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ON THE HETEROCHROMATIC NUMBER OF CIRCULANT DIGRAPHS

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Abstract

The *heterochromatic number* $hc(D)$ of a digraph D, is the minimum integer k such that for every partition of $V(D)$ into k classes, there is a cyclic triangle whose three vertices belong to different classes.

For any two integers s and n with $1 \leq s \leq n$, let $D_{n,s}$ be the oriented graph such that $V(D_{n,s})$ is the set of integers mod $2n+1$ and $A(D_{n,s}) = \{(i,j) : j - i \in \{1, 2, \ldots, n\} \setminus \{s\}\}$.

In this paper we prove that $hc(D_{n,s}) \leq 5$ for $n \geq 7$. The bound is tight since equality holds when $s \in \{n, \frac{2n+1}{3}\}$.

Keywords: circulant tournament, vertex colouring, heterochromatic number, heterochromatic triangle.

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1. Introduction

The heterochromatic number of an r-graph $H = (V, E)$ (hypergraph whose edges are sets of size r) is the minimum number k such that each vertex colouring of H using exactly k colours leaves at least one edge all whose vertices receive different colours.

The heterochromatic number of r-graphs has been studied in several papers including general and particular settings (see for instance [2] – [7]). An important instance of this invariant is the *heterochromatic number* $hc(D)$ (with respect to $\vec{C}_3$) of a digraph D, which is the minimum integer k such that for every partition of $V(D)$ into k classes, there is a cyclic triangle whose three vertices belong to different classes. The heterochromatic number is preserved under opposition (i.e., $hc(D^{op}) = hc(D)$ where D^{op} denotes the digraph obtained from D by reversing the direction of each arc of D).

Let $D_{n,s}$ be the oriented graph such that $V(D_{n,s})$ is the set of integers mod $2n+1$ and $A(D_{n,s}) = \{(i,j) : j-i \in \{1,2,\ldots n\} \setminus \{s\}\}$.

In this paper we prove that $hc(D_{n,s}) \leq 5$ for $n \geq 7$. The bound is tight since equality holds when $s \in \{n, \frac{2n+1}{3}\}$. Related results concerning the heterochromatic number of circulant tournaments were given in [5] and [7].

2. Preliminaries

For general concepts we refer the reader to [1]. If D is a digraph, $V(D)$ and $A(D)$ (or simply A) will denote the sets of vertices and arcs of D respectively. A vertex k-colouring of D is said to be *full* if it uses the k colours. We will denote by $c_1, c_2, \ldots, c_k$ the colours and by $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ the corresponding chromatic classes. A heterochromatic cyclic triangle (h. triangle) is a cyclic triangle whose vertices are coloured with 3 different colours.

Along this paper we will work in the ring Z_{2n+1} of integers mod $2n+1$. If J is a nonempty subset of $Z_{2n+1} \setminus \{0\}$ such that $|\{j,-j\} \cap J| \leq 1$ for every $j \in Z_{2n+1} \setminus \{0\}$ then the circulant oriented graph $\vec{C}_{2n+1}(J)$ is defined by $V(\vec{C}_{2n+1}(J)) = Z_{2n+1}$, $A(\vec{C}_{2n+1}(J)) = \{(i,j) : i,j \in Z_{2n+1}$ and $j-i \in J\}$ and $C_{2n+1}(J)$ is its underlying graph. In particular, $\vec{C}_{2n+1} = \vec{C}_{2n+1}(\{1\})$ is the oriented cycle of length $2n+1$ and C_{2n+1} is its underlying graph. Finally, for $S \subseteq I_n = \{1,2,\ldots,n\} \subseteq Z_{2n+1}$, $\vec{C}_{2n+1}\langle S\rangle$ will denote the circulant tournament $\vec{C}_{2n+1}(J)$ where $J = (I_n \cup (-S)) \setminus S$ (when $S = \{s\}$ we will denote $\vec{C}_{2n+1}\langle S\rangle$ by $\vec{C}_{2n+1}\langle s\rangle$).

The following statement is relevant in our approach.

Remark. Given any two different elements i, j of Z_{2n+1}, the reflection $\alpha_{i,j}$ of C_{2n+1} defined by $\alpha_{i,j}(x) = i+j-x$ is an antiautomorphism of $\vec{C}_{2n+1}(J)$ which interchanges i and j.

Although the aim of this work is to determine a tight upper bound for $hc(D_{n,s})$, for technical reasons we prefer dealing with $\vec{C}_{2n+1}\langle s\rangle$; so we define a normal triangle (n. triangle) of $\vec{C}_{2n+1}\langle s\rangle$ to be a cyclic triangle in $\vec{C}_{2n+1}\langle s\rangle$ avoiding the arcs of the form $(i+s, i)$, (i.e., a cyclic triangle of $D_{n,s}$).

We will write $(i \in \mathcal{C}_1 \cup \mathcal{C}_2, (i,j,k,i))$ to express that we may assume that $i \in \mathcal{C}_1 \cup \mathcal{C}_2$ because (i,j,k,i) is an heterochromatic normal triangle (h. n. triangle) whenever $i \notin \mathcal{C}_1 \cup \mathcal{C}_2$.

Let (j,k) be an arc of $\vec{C}_{2n+1}\langle s\rangle$, along the proofs we will write $(j,k)\sim s$ or $s\sim(j,k)$ (resp. $(j,k)\not\sim s$ or $s\not\sim(j,k)$) to mean that $(j,k)\in\{(i+s,i)\,|\,i\in Z_{2n+1}\}$ (resp. $(j,k)\notin\{(i+s,i)\,|\,i\in Z_{2n+1}\}$). For a pair (j,k), we write $s\not\sim(j,k)\in A$ to mean that $(j,k)\in A$ and $(j,k)\not\sim s$.

In what follows $\gamma_n(i,j)$ (or simply $\gamma(i,j)$) will denote the ij-path $(i,i+1,\dots,j)$ (notation $\bmod 2n+1$) in C_{2n+1} as well as the set of its vertices; $\ell(\gamma(i,j))$ will be the length of $\gamma(i,j)$, i.e., the number of edges of $\gamma(i,j)$.

Two vertex colourings f and f' of a digraph D is said to be equivalent, in symbols: $f\equiv f'$ when there exists either an automorphism or an anti-automorphism α of D such that $f'=f\circ\alpha$. Clearly $\equiv$ is an equivalence relation and f and f' use the same colours whenever $f\equiv f'$.

We will need the following two lemmas.

Lemma 2.1. *Let f and f' be vertex colourings of $\vec{C}_{2n+1}\langle s\rangle$.*

(i) *If $f\equiv f'$ and f leaves an h. n. triangle of $\vec{C}_{2n+1}\langle s\rangle$ then f' leaves an h. n. triangle of $\vec{C}_{2n+1}\langle s\rangle$.*

(ii) *If $f'=f\circ\alpha_{i,j}$, then $f'(\alpha_{i,j}(x))=f(x)$.* ■

Lemma 2.2. *Let f be a full vertex r-colouring of C_{2n+1}.*

(i) *Suppose $r\ge 4$. If (1) there exist two vertices $a,b\in V(C_{2n+1})$ with $\ell(\gamma(a,b))=n$ (resp. $n-1$) such that $a\in\mathcal{C}_2$, $b\in\mathcal{C}_1$, $\mathcal{C}_3\cap\gamma(a,b)\neq\emptyset$ and $\mathcal{C}_4\cap\gamma(a,b)\neq\emptyset$, then (2) there exist two vertices $a',b'\in V(C_{2n+1})$ with $\ell(\gamma(a',b'))=n$ (resp. $n-1$) such that $a'\in\mathcal{C}_i$, $b'\in\mathcal{C}_j$, $\mathcal{C}_k\cap\gamma(b',a')\neq\emptyset$ and $\mathcal{C}_\ell\cap\gamma(b',a')\neq\emptyset$ ($\{i,j,k,\ell\}=\{1,2,3,4\}$).*

(ii) *If $r\ge 5$, then (2) holds.*

Proof. To prove (i), take $c\in\mathcal{C}_3\cap\gamma(a,b)$ and $d\in\mathcal{C}_4\cap\gamma(a,b)$, and suppose that $c<d$ (c and d considered as integers).

First consider $b+n$ (resp. $b+n-1$). Since $\mathcal{C}_2\cap\gamma(b+n,b)\neq\emptyset$, $\mathcal{C}_3\cap\gamma(b+n,b)\neq\emptyset$ and $\mathcal{C}_4\cap\gamma(b+n,b)\neq\emptyset$ we may assume $b+n$ (resp. $b+n-1)\in\mathcal{C}_1$ (in other case we take $a'=b$ and $b'=b+n$ (resp. $b+n-1$)). Now, since $c<d$ we have that colours c_1, c_2 and c_3 appear in $\gamma(d+n,d)$; so we may assume $d+n\in\mathcal{C}_4$. Finally we have that colours c_4, c_1 and c_2 appear in $\gamma(c+n,c)$ so we may assume $c+n\in\mathcal{C}_3$ and we obtain the vertices a,b with $c+n\in\mathcal{C}_3\cap\gamma(b,a)$ and $d+n\in\mathcal{C}_4\cap\gamma(b,a)$ (resp. $d+n-1$ and $c+n-1$).

In order to prove (ii), recall that the number of connected components of $C_{2n+1}(\{s\})$ is the maximum common divisor of s and $2n+1$. In particular,

$C_{2n+1}(\{n\})$ is connected and $C_{2n+1}(\{n-1\})$ has either 1 or 3 connected components depending on whether $n \not\equiv 1$ or $n \equiv 1 \bmod 3$. Since $r = 5$, C_{2n+1} has a vertex i such that i and $i+n$ (resp. $i+n-1$) have different colours. Applying (i) the proof ends. ∎

3. An Upper Bound for $h_c(D_{n,s})$.

In this section we give a tight upper bound for $h_c(D_{n,s})$.

Theorem 3.1. *For $n \geq 7$, every full vertex 5-colouring of the circulant tournament $\vec{C}_{2n+1}\langle s\rangle$ leaves an h. n. triangle; in other words $hc(D_{n,s}) \leq 5$ and equality holds whenever $s \in \{n, \frac{2n+1}{3}\}$.*

Proof. Consider any full vertex 5-colouring and suppose that no h. n. triangle is produced. We divide the proof into two cases.

Case 1. $s \neq n$.
Because of Lemmas 2.2(ii) and 2.1, we may assume that $0 \in \mathcal{C}_1$ and $n+1 \in \mathcal{C}_2$, $\mathcal{C}_3 \cap \gamma(0, n+1) \neq \emptyset$ and $\mathcal{C}_4 \cap \gamma(0, n+1) \neq \emptyset$.

Let $i \in \mathcal{C}_3 \cap \gamma(0, n+1)$ and $j \in \mathcal{C}_4 \cap \gamma(0, n+1)$; we may assume that $|\{(n+1, i), (i, 0)\} \cap A| = 1$ and $|\{(n+1, j), (j, 0)\} \cap A| = 1$. If $|\{(n+1, i), (i, 0)\} \cap A| = 0$, then $(0, i, n+1, 0)$ is an h. n. triangle and if $|\{(n+1, i), (i, 0)\} \cap A| = 2$, then $(0, j, n+1, 0)$ is an h. n. triangle. Similarly $|\{(n+1, j), (j, 0)\} \cap A| = 1$. Moreover $|\{(n+1, j), (n+1, i)\} \cap A| = 1$ and $|\{(i, 0), (j, 0)\} \cap A| = 1$. We may assume w.l.o.g. that $(i, 0) \in A$ (with $(i, 0) \sim s$) and $(n+1, j) \in A$ (with $(n+1, j) \sim s$). Now observe that when $\mathcal{C}_5 \cap \gamma(0, n+1) \neq \emptyset$, $(0, k, n+1, 0)$ is an h. n. triangle, where $k \in \mathcal{C}_5 \cap \gamma(0, n+1)$. So we may assume that $\mathcal{C}_5 \cap \gamma(0, n+1) = \emptyset$ and then $\mathcal{C}_5 \cap \gamma(n+1, 0) \neq \emptyset$.

Let $k \in \mathcal{C}_5 \cap \gamma(n+1, 0)$. We will analyze several possible cases.

Subcase 1.a. $s \not\sim (j, k) \in A$.
$s \sim (0, k) \in A$. In other case $(0, j, k, 0)$ is an h. n. triangle ($s \not\sim (0, j) \in A$ as $(i, 0) \sim s$).

When $(i, k) \in A$ we have $(i, k) \not\sim s$ (because $(i, 0) \sim s$), also we have $2s \geq n+1$ (as $(i, 0) \sim s$, $(0, k) \sim s$ and $(i, k) \in A$ with $(i, k) \not\sim s$); so $s > 1$; $(1 \in \mathcal{C}_1 \cup \mathcal{C}_2$, $(0, 1, n+1, 0))$ (notice $1 \neq s$, $n \neq s$) and then $(i, k, 1, i)$ is an h. n. triangle. When $(k, i) \in A$ with $(k, i) \not\sim s$ we have $2s < n$ and hence $i < j$; also we observe that $s \sim (j, i) \in A$ (in other case (j, k, i, j) is an h. n. triangle and $s \sim (k, n+1) \in A$ (otherwise $(k, i, n+1, k)$ is an h. n. triangle;

so we obtain: $3s = n+1$ ($(n+1, j) \sim s$, $(j, i) \sim s$ and $(i, 0) \sim s$), $2s = n$ ($(0, k) \sim s$ and $(k, n+1) \sim s$), so $s = 1$ and $2n+1 = 5$ contradicting $n \geq 7$. When $s \sim (k, i) \in A$ we have $j < i$ (because $(n+1, j) \sim s$); in this case also we have $2s > n+1$, so $s > 1$ and ($1 \in \mathcal{C}_1 \cup \mathcal{C}_2$, $(0, 1, n+1, 0)$); we conclude that $(j, k, 1, j)$ is an h. n. triangle.

Subcase 1.b. $s \sim (j, k) \in A$.
Since $(n+1, j) \sim s$ and $(j, k) \sim s$ with $k \in \gamma(n+1, 0)$ we have $2s > n+1$ and hence $i > j$. Observe $(k, j+1) \in A$ (because $(j, k) \sim s < n$). Now $n \in \mathcal{C}_1$; ($n \in \mathcal{C}_1 \cup \mathcal{C}_2$, $(0, n, n+1, 0)$) and ($n \in \mathcal{C}_1 \cup \mathcal{C}_5$, $(0, n, k, 0)$). Consider $j+1$; when $j+1 = i$ we get the h. n. triangle $(k, j+1, n+1, k)$ (Notice that $(n+1, k) \not\sim s$ as $(j, k) \sim s$ and $n+1 \neq j$ since $(n+1, j) \sim s$). When $j+1 \neq i$ we obtain ($j+1 \in \mathcal{C}_1 \cup \mathcal{C}_2$, $(0, j+1, n+1, 0)$), now if $j+1 \in \mathcal{C}_1$ then $(j+1, n+1, k, j+1)$ is an h. n. triangle (we have observed that $(n+1, k) \not\sim s$) and if $j+1 \in \mathcal{C}_2$ then $(j+1, n, k, j+1)$ is an h. n. triangle (notice that $(n, k) \not\sim s$ because $(j, k) \sim s$ and $j \neq n$ as $j < i \in \gamma(0, n+1) \bigcap \mathcal{C}_3$ and $n+1 \in \mathcal{C}_2$).

Subcase 1.c. $(k, j) \in A$ (In this case $(k, j) \not\sim s$ because $(n+1, j) \sim s$).
$s \neq 1$. If $s = 1$ then $j = n$ but $(k, n) \notin A$ for every $k \in \gamma(n+1, 0)$; so, ($n \in \mathcal{C}_1 \cup \mathcal{C}_2$, $(0, n, n+1, 0)$) and hence $(k, n) \sim s$ (when $(n, k) \in A$, (k, j, n, k) is an h. n. triangle). Now consider $n-1$ if $n-1 = i$ then (j, i, k, j) is an h. n. triangle and when $n-1 \neq i$ we have ($n-1 \in \mathcal{C}_1 \cup \mathcal{C}_2$, $(0, n-1, n+1, 0)$). (observe that since $(k, n) \sim s$, $(n+1, j) \sim s$ and $s \not\sim (k, j) \in A$ we have $2s - 1 > n+1 \geq 8$ so $s > 2$) and then $(k, j, n-1, k)$ is an h. n. triangle.

Finally, if $s = \frac{2n+1}{3}$, the vertex 4-colouring defined by ($0 \in \mathcal{C}_1, s \in \mathcal{C}_2, 2s \in \mathcal{C}_3$ and $x \in \mathcal{C}_4$ for $x \notin \{0, s, 2s\}$) leaves no h. n. triangle and, since $s \neq n$, we obtain $hc(D_{n,s}) = 5$.

Case 2. $s = n$.
Because of Lemmas 2.2(ii) and 2.1, we may assume that $n+2 \in \mathcal{C}_2$, $0 \in \mathcal{C}_1$, $\mathcal{C}_3 \cap \gamma(0, n+2) \neq \emptyset$ and $\mathcal{C}_4 \cap \gamma(0, n+2) \neq \emptyset$.

For every $x \in \gamma(3, n-1)$, $x \in \mathcal{C}_1 \cup \mathcal{C}_2$. In other case $(0, x, n+2, 0)$ is an h. n. triangle.

We may assume: (1) $(\mathcal{C}_3 \cup \mathcal{C}_4) \cap \{1, 2\} \neq \emptyset$ (when $(\mathcal{C}_3 \cup \mathcal{C}_4) \cap \{1, 2\} = \emptyset$ we obtain $(\mathcal{C}_3 \cup \mathcal{C}_4) \cap \{n, n+1\} \neq \emptyset$ and such a colouring is equivalent to another one satisfying (1) by Lemma 2.1(ii) where $\alpha_{i,j} = \alpha_{0,n+2}$). Suppose $\mathcal{C}_5 \cap \{1, 2, n, n+1\} = \emptyset$, then $\mathcal{C}_5 \cap \gamma(n+2, 0) \neq \emptyset$, let $k \in \mathcal{C}_5 \cap \gamma(n+2, 0)$ and let $i \in \{1,2\} \cap (\mathcal{C}_3 \cup \mathcal{C}_4)$. If $k \in \gamma(n+4, 2n-2)$ or if $(k = 3$ and $i = 1)$ then $(i, n-1, k, i)$ is an h. n. triangle; now suppose $k = n+3$ and $i = 2$; clearly we may assume $1 \in \mathcal{C}_1 \cup \mathcal{C}_2$ (otherwise $(1, n-1, n+3, 1)$ is an h. n. triangle), also we may assume $n+1 \in \mathcal{C}_4$ (otherwise $n \in \mathcal{C}_4$ and $(1, n, n+3, 1)$ is an h. n. triangle), moreover $n+5 \in \mathcal{C}_3$ $((n+5 \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3, (2, n-1, n+5, 2))$ and $(n+5 \in \mathcal{C}_3 \cup \mathcal{C}_4, (2, n+1, n+5, 2)))$, so $(2, n+1, n+5, 2)$ is an h. n. triangle. Hence $k \in \{2n, 2n-1\}$ (notice that $k \neq n+2$ as $n+2 \in \mathcal{C}_2$ and $k \in \mathcal{C}_5$). If $i = 2$ we have $(n+1 \in \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5, (i, n+1, k, i))$ (notice that $i \in \mathcal{C}_3 \cup \mathcal{C}_4$ and $k \in \mathcal{C}_5$); when $n+1 \in \mathcal{C}_5$ we are done, so $n+1 \in \mathcal{C}_3 \cup \mathcal{C}_4$ and then $(n+1, k, 3, n+1)$ is an h. n. triangle; we conclude that $i = 1$ and $2 \notin \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$, and so $\{n, n+1\} \cap (\mathcal{C}_3 \cup \mathcal{C}_4) \neq \emptyset$; moreover, again by Lemma 2.1(ii) we may assume that $n \notin (\mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5)$, $1 \in \mathcal{C}_3$ and $n+1 \in \mathcal{C}_4$ and then $(n+1, k, 3, n+1)$ is an h. n. triangle.

Suppose now that $\mathcal{C}_5 \cap \{1, 2, n, n+1\} \neq \emptyset$ it follows that there exists an arc (a, b) with $a \in \{1, 2\}$, $b \in \{n, n+1\}$, $\ell(\gamma(a, b)) = n-1$, $a \in \mathcal{C}_i$, $b \in \mathcal{C}_j$ and $\{i, j\} \in \{3, 4, 5\}$ without loss of generality assume $1 \in \mathcal{C}_3$ and $n \in \mathcal{C}_4$ (the other possible cases are completly analogous). Now $(n+5 \in \mathcal{C}_3 \cup \mathcal{C}_4$, $(1, n, n+5, 1))$ (remember $n \geq 7$) and $\{2, n+1\} \cap \mathcal{C}_5 \neq \emptyset$. When $2 \in \mathcal{C}_5$ we get $(n+5, 2, n-1, n+5)$ an h. n. triangle and when $n+1 \in \mathcal{C}_5$ we obtain the h. n. triangle $(n+1, n+5, 3, n+1)$. ■

Finally, since the vertex 4-colouring of $D_{n,n}$ defined by $(0 \in \mathcal{C}_1, n \in \mathcal{C}_2, n+1 \in \mathcal{C}_3$ and $x \in \mathcal{C}_4$ for $x \notin \{0, n, n+1\})$ leaves no h. n. triangle, we obtain $hc(D_{n,n}) = 5$. ■

4. Final Comment

It can be proved that $hc(D_{n,s}) = 4$ whenever $n \geq 7$ and $s \notin \{n, (2n+1)/3\}$. The complete determination of $hc(D_{n,s})$, which is a useful tool in studying 4-heterochromatic cycles in circulant tournaments, requires an extense proof and will be given elsewhere.

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