then $\delta(G) \leq 6$.

A theorem of Kotzig [15] states that every 3-connected planar graph contains an edge with degree-sum of its endvertices being at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs. For example Ivančo [6] has proved that every polyhedral map on $\mathbb{S}_g$ contains an edge with degree sum of their end vertices being at most $2g+13$ if $0 \leq g \leq 3$ and at most $4g+7$, if $g \geq 4$. The bounds are best possible. For other results in this topic see e.g. [1, 4, 14, 21].

2. The General Problem

In the past subgraphs have been investigated which are light in a family of graphs (see our survey article [14]). There we have generalized this concept to a light class $\mathcal{L}$ of subgraphs in a family $\mathcal{H}$ of graphs.

Problem. Let $\mathcal{H}$ be a family of graphs and $\mathcal{L}$ be a finite class of connected graphs having the property that every member of $\mathcal{L}$ is isomorphic to a proper subgraph of at least one member of $\mathcal{H}$. Let $\varphi(\mathcal{L},\mathcal{H})$ be the smallest integer with the property that every graph $G \in \mathcal{H}$, which has a subgraph isomorphic with a member of $\mathcal{L}$, also contains a subgraph $K, K \simeq H, H \in \mathcal{L}$, such that for every vertex $A \in V(K)$

$$\deg_G(A) \leq \varphi(\mathcal{L},\mathcal{H}).$$

If such a $\varphi(\mathcal{L},\mathcal{H})$ does not exist we write $\varphi(\mathcal{L},\mathcal{H}) = +\infty$. If $\varphi(\mathcal{L},\mathcal{H}) < +\infty$ we call the class $\mathcal{L}$ *light in the family* $\mathcal{H}$. Obviously, if $\mathcal{L}' \subseteq \mathcal{L}$ then $\varphi(\mathcal{L},\mathcal{H}) \leq \varphi(\mathcal{L}',\mathcal{H})$. The corresponding problem of a light subgraph H is again obtained if $\mathcal{L} = \{H\}$ is chosen. In this case let $\varphi(\{H\},\mathcal{H}) = \varphi(H,\mathcal{H})$.

3. Results

A. Polyhedral maps

Let $\mathcal{G}(\delta,\rho;\mathbb{M})$ denote the set of all polyhedral maps on the surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$ having minimum vertex degree at least δ and minimum face degree at least ρ. The following theorem has been proved for the planes $\mathbb{S}_0$ and $\mathbb{N}_1$ by Fabrici and Jendrol' [1] and for 2-dimensional manifolds $\mathbb{M}$ other than the planes by Jendrol' and Voss [8].

Theorem 1 ([1], [8]). *Let k be an integer, $k \geq 1$, and $\mathbb{M}$ a surface with Euler characteristic $\chi(\mathbb{M})$. Then*

(i) $\varphi(P_k, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(P_k, \mathcal{G}(3,3;\mathbb{N}_1)) = 5k$,

(ii) $2\lfloor \frac{k}{2} \rfloor \cdot m(\mathbb{M}) \le \varphi(P_k, \mathcal{G}(3,3;\mathbb{M})) \le k \cdot m(\mathbb{M})$, *if* $\mathbb{M} \notin \{\mathbb{S}_0, \mathbb{N}_1\}$,

(iii) $\varphi(H, \mathcal{G}(3,3;\mathbb{M})) = \infty$ *for any connected graph* $H \ne P_k$.

By the same arguments used in the proof of (iii) for the sphere $\mathbb{S}_0$ and the projective plane $\mathbb{N}_1$ by Fabrici and Jendrol' [1] it can be proved that a class $\mathcal{L}$ of plane graphs is light in $\mathcal{G}(3,3;\mathbb{M})$ if and only if $\mathcal{L}$ contains a path. So, if $\mathcal{L}$ contains P_k then $\varphi(\mathcal{L}, \mathcal{G}(3,3;\mathbb{M})) \le \varphi(P_k, \mathcal{G}(3,3;\mathbb{M}))$. We will study how small can $\varphi(\mathcal{L}, \mathcal{G}(3,3;\mathbb{M}))$ be if besides P_k the class $\mathcal{L}$ contains some trees different from P_k.

The class $\mathcal{T}_k$ of all trees of order k contains a k-path. Obviously, $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M})) = \varphi(P_k, \mathcal{G}(3,\ 3;\mathbb{M}))$ for $k \in \{1,2,3\}$. For the sphere $\mathbb{S}_0$ Fabrici and Jendrol' [2] and for each surface $\mathbb{M}, \mathbb{M} \ne \mathbb{S}_0$, Jendrol' and Voss [13] proved

Theorem 2 ([2], [13]). *Let k be an integer, $k \ge 4$, and $\mathbb{M}$ a surface with Euler characteristic $\chi(\mathbb{M})$. Then*

(i) $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{N}_1)) = 4k+3$,

(ii) $\left\lfloor \frac{2k+2}{3} \right\rfloor \left(\left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor - \frac{3}{2} \right) \le \varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M})) \le$
$\left\lfloor (k+1)\frac{5+\sqrt{49-24\chi(\mathbb{M})}}{3} \right\rfloor$ *if* $\mathbb{M} \notin \{\mathbb{S}_0, \mathbb{N}_1\}$.

In the Theorem 1(i) not all vertices of a P_k must have the degree $5k$. Really, Madaras [16] improved Theorem 1(i) by showing

Theorem 3 ([16]). *Let k be an integer, $k \ge 2$. Then each map of $\mathcal{G}(3,3;\mathbb{S}_0)$ containing a path P_k has also a path P_k such that one vertex has a degree $\le 5k$ and all other $k-1$ vertices have a degree $\le \frac{5k}{2}$.*

Let $\mathbb{M}$ be a surface of Euler characteristic $\chi(\mathbb{M})$ and $m(\mathbb{M})$ as defined in (1). Using the arguments of Madaras [16] we can show that G contains at least one tree from a family of specified trees with given degree constraints.

Theorem 4. *Let $\mathbb{M}$ be a surface of Euler characteristic $\chi(\mathbb{M})$. Let $k \ge 1$ and $d \ge m(\mathbb{M})$ be integers. Let $G \in \mathcal{G}(3,3;\mathbb{M})$ contain a k-path. Then G contains at least one of the following subgraphs:*

(i) *a k-path of maximum degree $\le d$ in G, or*

(ii) *a generalized s-star $T, s \le m(\mathbb{M})$, on $d+2-m(\mathbb{M})$ vertices with root Z, where Z has a degree $\le k \cdot m(\mathbb{M})$ in G and the maximum degree of $T \setminus \{Z\}$ is $\le d$ in G.*

The generalized star T contains a path with $2\frac{d+1-m(\mathbb{M})}{m(\mathbb{M})}+1$ vertices.

If $d = \lfloor \frac{k}{2} m(\mathbb{M}) \rfloor$ then the generalized star T contains a P_k. Hence Theorem 4 implies the validity of the following result.

Theorem 5. *Let $\mathbb{M}$ be a surface of Euler characteristic $\chi(\mathbb{M})$ and $k \ge 1$ an integer. Then each map $G \in \mathcal{G}(3,3;\mathbb{M})$ containing a k-path, has a k-path P_k with the property: besides one vertex Z all vertices have a degree $\le \frac{k}{2} \cdot m(\mathbb{M})$ and the vertex Z has a degree $\le k \cdot m(\mathbb{M})$ in G.*

For the sphere $m(\mathbb{S}_0) = 5$ holds and Theorem 5 implies the validity of Theorem 3. If $d = k \cdot m(\mathbb{M})$ then the generalized star T contains a P_k not meeting the root of T. Hence Theorem 4 implies the validity of the upper bound in Theorem 1. Interesting special variants of Theorem 4 are also obtained for $d = k$ and $d = k+4$.

B. Large polyhedral maps

Let $\chi(\mathbb{M}) \le 0$ throughout section B.

For large maps of $\mathcal{G}(3,3;\mathbb{M})$ we await a smaller bound for the maximum degree of light paths. A *large* polyhedral map is one with a large number of vertices or a large positive charge. A positive *k-charge* $ch_k(G)$ is defined $ch_k(G) := \sum_{\deg_G(A)>6k}(\deg_G(A)-6k)$. Let $\mathcal{G}(3,3;\mathbb{M},n(a))$ and $\mathcal{G}(3,3;\mathbb{M},c_k(b))$ denote the sets of the graphs G of $\mathcal{G}(3,3;\mathbb{M})$ with $> a$ vertices or a k-charge $ch_k(G) > b$, respectively. Let b_k denote the largest number of vertices in a connected graph with maximum degree $\le 6k$ containing no path of k vertices. Obviously, $b_k \le (6k)^{k/2+2}$.

Let $l_k(\mathbb{M}) := 3 \cdot 10^4(|\chi(\mathbb{M})|+1)^3(b_k + 3(|\chi(\mathbb{M})|+1))$. We have proved

Theorem 6 ([9], [10]). *For any surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \le 0$, any integer $k \ge 1$, any integer $a > l_k(\mathbb{M})$ and any integer $b > 6k|\chi(\mathbb{M})|$,*

(i) $\varphi(P_k, \mathcal{G}(3,3;\mathbb{M},n(a))) = \begin{cases} 6k, & \text{if } k=1 \text{ or } k \text{ is even}, \\ 6k-2, & \text{if } k \ge 3 \text{ is odd}, \end{cases}$

(ii) $\varphi(P_k, \mathcal{G}(3,3;\mathbb{M},c_k(b))) = 5k$,

(iii) $\varphi(H, \mathcal{G}(3,3;\mathbb{M}, n(a))) = \varphi(H, \mathcal{G}(3,3;\mathbb{M}, c_k(a))) = \infty$ *for any* $H \not\cong P_k$ *and any* a.

In [9] we could show that $\varphi(P_k, \mathcal{G}(3,3;\mathbb{M}, n(a))) \leq 6k$ even for the smaller bound $a > (14(k-1)b_k + 6)|\chi(\mathbb{M})|$. For the class $\mathcal{T}_k$ of all trees of order k we could prove [11]

Theorem 7 ([11])**.** *For any surface* $\mathbb{M}$ *with Euler characteristic* $\chi(\mathbb{M}) \leq 0$*, any integer* $k \geq 3$*, and any integer* $a > (8k^2 + 6k - 6)|\chi(\mathbb{M})|$

(i) $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M}, n(a))) = 4k + 4$*, and*

(ii) $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M}, c_k(a))) = 4k + 3$.

With the arguments of Madaras [16] we will prove: for the graphs of $\mathcal{G}(3,3;\mathbb{M}, n(a))$ and $\mathcal{G}(3,3;\mathbb{M}, c_k(b))$ with large a and b the Theorem 4 is again valid, if $m(\mathbb{M})$ is replaced by 6 or 5, respectively.

Theorem 8. *Let* $\mathbb{M}$ *be a surface of Euler characteristic* $\chi(\mathbb{M}) \leq 0$*. Let* k, d *and* a, b *be integers with* $k \geq 1, d \geq 6$, $a > (14(k-1)b_k + 6)|\chi(\mathbb{M})|$*, and* $b > 6k|\chi(\mathbb{M})|$*. Let* $G_1 \in \mathcal{G}(3,3;\mathbb{M}, n(a))$ *and* $G_2 \in \mathcal{G}(3,3;\mathbb{M}, c_k(b))$ *contain a* k*-path. Let* $m_1 := 6$ *and* $m_2 := 5$*.*
Then for $i = 1, 2$ *the map* G_i *contains at least one of the following subgraphs:*

(i) *a* k*-path of maximum degree* $\leq d$ *in* G_i*, or*

(ii) *a generalized* s*-star* $T, s \leq m_i$ *on* $d + 2 - m_i$ *vertices with root* Z*, where* Z *has a degree* $\leq k \cdot m_i$ *in* G_i *and the maximum degree of* $T \setminus \{Z\}$ *is* $\leq d$ *in* G_i*.*

Finally we deal with light classes $\mathcal{H} \neq \mathcal{T}_k,\ k \geq 1$.

Since by Theorem 7 each polyhedral map G on $\mathbb{M}$ of large order contains a tree of order k such that each vertex has a degree at most $4k+4$, if $k \geq 3$, the map G also contains a P_k or a $K_{1,3}$ with the same bound. Examples in [11] show that the bound is best possible.

Theorem 9. *For any surface* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M}) \leq 0$ *and any integer* $k \geq 3$ *let* $a > (8k^2 + 6k - 6)|\chi(\mathbb{M})|$*. Then*

(i) $\varphi(\{P_k, K_{1,3}\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) = 4k + 4$*, and*

(ii) $\varphi(\{P_k, K_{1,3}\}, \mathcal{G}(3,3;\mathbb{M}, c_k(a))) = 4k + 3$.

Next the class $\{P_k, S_i\}$, $i \geq 2$, will be considered. If $\chi(\mathbb{M}) \leq 0$, $a > 6k(2b_k+1)|\chi(\mathbb{M})|$ and $i \geq \frac{k}{2}$ then $P_k \subseteq S_i$ and $\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) = \varphi(\{P_k\}, \mathcal{G}(3,3;\mathbb{M}, n(a)))$. For different i we will prove the following theorem.

Theorem 10. *Let $\mathbb{M}$ be a surface with Euler characteristic $\chi(\mathbb{M})$, and let $k \geq 3, i \geq 1$ be integers.*

(i) *If $\mathbb{M}$ is the sphere $\mathbb{S}_0$ or the projective plane $\mathbb{N}_1$, then*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{N}_1)) \leq 4(k+i) - 1.$$

(ii) *If $\chi(\mathbb{M}) \leq 0$, then for each integer $b > 4(k+i)|\chi(\mathbb{M})|$ it holds*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, c_k(b))) \leq 4(k+i) - 1.$$

(iii) *If $\chi(\mathbb{M}) \leq 0$, then for each integer $a > (6k+1)(2b_k+1)|\chi(\mathbb{M})|$ it holds*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a)) \leq 4(k+i).$$

Taking into the consideration the Theorems 1 and 6 we obtain tight bounds in some subclasses of $\mathcal{G}(3,3;\mathbb{M})$.

Theorem 11. *Let k and i be integers with $k \geq 3$ and $i \geq 1$. If $\mathbb{M}$ is the sphere $\mathbb{S}_0$ or the projective plane $\mathbb{N}_1$, then*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{N}_1)) = \min\{4(k+i) - 1; 5k\}.$$

Theorem 12. *For any surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$, any integers $k \geq 3, i \geq 1$, and $b > 6k|\chi(\mathbb{M})|$ it holds:*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, c_k(b))) = \min\{4(k+i) - 1; 5k\}.$$

Theorem 13. *For any surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$, any integers $k \geq 3, i \geq 1$, and $a > l_k(\mathbb{M})$ it holds:*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) = \begin{cases} \min\{4(k+i); 6k\} & for\ even\ k, \\ \min\{4(k+i); 6k-2\} & for\ odd\ k. \end{cases}$$

4. PROOF OF THEOREMS 4 AND 8

Assume there is a counterexample G to Theorem 4 or 8 having $v = |V(G)|$ vertices, where in Theorem 8 the number of vertices $v > (14(k-1)b_k + 6)|\chi(\mathbb{M})|$ or the positive k-charge

$$ch_k(G) = \sum_{\deg_G(A) > 6k} (\deg_G(A) - 6k) > 6k|\chi(\mathbb{M})|.$$

Let G be a counterexample with the maximum number of edges among all counterexamples having v vertices. A vertex A of the graph G is *major* (*minor*) if its degree is $\geq d + 1 (\leq d$, respectively). The assertions (1) $-$ (3) can be found in the introduction.

(4) Each path P_k of k vertices contains a major vertex.

Hence G contains at least one major vertex.

(5) Each r-face $\alpha, r \geq 4$, contains at most two major vertices; if α has precisely two major vertices then they are adjacent.

***Proof of* (5).**
Suppose G has an r-face with two nonadjacent vertices of degree $\geq d+1$. Since G is a polyhedral map we can join these two vertices by an edge. The resulting embedding is again a counterexample but with one edge more, a contradiction. ∎

Let H denote the subgraph of G induced by the major vertices, and let $v(H)$ be the number of vertices of H.

(6) The subgraph H contains a vertex Z of degree $s := \deg_H(Z)$ with

(i) $s \leq m(\mathbb{M})$ if $G \in \mathcal{G}(3,3;\mathbb{M})$, or

(ii) $s \leq 6$, if $G \in \mathcal{G}(3,3;\mathbb{M}, n(a)), \chi(\mathbb{M}) \leq 0$, or

(iii) $s \leq 5$, if $G \in \mathcal{G}(3,3;\mathbb{M}, c_k(b)), \chi(\mathbb{M}) \leq 0$.

***Proof of* (6).**

(i) This assertion follows from (1)(see the introduction).

(ii) Suppose there is a $G \in \mathcal{G}(3,3;\mathbb{M}, n(a))$ with the subgraph H of major vertices of G with minimum degree $\delta(H) > 6$. In Lemma 5 of [9] we have proved that $v(H) > 6|\chi(\mathbb{M})|$. By (3) the subgraph H has $v(H) \leq 6|\chi(\mathbb{M})|$ vertices. This contradiction completes the proof of (ii).

(iii) Suppose there is a $G \in \mathcal{G}(3,3;\mathbb{M},c_k(b))$ with the subgraph H of major vertices of G with minimum degree $\delta(H) > 5$. By (2) we have $\sum(\deg_H(A) - 6) \leq 6|\chi(\mathbb{M})|$ where the sum is taken over all vertices A of H with $\deg_H(A) > 6$.

Since G is a polyhedral map the union of all faces incident with Z forms a wheel with nave Z and cycle $\mathcal{C}_Z$; it may be that some vertices of the cycle $\mathcal{C}_Z$ are not joined with Z by an edge. By (5) each vertex of $\mathcal{C}_Z$ not adjacent with Z is a minor vertex. Hence all major vertices of $\mathcal{C}_Z$ are neighbours of Z. These neighbours partition $\mathcal{C}_Z$ into $\deg_H(Z)$ paths which have at most $k-1$ minor vertices according (4). Therefore, $\deg_G(Z) \leq k\ \deg_H(Z)$. This together with $ch_k(G) > 6k|\chi(\mathbb{M})|$, $\delta(H) \geq 6$ and $\sum_{\deg_H(A)\geq 6}(\deg_H(A) - 6) \leq 6|\chi(\mathbb{M})|$ implies:

$$6k|\chi(\mathbb{M})| < ch_k(G) = \sum_{\deg_G(A)>6k} (\deg_G(A) - 6k) \leq \sum_{\deg_G(A)>6k} (k\ \deg_H(A) - 6k)$$

$$\leq k \sum_{A\in V(H)} (\deg_H(A) - 6) \leq 6k|\chi(\mathbb{M})|.$$

This contradiction completes the proof of assertion (iii). ■

The s neighbours $Y_1, Y_2, \ldots, Y_s$ of Z in H are the only major vertices on the cycle $\mathcal{C}_Z$. An upper bound for s is known by (6). If $\mathcal{C}_Z$ has no major vertices then let $s := 1$ and Y_1 be an arbitrary neighbour of Z on $\mathcal{C}_Z$. The cycle $\mathcal{C}_Z$ contains altogether $\deg_G(Z) \geq d+1$ neighbours of Z. Next $\mathcal{C}_Z\backslash\{Y_1,\ldots,Y_s\}$ consists of s paths $p_1, p_2, \ldots, p_s$ of minor vertices. These paths and Z induce a subgraph which contains a generalized star T with root Z of degree $\deg_T(Z) \leq s$ and containing all $\geq \deg_G(Z) - s \geq d+1-s$ minor neighbours of Z. By (4) each path p_i has at most $k-1$ vertices. Hence the cycle $\mathcal{C}_Z$ has at most $s \cdot k$ vertices and $\deg_G(Z) \leq s \cdot k$. Consequently, G contains a generalized star T of order $d+2-s$ with root Z of degrees $\deg_T(Z) \leq s$ and $\deg_G(Z) \leq s \cdot k$, all other vertices of T have a degree $\leq d$ in G. This contradicts our assumption that G is a counterexample to Theorem 4 or 8. Thus the proof of the Theorems 4 and 8 is complete. ■

5. Proof of Theorems 10–13 — Upper Bounds

Theorem 1, 6, and 10 imply the validity of the upper bounds in Theorems 11–13. Hence it suffices to prove Theorem 10.

If $i \geq \frac{k}{2}$ then $P_k \subseteq S_i$ and $\varphi(\{P_k, S_i\}, \mathcal{L}) = \varphi(\{P_k\}, \mathcal{L})$ for each class $\mathcal{L}$ of graphs. This bound is $\leq 6k$ in each class $\mathcal{L}$ we have considered. Hence $\varphi(\{P_k, S_i\}, \mathcal{L}) \leq 6k \leq 4(k+i)$, and it suffices to accomplish the proof for all $i \leq \frac{k-1}{2} \leq \frac{k}{2}$.

The proof follows the ideas of [1] and [13]. Suppose that there is a counterexample to one version of our theorem having v vertices. Let G be a counterexample with the maximum number of edges among all counterexamples having v vertices. Obviously, G contains a P_k or an S_i.

(A) If G is a counterexample to Theorem 10(i) then $\mathbb{M}$ is the sphere $\mathbb{S}_0$ or is the projective plane $\mathbb{N}_1$ and each P_k and each S_i of G contains a vertex of degree $\geq 4(k+i)$.

(B) If G is a counterexample to Theorem 10(ii) then $\chi(\mathbb{M}) \leq 0$, the map G has a positive k-charge $ch_k(G) > 4(k+i)|\chi(\mathbb{M})|$, and each P_k and each S_i of G contains a vertex of degree $\geq 4(k+i)$.

(C) If G is a counterexample to Theorem 10(iii) then $\chi(\mathbb{M}) \leq 0$, the map G has an order $v(G) > (6k+1)(2b_k+1)|\chi(\mathbb{M})|$ and each P_k and each S_i of G contains a vertex of degree $\geq 4(k+i)+1$.

In the cases (A) and (B) a vertex A is a *minor vertex* if $\deg_G(A) \leq 4(k+i)-1$ and is a *major* vertex if $\deg_G(A) \geq 4(k+i)$. In case (C) a vertex A is a *minor vertex* if $\deg_G(A) \leq 4(k+i)$ and is a *major vertex* if $\deg_G(A) \geq 4(k+i)+1$. Since G is a counterexample it holds.

(1) Each k-path and each generalized star S_i in G contains a major vertex.

(2) Every r-face $\alpha, r \geq 4$, of G is incident only with minor vertices.

***Proof of* (2).** Suppose there is a major vertex B incident with an r-face $\alpha, r \geq 4$. Let C be a diagonal vertex on α with respect to B i.e., BC is no edge of the boundary of α. Because G is a polyhedral map we can insert the edge BC into the r-face α The resulting embedding is again a counterexample but with one edge more, a contradiction. ∎

Let $H = H(G)$ and $H' = H'(G)$ be the subgraphs of G induced on all major or minor vertices of G, respectively.

(3) H is not empty.

***Proof of* (3).** Since G is a counterexample it contains a k-path P_k or a 3-star S_i. By (1) P_k or S_i contains a major vertex. ∎

(2) directly implies:

(4) All faces incident with a major vertex X induce a wheel with nave X and a cycle $\mathcal{C}$ of length $\geq 4(k+i)$ consisting of all neighbours of X. The cycle $\mathcal{C}$ contains at least 5 major vertices.

***Proof of* (4).** The first assertion is clear. If $\mathcal{C}$ would contain at most 4 major vertices then $\mathcal{C}$ would also contain at most 4 paths of $\leq k-1$ minor vertices and $\mathcal{C}$ would have a length $\leq 4+4(k-1) = 4k < 4(k+i)$. This contradiction proves (4). ■

(5) By (4) the minimum degree of H is at least 5.

Note that a triangle is always a 3-face. For the following we need Lemma 1.

■

Lemma 1. *The three vertices of each triangle D of H are joint with the minor vertices inside D by at most* $2(k-1+i)-1$ *edges.*

Proof. Let $D = [PQR]$ be a triangle of H. Let K denote the subgraph of G induced by the minor vertices of G lying in the interior of $[PQR]$. By (2) all faces incident with P induce a wheel W_P with nave P and a cycle containg all neighbours of P. Correspondingly, Q and R are the naves of a wheel W_Q and W_R, respectively. Let p, q, and r denote the path of $W_P \cap K, W_Q \cap K$, and $W_R \cap K$, respectively. Then p, q and q, r and r, p have a common endvertex Q', R', and P', respectively (a sketch of the situation is depicted in Figure 1).

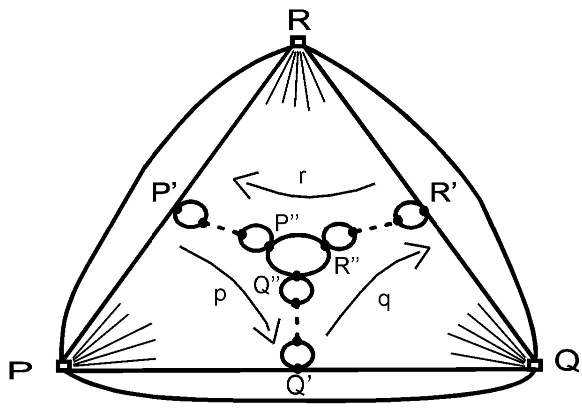

Figure 1

Case 1. Let p and q have precisely one common vertex, namely, Q'. Then $p \cup q$ is a path of K having $\leq k-1$ vertices. Hence the paths $p \cup q$ and r having at most $k-1$ vertices each, and P, Q, R are joined by $\leq 2k-1 < 2(k-1+i)$ edges with K.

Case 2. Let p, q and q, r and r, p have a second common vertex. Let Q'', and R'', and P'' denote the last common vertex of p, q, and q, r, and r, p by walking along p, or q, or r with start in Q', or R', or P', respectively. Since K does not contain a generalized star S_i, w.l.o.g. the paths $R''qR'$ and $R'rR''$ have at most i vertices. The paths $P'rR''$ and $R''qQ'$ have precisely one common vertex, namely, R''. Hence $P'rR''qQ'$ is a path of K with $\leq k-1$ vertices. The path p has also $\leq k-1$ vertices, and $R'rR'' - \{R''\}$ and $R''qR' - \{R''\}$ have at most $i-1$ vertices each. Therefore, P, Q, R are joined with K by $\leq (k-1)+(k-1)+1+(i-1)+(i-1) = 2(k-1+i)-1$ edges with K. ■

Let α be a face of H. Let $D_1, D_2, \cdots, D_s$ be the components of the boundary of α. Let W_i be the shortest closed walk induced by all edges of D_i, and let $\partial(W_i)$ be its length. Then the degree of the face α is $\deg_H(\alpha) = \sum_{i=1}^{s} \partial(W_i)$. Since G is a polyhedral map any three consecutive vertices on the boundary of α (i.e., in the walk W_i for some i) are pairwise different. Hence $\partial(W_i) \geq 3$, and

(6) $\deg_H(\alpha) \geq 3s \geq 3$.

Let X, Y, Z be three consecutive vertices on the boundary of α. We call XYZ a *corner* of α at the vertex Y. Assertion (4) implies: In G the vertices X and Z are joined by a path q completely lying in α and containing all minor neighbours of Y in this corner (Y can have some other minor neighbours at some other corners of α at the vertex Y, because Y can appear on the boundary of α more than once).

The path $q \setminus \{X, Z\}$ consists of all minor neighbours of Y in this corner.

(7) In each corner XYZ of α at Y the vertex Y has at most $k-1$ minor neighbours. They form a path of $H'(G)$.

It is obvious that

(8) each face α of H has precisely $\deg_H(\alpha)$ corners.

Let $w(\alpha)$ denote the number of edges joining the minor vertices inside α with all major vertices of H (i.e., the major vertices on the boundary of α).

With (8) it follows:

(9) The minor vertices inside α are joined with all major vertices by $w(\alpha) \leq (k-1)\deg_H(\alpha)$ edges.

Thus the number w of all edges of G joining minor vertices with major vertices is

(10) $w = \sum_{\alpha \in F(H)} w(\alpha) \leq \sum_{\alpha \in F(H)} (k-1)\deg_H(\alpha)$.

By Lemma 1 we have a better bound if α is a 2-cell 3-face (triangle).

(11) If α is a triangle of H then $w(\alpha) \leq 2(k-1+i)-1$.

We proceed in three steps. First, we assign to each face α of H the charge $w(\alpha)$. Next, we triangulate each face α of H by introducing diagonals into the face α (a diagonal is an edge joining two vertices of the boundary of α such that no 1-face or 2-face is generated). By this method α is splitted into at least $t-2$ triangles, $t = \deg_H(\alpha)$. The obtained semitriangulation H^* can have loops or muliple edges (A *triangulation* (*semitriangulation*) is an embedding of a graph (multigraph) such that each face is a triangle). In the third step the charge $w(\alpha)$ is equally distributed to the triangles inside α. The charge of a triangle D of H^* is denoted by $w^*(D)$. Distributing the old charges no charge has been lost. Hence,

(12) $w = \sum_{\alpha \in F(H)} w(\alpha) = \sum_{D \in F(H^*)} w^*(D)$.

(13) Each triangle D of H^* has a charge $w^*(D) \leq 2(k-1+i)-1$.

***Proof of* (13).** Let α be a face of H. We consider two cases. ■

Case 1. Let $t := \deg_H(\alpha) \geq 4$. Then with (9) each triangle D inside α has a charge

$$\begin{aligned} w^*(D) &\leq \frac{w(\alpha)}{t-2} \leq \frac{t(k-1)}{t-2} \leq \left(1 + \frac{2}{t-2}\right)(k-1) \\ &\leq 2(k-1) < 2(k-1+i)-1. \end{aligned}$$

(Note $i \geq 1$).

Case 2. Let $t = \deg_H(\alpha) = 3$.
If α is a triangle (2-cell 3-face) of H then α is also a triangle of H^*. Hence with (11) the charge $w^*(\alpha) = w(\alpha) \le 2(k-1+i)$.

Next let α not be a triangle (2-cell 3-face). Then at least one diagonal d can be added so that no new face is created. The diagonal is counted twice on the boundary of α, i.e., $\deg_{H+d}(\alpha) = 5$. The charge $w(\alpha) \le 3(k-1)$ is equally distributed to at least three (new) triangles of H^*, each receiving a charge $\le \frac{w(\alpha)}{3} \le \frac{3(k-1)}{3} = k-1 \le 2(k-1+i)-1$. Thus the proof of (13) is complete. ■

Properties (12) and (13) imply:

(14) $w = \sum_{\alpha \in F(H)} w(\alpha) = \sum_{D \in F(H^*)} w^*(D) \le (2(k-1+i)-1) f(H^*)$.

where $f(H^*) = |F(H^*)|$.

The semitriangulation H^* satisfies the equation

$$2e(H^*) = 3f(H^*),$$

and Euler's formula

$$v(H^*) - e(H^*) + f(H^*) = \chi(\mathbb{M}).$$

Hence

$$f(H^*) = 2(v(H^*) - \chi(\mathbb{M})), \tag{15}$$

and

$$e(H^*) = 3(v(H^*) - \chi(\mathbb{M})). \tag{16}$$

The number of the edges joining vertices of H and the number w of the edges joining minor vertices with major vertices in G contribute to the degree sum $\sum_{A \in V(H)} \deg_G(A)$. Consequently, with (14) it holds

$$\sum_{A \in V(H)} \deg_G(A) = \sum_{A \in V(H)} \deg_H(A) + w$$

$$\le \sum_{A \in V(H)} \deg_H(A) + (2(k-1+i)-1) f(H^*).$$

With (15)

$$\sum_{A\in V(H)} \deg_G(A) \le 2e(H) + (4(k-1+i)-2)(v(H^*) - \chi(\mathbb{M})). \tag{17}$$

With $e(H) \le e(H^*)$ and (16) we have

$$\sum_{A\in V(H)} \deg_G(A) \le (6+4(k-1+i)-2)(v(H^*) - \chi(\mathbb{M})), \quad \text{and}$$

$$\sum_{A\in V(H)} \deg_G(A) \le 4(k+i)(v(H^*) - \chi(\mathbb{M})). \tag{18}$$

The inequality (18) implies with $v(H) = v(H^*)$ the existence of a major vertex B of degree

$$\deg_G(B) \le 4(k+i)\left(1 - \frac{\chi(\mathbb{M})}{v(H^*)}\right). \tag{19}$$

If $\mathbb{M} = \mathbb{S}_0$ or $\mathbb{N}_1$ then $\chi(\mathbb{M}) \ge 1$ and by (5) H has at least 6 vertices. Moreover (18) implies the existence of a major vertex B of degree $\deg_G(B) \le 4(k+i)-1$. But by condition (A) each major vertex has a degree $\ge 4(k+i)$. This contradiction completes the proof of Theorem 10(i). ■

Next Theorem 10(ii) can be proved in the following way. By condition (B) with $6k \ge 4(k+i)$ (i.e., with $i \le \frac{k}{2}$) and $\chi(\mathbb{M}) \le 0$

$$\begin{aligned}\sum_{\deg_G(A)\ge 4(k+i)} (\deg_G(A) - 4(k+i)) \ &\ge \sum_{\deg_G(A)>6k} (\deg_G(A) - 6k) \\ &= \ ch_k(G) > 4(k+i)|\chi(\mathbb{M})|.\end{aligned}$$

With

$$\sum_{\deg_G(A)\ge 4(k+i)} (\deg_G(A) - 4(k+i)) = \left(\sum_{\deg_G(A)\ge 4(k+i)} \deg_G(A)\right) - 4(k+i)v(H^*)$$

this implies

$$\begin{aligned}\sum_{\deg_G(A)\ge 4(k+i)} \deg_G(A) \ &> \ 4(k+i)(v(H^*) + |\chi(\mathbb{M})|) \\ &= \ 4(k+i)(v(H^*) - \chi(\mathbb{M})).\end{aligned} \tag{20}$$

The assertion (20) contradicts (18). Thus the proof of Theorem 10(ii) is complete. ■

Finally Theorem 10(iii) will be proved. For these purposes we need an upper bound for the number $v(H')$ of vertices of H' in dependence on $f(H^*)$. Recall that H' is the subgraph of G induced by the minor vertices of G. Let l denote the number of components of H'. Since G is 3-connected each component K of H' contains the minor vertices of at least three corners of a face of H. The number of corners of H is not greater than the number of corners of H^*, and H^* has at most $3f(H^*)$ corners. Hence $3l \leq 3f(H^*)$, and $l \leq f(H^*)$. Since each component K of H' has no path with k vertices and each (minor) vertex of K has a degree $\leq 4(k+i) \leq 6k$ in G the number of vertices of K is $v(K) \leq b_k$, and the number of vertices of H' is $v(H') \leq l \cdot b_k \leq f(H^*) \cdot b_k$. Therefore,

$$v(G) = v(H) + v(H') \leq v(H^*) + f(H^*) \cdot b_k.$$

Assertion (15) implies

$$v(G) \leq v(H^*) + 2(v(H^*) + |\chi(\mathbb{M})|) \cdot b_k \leq 2(v(H^*) + |\chi(\mathbb{M})|) \left(b_k + \frac{1}{2}\right).$$

With the hypothesis $v(G) > (6k+1)(2b_k+1)|\chi(\mathbb{M})|$ the number of vertices of H^* is

$$v(H^*) \geq \frac{v(G)}{2b_k+1} - |\chi(\mathbb{M})| > \frac{12k(b_k+\frac{1}{2})|\chi(\mathbb{M})|}{2b_k+1} = 6k|\chi(\mathbb{M})|,$$

and

$$v(H^*) > 6k|\chi(\mathbb{M})| \geq 4(k+i)|\chi(\mathbb{M})|. \tag{21}$$

(19) and (21) imply: there is a vertex $B \in V(H)$ such that its degree

$$\begin{aligned} \deg_G(B) &\leq 4(k+i) + \frac{4(k+i)|\chi(\mathbb{M})|}{v(H^*)} \\ &< 4(k+i) + \frac{4(k+i)|\chi(\mathbb{M})|}{4(k+i)|\chi(\mathbb{M})|} = 4(k+i)+1. \end{aligned} \tag{22}$$

Therefore, the degree of the major vertex B in G is $\leq 4(k+i)$. But by the condition (C) each major vertex has a degree $\geq 4(k+i)+1$. This contradiction completes the proof of Theorem 10(iii). ■

6. Proof of Theorem 13 for Polyhedral Maps — Lower Bound

The main goal of this part is to prove that $\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) \geq 4k+4i,\ k \geq 3,\ \chi(\mathbb{M}) \leq 0$, that is to construct a large polyhedral map G on surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$ so that each path P_k with k vertices and each generalized 3-star S_i contains a vertex of degree at least $4k+4i$. This construction is very similar to our construction presented in Sections 3 and 4 of [11].

Let $P_n \times P_n$ be the cartesian product of two n-paths with vertex set $\{(x,y)|x,y \in \mathbb{Z}, 1 \leq x \leq n, 1 \leq y \leq n\}$ and edge set $\{\{(x,y),(x,y+1)\}|1 \leq x \leq n, 1 \leq y \leq n-1\} \cup \{\{(x,y),(x+1,y)\}|1 \leq x \leq n-1, 1 \leq y \leq n\}$. Add the edge set $\{\{(x,y),(x+1,y-1)\}|1 \leq x \leq n-1, 2 \leq y \leq n\}$. The so obtained plane graph with $2(n-1)^2$ triangles and an outer $4(n-1)$-face is denoted by R_n.

Into each triangle D of the obtained graph we insert a generalized 3-star $S(r)$, $0 \leq r \leq k-2i$, consisting of a central vertex Z and three paths p_1, p_2 and p_3 starting in Z, the path p_1 of length $k-(i+r)$, the path p_2 of length $i+r$, and the path p_3 of length i. Let the paths p_1, p_2, and p_3 be in this anticlockwise cyclic order in D. If $D_{x,y} = ((x,y),(x+1,y),(x,y+1))$ then (x,y) is joined to all vertices of p_1 and $p_2, (x+1,y)$ is joined to all vertices of p_2 and p_3, and $(x,y+1)$ is joined to all vertices of p_3 and p_1 (see Figure 2). We do the same in $D'_{x,y} = ((x,y),(x-1,y),(x,y-1))$. The resulting plane graph is denoted by R_n^*.

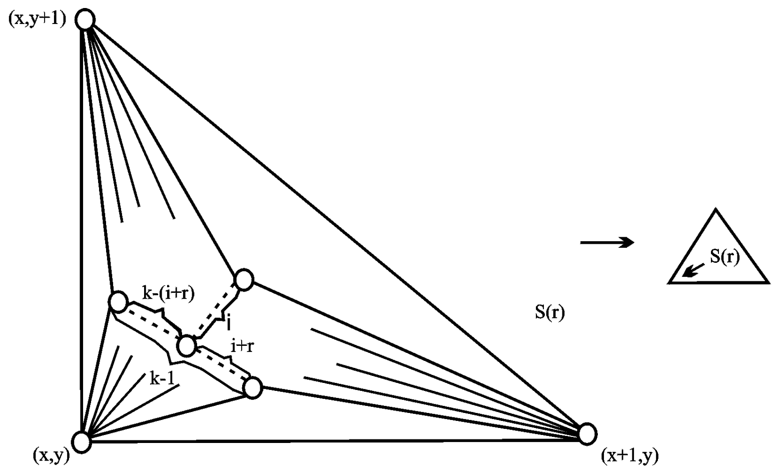

Figure 2

The situation is presented in Figure 3, where in each triangle D an arrow indicates which vertex of D is joined with all vertices of p_1 and p_2. In this part of the proof the labels 0 and $k-2i, \cdots$ have no meaning. For the further proof of the lower bound $4(k+i)$ choose a fixed r, $0 \leq r \leq k-2i$.

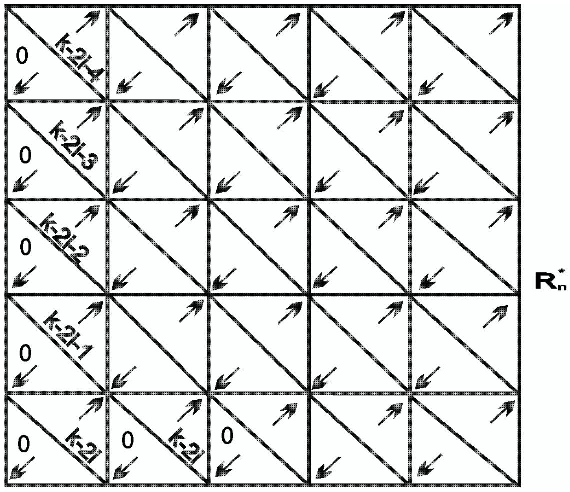

Figure 3

The inserted trees have $k-(i+r)+(i+r)+i-2 = k+i-2$ vertices, and the degree of each inner vertex (x,y), $2 \leq x, y \leq n-1$ is

$$\begin{aligned} \deg(x,y) = 6 \; &+ \; 2((k-(i+r))+(i+r)-1) \\ &+ \; 2((i+r)+i-1) \\ &+ \; 2(i+(k-(i+r)-1) = 4k+4i. \end{aligned}$$

Deleting the outer face of R_n^* and identifying opposite sides of the "quadrangle" results in a toroidal map T_n, and reversing one side of this "quadrangle" and then identifying opposite sides of this "quadrangle" results in a map Q_n on the Klein bottle, respectively, both satisfying the degree requirements.

The required polyhedral map on an orientable surface $\mathbb{S}_g$ of genus $g \geq 2$ will be constructed from the toroidal triangulation T_n^* with the triangulation T_n. We choose $2g-2$ triangles of T_n so that any two of them have a distance ≥ 2 in T_n (this is possible if n is large enough). In T_n^* from each of these triangles D we delete the interior part so that the bounding 3-cycle of D

bounds now a hole of the torus. We join repeatedly two holes of T_n^* by a handle, and $g-1$ handles are added to the torus in this way.

The handles are triangulated in the following way: if $[X_1X_2X_3]$ and $[Y_1Y_2Y_3]$ are the bounding cycles of some handle which are around the handle in the same cyclic order then add the cycle $[X_1Y_1X_2Y_2X_3Y_3]$. In each of the new triangles a generalized 3-star $S(r)$ can be placed in such a manner that the obtained polyhedral triangulation of $\mathbb{S}_g$ fulfils also the degree requirements.

The required polyhedral map on an unorientable surface $\mathbb{N}_q$ of genus $q \geq 3$ will be constructed from the triangulation Q_n^* of the Klein bottle with triangulation Q_n. We choose $q-2$ triangles of Q_n so that any two of them have a distance ≥ 4 in Q_n.

Let D be one of these triangles with bounding cycle $[X_1X_2X_3]$ and D_1, D_2, D_3 the three neighbouring triangles in Q_n with bounding cyles $[Y_1X_3X_2]$, $[Y_2X_1X_3]$, and $[Y_3X_2X_1]$ (see Figure 4). In Q_n^* we delete the inserted trees of D, D_1, D_2, D_3 and the separating edges X_1X_2, X_2X_3 and X_3X_1. A greater face F with bounding 6-cycle $\mathcal{C} = [X_1Y_3X_2Y_1X_3Y_2]$ is obtained (for the notation see Figure 5).

In F a crosscap is placed and the edges X_1X_2, X_2X_3, and X_3X_1 are again added so that the "interior" of $\mathcal{C}$ is subdivided into three quadrangles (see Figure 5). These quadrangles are subdivided by the edges X_iY_i, $i = 1, 2, 3$ (see Figure 6). Finally in each of the new triangles a generalized 3-star $S(r)$ can be placed in such a manner that the obtained polyhedral triangulation of $\mathbb{N}_q$ fulfils the degree requirements.

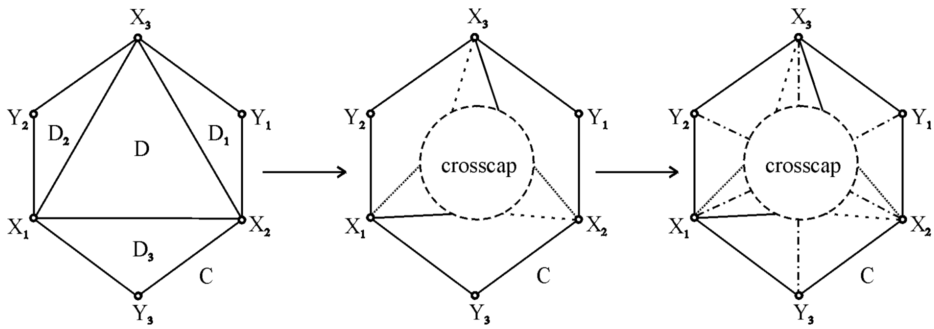

Figure 4 Figure 5 Figure 6

7. Proof of Theorems 11 and 12 for Polyhedral Maps — Lower Bounds

Let $\mathbb{M}$ be a surface with Euler characteristic $\chi(\mathbb{M})$. Firstly we construct a polyhedral graph of the plane with degree sum $\sum_{j>4k+4}(j-6k)v_j > (4k+4i)|\chi(\mathbb{M})|$, $k \geq 3$, such that each subgraph P_k and each subgraph S_i contains a vertex of degree at least $4(k+i)-1$, $k \geq 3$. Our method used here is very similar to that one used in [12]. We start with a plane graph R_{n+1} with $n > (k+1)|\chi(\mathbb{M})| + 3k$ as described in Section 6. Next the outer $4n$-face is deleted and the opposite "vertical sides" are identified, i.e., the two paths $(1,1),(1,2),\ldots,(1,n+1)$ and $(n+1,1),(n+1,2),\ldots,(n+1,n+1)$ are identified in the given order.

The result is a triangulated cylinder Z_n^*. A plane polyhedral graph Z_n is obtained by adding a bottom n-face F_1 and a top n-face F_2 which are the only n-faces of a Z_n, all other faces of Z_n are triangles. We use the same notation as in R_n. If in all triangles of Z_n a generalized 3-star $S(r)$ with a fixed r is inserted then all inner vertices of Z_n have the degree $4(k+i)$. For instance choose $r=0$. We want to increase the degrees of the vertices of the boundaries of Z_n, i.e., for the vertices $(1,1)$,$(1,2)$,...,$(1,n)$ and $(n+1,1)$,$(n+1,2)$,...,$(n+1,n)$. In order to do this we vary r so that from each inner vertex near to these boundaries a degree unit is transferred to one of these boundaries. We achieve this in the following way: According to Figure 3 insert the 3-star $S(k-2i)$ into $D'_{x,1}$, the 3-star $S(k-2i-1)$ into $D'_{x,2},\ldots,$ the 3-star $S(0)$ into $D'_{x,k-2i+1}$ for all $x, 1 \leq x \leq n$. According to Figure 3 insert the 3-star $S(k-2i)$ into $D_{x,n+1}$, the 3-star $S(k-2i-1)$ into $D_{x,n},\ldots,$ the 3-star $S(0)$ into $D_{x1,n-(k-2i)M}$, for all $2 \leq x \leq n+1$. Into all other triangles insert the 3-star $S(0)$ according to Figure 3.

By the construction the vertices (x,y), $1 \leq x \leq n$, $2+(k-2i) \leq y \leq n-(k-2i)$ have degree $4(k+i)$. The vertices $(x,y), 1 \leq x \leq n$, $2 \leq y \leq k-2i+1$ or $n-(k-2i)+1 \leq y \leq n$ have the degree $4(k+i)-1$. The vertices on the boundaries, i.e., the vertices $(x,1)$ and $(x,n+1)$, $1 \leq x \leq n$ have the degree $3k+1$. In order to complete our construction we put into F_i a new vertex X_i and join X_i with all bounding vertices of F_i, $i=1,2$. In each new triangle Δ a k-path p of F_1 and F_2 is inserted. One endvertex of p is joined with all three vertices of Δ, and all other vertices of p are joined with each of the two remaining vertices of Δ. In the obtained triangulation Z_n^{**} the vertices bounding F_i have degree $3k+1+3+2(k-1)-2=5k$, and X_i has degree $\deg X_i \geq 2n > 2(k+i)|\chi(\mathbb{M})| + 6k$. Thus $ch_k(Z_n^{**}) \geq (\deg X_1 - 6k) + (\deg X_2 - 6k) > 4(k+i)|\chi(\mathbb{M})|$.

Next the wanted polyhedral maps of $\mathbb{M}$ will be constructed. If $\mathbb{M}$ is an orientable 2-manifold $\mathbb{S}_g$ of genus g then g handles have to be added. If $\mathbb{M}$ is a nonorientable 2-manifold $\mathbb{N}_q$ of genus q then q crosscaps have to be added. In both cases this is accomplished in the same way as in Section 6. The addition of g handles or of q crosscaps causes no problems according Section 6.

Acknowledgement

1. Support of Slovak VEGA Grant 1/0424/03 is acknowledged.
2. The authors are gratitude to anonymous referees who read original versions of the paper very carefully. Their suggestions helped us to improve the presentation of the paper.

References

[1] I. Fabrici and S. Jendrol', *Subgraphs with restricted degrees of their vertices in planar 3-connected graphs*, Graphs and Combinatorics **13** (1997) 245–250.

[2] I. Fabrici and S. Jendrol', *Subgraphs with restricted degrees of their vertices in planar graphs*, Discrete Math. **191** (1998) 83–90.

[3] B. Grünbaum, *New views on some old questions of combinatorial geometry* (Int. Theorie Combinatorie, Rome, 1973) **1** (1976) 451–468.

[4] B. Grünbaum and G.C. Shephard, *Analogues for tiling of Kotzig's theorem on minimal weight of edges*, Ann. Discrete Math. **12** (1982) 129–140.

[5] J. Harant, S. Jendrol' and M. Tkáč, *On 3-connected plane graphs without trianglar faces*, J. Combin. Theory (B) **77** (1999) 150–61.

[6] J. Ivančo, *The weight of a graph*, Ann. Discrete Math. **51** (1992) 113–116.

[7] S. Jendrol', T. Madaras, R. Soták and Zs. Tuza, *On light cycles in plane triangulations*, Discrete Math. **197/198** (1999) 453–467.

[8] S. Jendrol' and H.-J. Voss, *A local property of polyhedral maps on compact 2-dimensional manifolds*, Discrete Math. **212** (2000) 111–120.

[9] S. Jendrol' and H.-J. Voss, *A local property of large polyhedral maps on compact 2-dimensional manifolds*, Graphs and Combinatorics **15** (1999) 303–313.

[10] S. Jendrol' and H.-J. Voss, *Light paths with an odd number of vertices in large polyhedral maps*, Annals of Combin. **2** (1998) 313–324.

[11] S. Jendrol' and H.-J. Voss, *Subgraphs with restricted degrees of their vertices in large polyhedral maps on compact* 2*-manifolds*, European J. Combin. **20** (1999) 821–832.

[12] S. Jendrol' and H.-J Voss, *Light subgraphs of multigraphs on compact* 2*-dimensional manifolds*, Discrete Math. **233** (2001) 329–351.

[13] S. Jendrol' and H.-J. Voss, *Subgraphs with restricted degrees of their vertices in polyhedral maps on compact* 2*-manifolds*, Annals of Combin. **5** (2001) 211–226.

[14] S. Jendrol' and H.-J. Voss, *Light subgraphs of graphs embedded in* 2*-dimensional manifolds of Euler characteristic* ≤ 0 *– a survey*, in: Paul Erdös and his Mathematics, II (Budapest, 1999) Bolyai Soc. Math. Stud., 11 (János Bolyai Math. Soc., Budapest, 2002) 375–411.

[15] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Math. Čas. SAV (Math. Slovaca) **5** (1955) 111–113.

[16] T. Madaras, *Note on weights of paths in polyhedral graphs*, Discrete Math. **203** (1999) 267–269.

[17] B. Mohar, *Face-width of embedded graphs*, Math. Slovaca **47** (1997) 35–63.

[18] G. Ringel, Map color Theorem (Springer-Verlag Berlin, 1974).

[19] N. Robertson and R. P. Vitray, *Representativity of surface embeddings*, in: B. Korte, L. Lovász, H.J. Prömel and A. Schrijver, eds., Paths, Flows and VLSI-Layout (Springer-Verlag, Berlin-New York, 1990) 293–328.

[20] H. Sachs, Einführung in die Theorie der endlichen Graphen, Teil II. (Teubner Leipzig, 1972).

[21] J. Zaks, *Extending Kotzig's theorem*, Israel J. Math. **45** (1983) 281–296.

Received 10 January 2002
Revised 20 May 2003