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# A NOTE ON STRONGLY MULTIPLICATIVE GRAPHS

Chandrashekar Adiga
H.N. Ramaswamy
and
D.D. Somashekara

*Department of Studies in Mathematics*
*University of Mysore*
*Manasagangotri*
*Mysore – 570 006, INDIA*

**Abstract**

In this note we give an upper bound for $\lambda(n)$, the maximum number of edges in a strongly multiplicative graph of order $n$, which is sharper than the upper bound obtained by Beineke and Hegde [1].

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## 1. Introduction

In an interesting paper [1] L.W. Beineke and S.M. Hegde have studied strongly multiplicative graphs. In fact, they showed that certain graphs like trees, wheels and grids are strongly multiplicative. They also obtained an upper bound for the maximum number of edges $\lambda(n)$ for a given strongly multiplicative graph of order $n$. Erdös [2] has earlier obtained an asymptotic formula for $\lambda(n)$.

In this note we obtain an upper bound for $\lambda(n)$ which is sharper than that upper bound given in [1], for large values of $n$.

## 2. Main Result

We recall the definition of strongly multiplicative graphs given in [1].

**Definition.** A graph with $n$ vertices is said to be strongly multiplicative if its vertices can be labeled $1, 2, \dots, n$ so that the values on the edges, obtained as the product of the labels of their end vertices, are all distinct.

Let $\lambda(n)$ denote the maximum number of edges in a strongly multiplicative graph of order $n$. Thus

$$\lambda(n) = \left|\{rs|1 \le r < s \le n\}\right| .$$

Also, let

$$\delta(n) = \lambda(n) - \lambda(n-1) .$$

The number theoretic functions $\lambda$ and $\delta$ were studied in [1]. We first give an upper bound for $\delta(n)$ which leads to an improvement of Theorem 3.1 [1].

**Lemma.** If $p(n)$ denotes the least prime divisor of $n$, then

$$\delta(n) \le \begin{cases} n - \dfrac{n}{p(n)} + 1 & for \;\; n \equiv 0, 1, 3 (\text{mod } 4), \\ n - \dfrac{n}{p(n)} & for \;\; n \equiv 2 (\text{mod } 4). \end{cases}$$

***Proof.*** Let $p(n) = p$. We observe that $n \cdot 1$ can be written as $p\left(\frac{n}{p}\right)$, $n \cdot 2$ can be written as $2p\left(\frac{n}{p}\right), \dots, n\left(\frac{n}{p} - 1\right)$ can be written as $\left\{\left(\frac{n}{p} - 1\right) p\right\} \frac{n}{p}$. This shows that $\delta(n) \le n - \frac{n}{p} + 1$. Note that if $n \equiv 2 (\text{mod } 4)$, then $p = 2$, so that $\frac{n}{p}$ is odd and $kp$ is even. Therefore $\delta(n) \le n - \frac{n}{p(n)}$.

Now we obtain an upper bound for $\lambda(n)$ using the above Lemma.

**Theorem.** $\lambda(n) \le \frac{n(n+1)}{2} + (n-2) - \left[\frac{n+2}{4}\right] - \sum_{i=2}^{n} \frac{i}{p(i)}$, where [x] denotes the largest integer less than or equal to $x$.

***Proof.*** Since $\delta(n) = \lambda(n) - \lambda(n-1)$, we have

$$\lambda(n) = \sum_{i=2}^{n} \delta(i).$$

On using the above Lemma, it follows that,

$$\begin{aligned}\lambda(n) &\leq \sum_{i=2}^{n}\left(i - \frac{i}{p(i)} + 1\right) - \left[\frac{n+2}{4}\right] \\ &= \left(\frac{n(n+1)}{2} - 1\right) - \sum_{i=2}^{n}\frac{i}{p(i)} + (n-1) - \left[\frac{n+2}{4}\right],\end{aligned}$$

and the result follows.

**Remark.** The following table shows that the upper bound for $\lambda(n)$ given in the above Theorem is sharper than the upper bound given by L.W. Beineke and S.M. Hegde [1] for large values of $n$.

| n | Upper bound for $\lambda(n)$ using our Theorem | Upper bound for $\lambda(n)$ given by Beineke and Hegde |
|---|---|---|
| 25 | 236 | 240 |
| 27 | 268 | 279 |
| 42 | 629 | 661 |
| 60 | 1,263 | 1,350 |

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## References

[1] L.W. Beineke and S.M. Hegde, *Strongly multiplicative graphs*, Discuss. Math. Graph Theory **21** (2001) 63–76.

[2] P. Erdös, *An asymptotic inequality in the theory of numbers*, Vestnik Leningrad, Univ. **15** (1960) 41–49.

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