

ON TRACEABILITY AND 2-FACTORS
IN CLAW-FREE GRAPHS

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Abstract

If G is a claw-free graph of sufficiently large order n , satisfying a degree condition $\sigma_k > n + k^2 - 4k + 7$ (where k is an arbitrary constant), then G has a 2-factor with at most $k - 1$ components. As a second main result, we present classes of graphs $\mathcal{C}_1, \dots, \mathcal{C}_8$ such that every sufficiently large connected claw-free graph satisfying degree condition $\sigma_6(k) > n + 19$ (or, as a corollary, $\delta(G) > \frac{n+19}{6}$) either belongs to $\cup_{i=1}^8 \mathcal{C}_i$ or is traceable.

Keywords: traceability, 2-factor, claw, degree condition, closure

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1. INTRODUCTION

We consider finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. We follow the most common terminology and notation and for concepts not defined here we refer e.g. to [1]. For any set $A \subset V(G)$ we denote by $\langle A \rangle_G$ the subgraph of G induced on A and $G - A$ stands for $\langle V(G) \setminus A \rangle$. A graph G is H -free (where H is a graph), if G does not contain an induced subgraph isomorphic to H . In the special case $H = K_{1,3}$ we say that G is *claw-free*. The *independence number* of G is denoted by $\alpha(G)$ and the *clique covering number* of G (i.e., the minimum number of cliques necessary for covering $V(G)$) by $\theta(G)$. We denote by $\delta(G)$ the *minimum degree* of G and by $\sigma_k(G)$ ($k \geq 1$) the *minimum degree sum* over all independent sets of k vertices in G (for $k > \alpha(G)$ we set $\sigma_k(G) = \infty$). The *circumference* of G , i.e., the length of a longest cycle in G , is denoted by $c(G)$, and the length of a longest path in G is denoted by $p(G)$. A graph G of order n is *hamiltonian* or *traceable* if $c(G) = n$ or $p(G) = n$, respectively.

The *line graph* of a graph H is denoted by $L(H)$. If $G = L(H)$, then we also denote $H = L^{-1}(G)$ and say that H is the *line graph preimage* of G (recall that for any line graph G nonisomorphic to K_3 , its line graph preimage is uniquely determined).

A vertex $x \in V(G)$ is said to be *locally connected* if its neighborhood $N(x)$ induces a connected graph. The *closure* of a claw-free graph G (introduced in [12] by the first author) is defined as follows: the closure $\text{cl}(G)$ of G is the (unique) graph obtained by recursively completing the neighborhood of any locally connected vertex of G , as long as this is possible. The closure $\text{cl}(G)$ remains a claw-free graph and its connectivity is at least equal to the

connectivity of G . The following basic properties of the closure $\text{cl}(G)$ were proved in [12], [3] and [13].

Theorem A. *Let G be a claw-free graph and $\text{cl}(G)$ its closure. Then*

- (i) [12] *there is a triangle-free graph H_G such that $\text{cl}(G) = L(H_G)$,*
- (ii) [12] *$c(G) = c(\text{cl}(G))$,*
- (iii) [3] *$p(G) = p(\text{cl}(G))$,*
- (iv) [13] *G has a 2-factor with at most k components if and only if $\text{cl}(G)$ has a 2-factor with at most k components.*

Consequently, G is hamiltonian (traceable) if and only if $\text{cl}(G)$ is hamiltonian (traceable). If G is a claw-free graph such that $G = \text{cl}(G)$, then we say that G is *closed*. It is apparent that a claw-free graph G is closed if and only if every vertex $x \in V(G)$ is either *simplicial* (i.e., $\langle N(x) \rangle_G$ is a clique), or is *locally disconnected* (i.e., $\langle N(x) \rangle_G$ consists of two vertex disjoint cliques).

In [12], the closure concept was used to answer an old question by showing that every 7-connected claw-free graph is hamiltonian. H. Li [10] extended this result as follows.

Theorem B [10]. *Every 6-connected claw-free graph with at most 34 vertices of degree 6 is hamiltonian.*

In [5], the following result on 2-factors with limited number of components was proved.

Theorem C [5]. *If G is a claw-free graph of order n and minimum degree $\delta \geq 4$, then G contains a 2-factor with at most $\frac{6n}{\delta+2} - 1$ components.*

This result was improved by Gould and Jacobson [8].

Theorem D [8]. *Let $k \geq 2$ be an integer and let G be a claw-free graph of order $n \geq 16k^3$ and minimum degree $\delta \geq \frac{n}{k}$. Then G contains a 2-factor with at most k components.*

In the first main result of this paper, Theorem 3, we give a strengthening of this result.

A trail T (closed or not) in a graph H is said to be *dominating* if every edge of H has at least one vertex on T . Harary and Nash-Williams [11]

proved the following result, showing that hamiltonicity of a line graph is equivalent to the existence of a dominating closed trail in its preimage.

Theorem E [11]. *Let H be a graph without isolated vertices. Then $L(H)$ is hamiltonian if and only if either H is isomorphic to $K_{1,r}$ (for some $r \geq 3$) or H contains a dominating closed trail.*

It is straightforward to verify the following analogue of Theorem E for traceability.

Theorem F. *Let H be a graph without isolated vertices. Then $L(H)$ is traceable if and only if either H is isomorphic to $K_{1,r}$ (for some $r \geq 3$) or H contains a dominating trail.*

Using the closure concept in claw-free graphs [12], Favaron, Flandrin, Li and Ryjáček [6] observed that there is a close relation between the minimum degree sum $\sigma_k(G)$ (or the minimum degree $\delta(G)$, respectively) of a closed claw-free graph G and its clique covering number. These connections are established in the following results [6].

Theorem G [6]. *Let $k \geq 2$ be an integer and let G be a claw-free graph of order n such that $\delta(G) > 3k - 5$ and $\sigma_k(G) > n + k^2 - 2k$. Then $\theta(\text{cl}(G)) \leq k - 1$.*

Corollary H [6]. *Let $k \geq 2$ be an integer and let G be a claw-free graph of order $n \geq 2k^2 - 3k$ and minimum degree $\delta(G) > \frac{n}{k} + k - 2$. Then $\theta(\text{cl}(G)) \leq k - 1$.*

The bounds on $\sigma_k(G)$ ($\delta(G)$) in the previous results are sharp (this can be easily seen considering the cartesian product of cliques).

It was shown in [6] and [9] that these results can be slightly strengthened under an additional assumption that G is not hamiltonian, and this result was used to obtain degree conditions for hamiltonicity (by characterizing the classes of all 2-connected nonhamiltonian closed claw-free graphs with small clique covering number). In the second main result of this paper, Theorem 6, we follow up with this study by considering analogous questions for traceability.

2. MAIN RESULTS

We begin with a structural result that can be considered, in a sense, as a strengthening of Theorem G.

Theorem 1. *Let $k \geq 2$ be an integer, let G be a claw-free graph of order n and let $\kappa = \kappa(\text{cl}(G))$. Suppose that G is such that $n \geq 3k^2 + k - (k+1)\kappa - 2$, $\delta(G) \geq 3k - 4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 2 + \kappa.$$

Then either $\theta(\text{cl}(G)) \leq k - 1$, or $\alpha(\text{cl}(G)) \leq \kappa$.

Before proving Theorem 1, we first recall the following auxiliary results that were proved in [6].

Lemma I [6]. *Let G be a closed claw-free graph of order n and $\{a_1, a_2, \dots, a_t\} \subset V(G)$ an independent set. Then*

- (i) $|N(a_i) \cap N(a_j)| \leq 2$, $1 \leq i < j \leq t$,
- (ii) $\sum_{i=1}^t d(a_i) \leq n + t^2 - 2t$.

Lemma J [6].

(i) *Any triangle-free graph H whose matching number $\nu(H)$ and vertex covering number $\tau(H)$ satisfy $\nu(H) < \tau(H)$, contains an edge xy such that $d(x) + d(y) \leq \nu(H) + \tau(H)$.*

(ii) *Let G be a closed claw-free graph. If $\alpha(G) < \theta(G)$, then $\delta(G) \leq \alpha(G) + \theta(G) - 2$.*

Lemma K [6]. *Let G be a closed claw-free graph. Then $\theta(G) \leq 2\alpha(G)$.*

Lemma L [6]. *Let G be a closed claw-free graph of order n and connectivity $\kappa(G)$ such that $1 \leq \kappa(G) < \alpha(G)$ and let $A = \{a_1, \dots, a_\alpha\}$ be a maximum independent set in G . Then*

$$\sum_{i=1}^{\alpha} d(a_i) \leq n + \alpha^2 - 4\alpha + 2 + \kappa(G).$$

Proof of Theorem 1. If G is a counterexample to Theorem 1 such that G satisfies the assumptions but $\kappa < \alpha(\text{cl}(G))$ and $\theta(\text{cl}(G)) \geq k$, then so is the closure $\text{cl}(G)$. Hence we can suppose that G is closed.

If $\alpha(G) \geq k + 1$, then by Lemma I we have $\sigma_{k+1}(G) \leq n + (k + 1)^2 - 2(k + 1) = n + k^2 - 1$, implying $\sigma_k(G) \leq \frac{k}{k+1}(n + k^2 - 1) \leq n + k^2 - 4k + 2 + \kappa$ for $n \geq 3k^2 + k - (k + 1)\kappa - 2$, a contradiction. Hence $\alpha(G) \leq k$.

If $\alpha(G) \leq k - 1$, then $\alpha(G) < \theta(G)$ and, by Lemma J and Lemma K, $\delta(G) \leq \alpha(G) + \theta(G) - 2 \leq (k - 1) + 2(k - 1) - 2 = 3k - 5$, a contradiction.

Hence we have $\alpha(G) = k$. Since $\kappa(G) < \alpha(G)$, Lemma L gives $\sigma_k(G) \leq n + k^2 - 4k + \kappa + 2$, a contradiction. ■

From Theorem 1 we obtain the following minimum degree result.

Corollary 2. *Let $k \geq 2$ be an integer, let G be a claw-free graph of order n and let $\kappa = \kappa(\text{cl}(G))$. Suppose that G is such that $n \geq 3k^2 - k - \kappa - 2$ and*

$$\delta(G) > \frac{n + k^2 - 4k + 2 + \kappa}{k}.$$

Then either $\theta(\text{cl}(G)) \leq k - 1$, or $\alpha(\text{cl}(G)) \leq \kappa$.

Proof. We can again suppose that G is closed. If $n \geq 3k^2 - k - \kappa - 2$, then obviously $\delta(G) > \frac{n+k^2-4k+2+\kappa}{k} \geq 3k - 5$ and hence $\delta(G) \geq 3k - 4$. The rest of the proof follows immediately from Theorem 1. ■

Now we can prove our first main result that gives a degree condition for the existence of a 2-factor with limited number of components.

Theorem 3. *Let $k \geq 2$ be an integer, let G be a claw-free graph of order n and let $\kappa = \kappa(\text{cl}(G))$. Suppose that G is such that $n \geq 3k^2 + k - (k + 1)\kappa - 2$, $\delta(G) \geq 3k - 4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 2 + \kappa.$$

Then G has a 2-factor with at most $k - \kappa$ components.

Proof. If G satisfies the assumptions of the theorem but has no 2-factor with at most $k - \kappa$ components, then, since $\delta(G) \leq \delta(\text{cl}(G))$ and by Theorem A(iv), so does its closure $\text{cl}(G)$. Since $\text{cl}(G)$ is nonhamiltonian, by the well-known theorem of Chvátal and Erdős (see [2]), $\alpha(\text{cl}(G)) > \kappa(\text{cl}(G))$. By Theorem 1, we have $\theta(\text{cl}(G)) \leq k - 1$.

Let $\mathcal{P} = \{K_1, \dots, K_\theta\}$ be a minimum clique covering of $\text{cl}(G)$ such that each of the cliques K_1, \dots, K_θ is maximal. We show that each clique of \mathcal{P} has at least $2k - 1$ vertices. This follows immediately from $\delta(\text{cl}(G)) \geq 3k - 4$ for those K_i 's that contain at least one simplicial vertex. Thus, suppose that (say) K_1 contains no simplicial vertex. By the minimality of \mathcal{P} , there is a clique K' with $K_1 \cap K' \neq \emptyset$ and $K' \notin \mathcal{P}$ (otherwise $\mathcal{P} \setminus \{K_1\}$ is also a clique covering of $\text{cl}(G)$). Since clearly every clique in $\text{cl}(G)$ that contains a simplicial vertex must be in \mathcal{P} , K' has no simplicial vertex. By the structure of the closure, $|K_1 \cap K'| = 1$.

Denote $K_1 \cap K' = \{x\}$, $|K_1| = t$ and $|K'| = r$. Then we have $d(x) = t - 1 + r - 1 \geq \delta(\text{cl}(G)) \geq 3k - 4$, implying $t + r \geq 3k - 2$. Since $K' \notin \mathcal{P}$, there are $r - 1$ further cliques $K_{i_1}, \dots, K_{i_{r-1}} \in \mathcal{P}$ having a common vertex with K' . By the structure of the closure, $K_{i_j} \neq K_{i_\ell}$, $j \neq \ell$, $j, \ell = 1, \dots, r - 1$, implying $\theta \geq r$. Since $\theta \leq k - 1$, we have $3k - 2 \leq t + r \leq t + k - 1$, from which $t \geq 2k - 1$.

Now, since $\theta \leq k - 1$, each clique of \mathcal{P} contains at least $2k - 1 - (k - 2) = k + 1$ vertices that are in no other clique of \mathcal{P} . Since $k \geq 2$, every $K_i \in \mathcal{P}$ contains a cycle C_i that is vertex-disjoint from all other cliques of \mathcal{P} . Let $x_i \in K_i$, $i = 1, \dots, \theta$ be such that each x_i is in no other clique of \mathcal{P} . Since $\kappa = \kappa(\text{cl}(G))$, by a well-known theorem by Dirac [4], there is a cycle C in $\text{cl}(G)$ containing all the vertices x_1, \dots, x_κ . Let \mathcal{C} be the collection of those of the cycles $C_{\kappa+1}, \dots, C_\theta$, which are vertex-disjoint with C . Then the collection of cycles $\{C\} \cup \mathcal{C}$ can be easily extended to a 2-factor of $\text{cl}(G)$ with at most $k - \kappa$ components. The result then follows by Theorem A(iv). ■

Corollary 4. *Let $k \geq 4$ be an integer and G be a connected claw-free graph of order $n \geq 3k^2 - 3$, $\delta(G) \geq 3k - 4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 7.$$

Then G has a 2-factor with at most $k - 1$ components.

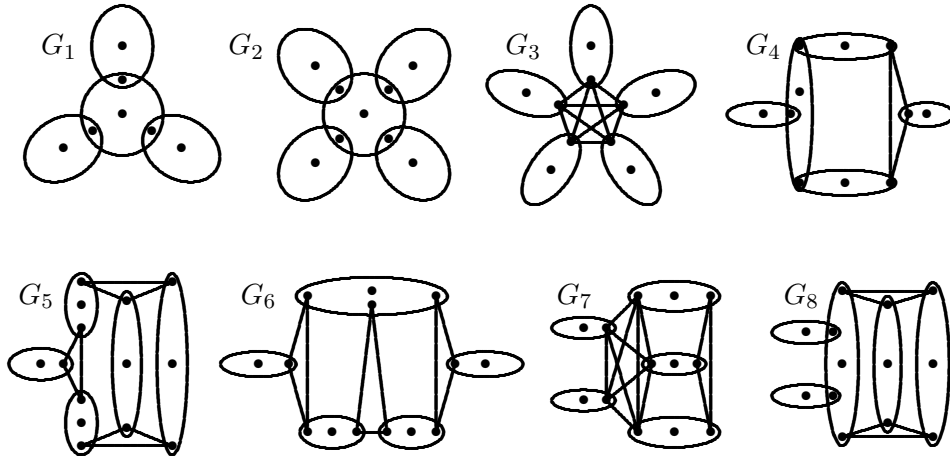
Proof. We can again suppose that G is closed. If $\kappa(\text{cl}(G)) \geq 6$, then G has a required 2-factor since G is hamiltonian by Theorem B (note that $\delta(G) \geq 3k - 4 \geq 8$). Hence suppose $1 \leq \kappa(\text{cl}(G)) \leq 5$. Then we have $n \geq 3k^2 - 3 \geq 3k^2 - (k + 1)\kappa + 2$ since $\kappa \geq 1$ and $\sigma_k(G) > n + k^2 - 4k + 7 \geq n + k^2 - 4k + \kappa + 2$ since $\kappa \leq 5$. Then G has a 2-factor with at most $k - 1$ components by Theorem 3. ■

Remark. It would be possible to formulate minimum degree results corresponding to Theorem 3 and Corollary 4. Details are left to the reader.

Next we turn our attention to traceability. Let \mathcal{C}_i , $i = 1, \dots, 8$, be the class of all spanning subgraphs of the graphs G_i , $i = 1, \dots, 8$, shown in Figure 1 (where the circular and elliptical parts represent cliques of arbitrary order). Using a technique similar to that of [6], we can prove the following result.

Theorem 5. *Let G be a connected closed claw-free graph with clique covering number $\theta \leq 5$. Then either $G \in \cup_{i=1}^8 \mathcal{C}_i$, or G is traceable.*

Proof of Theorem 5 is lengthy and it is therefore postponed to Section 3.



\mathcal{C}_i is the class of all spanning subgraphs of G_i , $i = 1, \dots, 8$.

Figure 1

Combining Theorem 5 and Theorem 1, we can now obtain the following theorem, which is the second main result of this paper.

Theorem 6. *Let G be a connected claw-free graph of order $n \geq 112 - 7\kappa(\text{cl}(G))$ such that $\delta(G) \geq 14$ and*

$$\sigma_6(G) > n + 14 + \kappa(\text{cl}(G)).$$

Then either $\text{cl}(G) \in \cup_{i=1}^8 \mathcal{C}_i$, or G is traceable.

Proof. If G is a nontraceable graph satisfying the assumptions of the theorem, then clearly so is $\text{cl}(G)$. Thus, suppose that G is closed. By a well-known consequence of a theorem by Chvátal and Erdős [2] (see e.g. [7], Part I, Corollary 4.17), nontraceability of G implies $\alpha(G) > \kappa + 1$. From Theorem 1 (for $k = 6$) we then obtain $\theta(G) \leq 5$. The rest of the proof follows from Theorem 5. ■

Corollary 7. *Let G be a connected claw-free graph of order $n \geq 105$ such that $\delta(G) \geq 14$ and*

$$\sigma_6(G) > n + 19.$$

Then either $\text{cl}(G) \in \cup_{i=1}^8 \mathcal{C}_i$, or G is traceable.

Proof. We can again suppose that G is closed. If G is nontraceable, then Theorem B implies $1 \leq \kappa(G) \leq 5$. Rest of the proof follows immediately from Theorem 6. ■

Corollary 8. *Let G be a connected claw-free graph of order $n \geq 105$ with minimum degree*

$$\delta(G) > \frac{n + 19}{6}.$$

Then either $\text{cl}(G) \in \cup_{i=1}^8 \mathcal{C}_i$, or G is traceable. ■

3. PROOF OF THEOREM 5

We basically follow the terminology and notation introduced in [6] and [9]. Let \mathcal{G}_θ be the class of all connected nontraceable closed claw-free graphs with clique covering number θ . By Theorem A, every $G \in \mathcal{G}_\theta$ is the line graph of some (unique) triangle-free graph H . Let $D_1(H)$ be the set of all degree 1 vertices of H and put $H' = H - D_1(H)$. Set $\mathcal{H}_\theta = \{L^{-1}(G) \mid G \in \mathcal{G}_\theta\}$ and $\mathcal{H}'_\theta = \{H - D_1(H) \mid H \in \mathcal{H}_\theta\}$.

In every $G \in \mathcal{G}_\theta$ choose a fixed minimum clique covering $\mathcal{P}_G = \{B_1, \dots, B_\theta\}$ of G such that each clique B_i is maximal. Since \mathcal{P}_G is minimum, every B_i contains at least one *proper vertex*, i.e., a vertex belonging to no other clique of \mathcal{P}_G . The centers B_1, \dots, B_θ of the stars of $H = L^{-1}(G)$ that correspond to the cliques of G will be called the *black vertices* of H . The other vertices of H are called *white*. The set of black (white) vertices of H is denoted by $B(H)$ ($W(H)$), respectively. Since $B(H)$ is a vertex covering

of H (i.e., every edge of H has at least one vertex in $B(H)$), the set $W(H)$ is independent.

It is easy to see that for any $G \in \mathcal{G}_\theta$, any graph obtained from G by adding/removing simplicial vertices to/from cliques of \mathcal{P}_G also belongs to \mathcal{G}_θ as long as (in the case of removal) at least one simplicial vertex in the clique remains (while the removal of the last simplicial vertex of a clique can turn G into a traceable graph). Hence we can without loss of generality denote for any $H' \in \mathcal{H}'_\theta$ by $L(H')$ the graph obtained from the line graph of H' by adding one simplicial vertex to every clique corresponding to a black vertex of H' .

Let $G_1, G_2 \in \mathcal{G}_\theta$. We say that G_1 is an *ss-subgraph* of G_2 , if G_1 is isomorphic to a spanning subgraph of a graph, which is obtained from G_2 by adding an appropriate number of simplicial vertices to some cliques of \mathcal{P}_{G_2} , and that G_1 is a *proper ss-subgraph* of G_2 if G_1 is an *ss-subgraph* of G_2 and G_1, G_2 are nonisomorphic. In the following we present a method for finding a subset $\mathcal{F}_\theta \subset \mathcal{H}'_\theta$ such that

- (i) every $G \in \mathcal{G}_\theta$ is an *ss-subgraph* of $L(F)$ for some $F \in \mathcal{F}_\theta$,
- (ii) for any $F_1, F_2 \in \mathcal{F}_\theta$, $L(F_1)$ is not an *ss-subgraph* of $L(F_2)$.

By the previous observations, the class \mathcal{G}_θ is fully characterized by \mathcal{F}_θ .

If, for some $H \in \mathcal{H}_\theta$, the corresponding $H' \in \mathcal{H}'_\theta$ has a *black trail* (abbreviated BT), i.e., a trail containing all black vertices of H' , then clearly H has a dominating trail. Since, by Theorem F, no $H \in \mathcal{H}_\theta$ has a dominating trail, no $H' \in \mathcal{H}'_\theta$ has a BT.

For a trail T in $H' \in \mathcal{H}'_\theta$ we denote by $\text{bla}(T)$ the black length of T , i.e., the number of black vertices of H' that are on T , and by $\text{cro}(T)$ the number of “crossings” of T , i.e., the number of vertices of H' that are visited by T at least twice.

Two vertices of H' are said to be *related* if they are adjacent or if they are both black and have a common white neighbor. If T is a (fixed) trail in H' and x, y are vertices of H' , then we say that x, y are \bar{T} -*related* (denoted $x \sim y$) if $xy \in E(H') \setminus E(T)$ or x and y have a white common neighbor outside T .

Let now $H' \in \mathcal{H}'_\theta$, and let T be a trail in H' such that

- (i) $\text{bla}(T)$ is maximum,
- (ii) subject to (i), $\text{cro}(T)$ is minimum,
- (iii) subject to (i) and (ii), the length of T is minimum.

Then T has two black vertices of degree 1. We will always denote by b_1, \dots, b_k the black vertices of T labelled along T , and by w_i the white successor of b_i on T , if it exists. Note that, since T is a trail, possibly $b_i = b_j$ or $w_i = w_j$ for some $i \neq j$. If $b_i \sim b_j$, then the (possible) white common neighbor of b_i, b_j outside T will be denoted by w_{ij} .

Case $\theta = 3$.

Let $B = \{b_1, b_2, b_3\}$. Then, clearly, $\text{bla}(T) = 2$, $\text{cro}(T) = 0$ and $T = b_1(w_1)b_2$. Since H' is connected and the set $\{b_1, b_2, b_3\}$ is dominating, b_3 is \bar{T} -related to some vertex of T . Clearly both $b_3 \sim b_1$ and $b_3 \sim b_2$ imply traceability of $L(H')$, hence $b_3 \sim w_1$, implying $b_3w_1 \in E(H')$. The existence of any further relation implies traceability of $L(H')$, hence $V(H') = \{b_1, b_2, b_3, w_1\}$ and $E(H') = \{w_1b_1, w_1b_2, w_1b_3\}$, implying $L(H') \in \mathcal{C}_1$.

Case $\theta = 4$.

Let $B = \{b_1, b_2, b_3, b_4\}$. Then obviously $2 \leq \text{bla}(T) \leq 3$ and $\text{cro}(T) = 0$, i.e., T is a path. We have two subcases.

Subcase $\text{bla}(T) = 3$.

Then $T = b_1(w_1)b_2(w_2)b_3$. Suppose first that b_4 is \bar{T} -related to some black vertex. Then necessarily $b_4 \sim b_2$. If $b_4 \sim x$ for some $x \in \{b_1, (w_1), b_3, (w_2)\}$, then we immediately have a trail T' with $\text{bla}(T') = 4$. Similarly, $b_1 \sim b_2$ yields $T' = b_3(w_2)b_2(w_1)b_1(w_{12})b_2(w_{24})b_4$, $b_1 \sim w_2$ gives $T' = b_3w_2b_1(w_1)b_2(w_{24})b_4$, and for $b_1 \sim b_3$ we have $T' = b_3(w_{13})b_1(w_1)b_2(w_{24})b_4$. By symmetry and since H' is triangle-free, these are all possibilities. Hence there are no further \bar{T} -relations, implying $L(H') \in \mathcal{C}_1$.

Hence b_4 is related to white vertices only. If both $b_4 \sim w_1$ and $b_4 \sim w_2$, then H' contains no more relations, implying $L(H') \in \mathcal{C}_2$. If (by symmetry) $b_4 \sim w_1$ and $b_4 \not\sim w_2$, then the only possible additional relation that does not create a trail T' with $\text{bla}(T) = 4$ is $b_1 \sim w_2$. Then for $b_1 \not\sim w_2$ we have $L(H') \in \mathcal{C}_1$ and for $b_1 \sim w_2$ we have $L(H') \in \mathcal{C}_4$.

Subcase $\text{bla}(T) = 2$.

Then $T = b_1(w_1)b_2$. Then immediately $b_3 \sim w_1$ and $b_4 \sim w_1$. If $b_3 \sim b_4$, then $L(H')$ is traceable; hence $b_3 \not\sim b_4$, implying $L(H') \in \mathcal{C}_2$.

Case $\theta = 5$.

Let $B = \{b_1, b_2, b_3, b_4, b_5\}$. We have obviously $2 \leq \text{bla}(T) \leq 4$. If $2 \leq \text{bla}(T) \leq 3$, then $\text{cro}(T) = 0$ (since H' is triangle-free), for $\text{bla}(T) = 4$

we have $0 \leq \text{cro}(T) \leq 1$. We will denote these subcases by k/ℓ , where $k = \text{bla}(T)$ and $\ell = \text{cro}(T)$. Thus, we have subcases 4/0, 4/1, 3/0 and 2/0. The subcase 4/1 splits into two subcases 4/1w and 4/1b according to whether the vertex visited twice by T is white or black, respectively. We consider these subcases separately.

Subcase 4/0.

Then $T = b_1(w_1)b_2(w_2)b_3(w_3)b_4$ is a path. It is straightforward to check that b_5 can be \bar{T} -related to at most one black vertex of T (for otherwise $L(H')$ is traceable). Thus, we have two possibilities.

Subcase 4/0-1. b_5 is \bar{T} -related to exactly one black vertex of T .

By symmetry, we can suppose $b_5 \sim b_2$. We distinguish two subcases.

Subcase 4/0-1-1. b_5 is \bar{T} -related to some white vertex on T .

Then the only possibility that does not imply $L(H')$ is traceable is $b_5 \sim w_3$. Then it is straightforward to check that any further \bar{T} -relation between vertices of T implies $L(H')$ is traceable, but then $L(H') \in \mathcal{C}_4$.

Subcase 4/0-1-2. b_2 is the only \bar{T} -relation of b_5 on T .

We consider possible \bar{T} -relations between vertices of T .

If $b_1 \sim w_3$, then we are in a situation symmetric to the subcase 4/0-1-1 and hence $L(H') \in \mathcal{C}_4$. All the other relations of b_1 on T imply $L(H')$ is traceable. Hence we can assume b_1 has no \bar{T} -relation on T . Now, if also b_4 has no \bar{T} -relation on T , then we have $L(H') \in \mathcal{C}_1$. Hence we can suppose $b_4 \sim x$ for some $x \in V(T)$. If $x \in \{b_1, w_1, b_2\}$, then $L(H')$ is traceable. Hence $x \in \{w_2, b_3\}$. Now, if there is no \bar{T} -relation $y \sim z$ for any $y \in \{w_1, b_2\}$, $z \in \{w_2, b_3, w_3, b_4\}$, then we have $L(H') \in \mathcal{C}_1$. It is straightforward to check that all such relations $y \sim z$ imply $L(H')$ is traceable.

Subcase 4/0-2. b_5 is \bar{T} -related only to white vertices on T .

In this subcase, we have three further possibilities.

Subcase 4/0-2-1. b_5 is \bar{T} -related to w_1, w_2 and w_3 .

Then there is no further \bar{T} -relation on T and $L(H') \in \mathcal{C}_6$.

Subcase 4/0-2-2. b_5 is \bar{T} -related to two white vertices on T .

By symmetry, we can suppose that $b_5 \sim w_1$ and either $b_5 \sim w_2$ or $b_5 \sim w_3$.

Let first $b_5 \sim w_2$. If no vertices on T are \bar{T} -related, then $L(H') \in \mathcal{C}_4$ (with $b_3, (w_3), b_4$ in one clique of $L(H')$). Hence suppose there is a \bar{T} -relation between some vertices of T . Clearly $b_1 \not\sim b_2, b_1 \not\sim w_2, b_1 \not\sim b_3, b_1 \not\sim b_4, w_1 \not\sim b_3, w_1 \not\sim b_4, b_2 \not\sim b_3, b_2 \not\sim b_4$ and $w_2 \not\sim b_4$, since any of these relations implies $L(H')$ is traceable. It remains to consider the possibilities $b_1 \sim w_3, b_2 \sim w_3$ and $b_3 \sim b_4$.

If $b_1 \sim w_3$, then both $b_2 \not\sim w_3$ and $b_3 \not\sim b_4$ (otherwise $L(H')$ is traceable), and then $L(H') \in \mathcal{C}_5$; if $b_2 \sim w_3$, then $b_1 \not\sim w_3$ and $b_3 \not\sim b_4$, implying $L(H') \in \mathcal{C}_6$; and if $b_3 \sim b_4$, then similarly $b_1 \not\sim w_3, b_2 \not\sim w_3$ and $L(H') \in \mathcal{C}_4$ (in which b_3, b_4 and their common neighbors are in one clique).

Hence suppose $b_5 \sim w_3$. Similarly as before, no \bar{T} -relation between vertices of T implies $L(H') \in \mathcal{C}_4$ with $b_2, (w_2), b_3$ in one clique. Thus, suppose some vertices of T are \bar{T} -related. Immediately $b_1 \not\sim b_2, b_1 \not\sim b_3, b_1 \not\sim w_3, b_1 \not\sim b_4$ and $w_1 \not\sim b_3$, since any of these relations implies $L(H')$ is traceable. By symmetry, it remains to consider the possibilities $b_1 \sim w_2$ and $b_2 \sim b_3$. If $b_1 \sim w_2$, then $b_2 \not\sim b_3$ (otherwise $L(H')$ is traceable), implying $L(H') \in \mathcal{C}_5$; if $b_2 \sim b_3$, then similarly $b_1 \not\sim w_2$ and $L(H') \in \mathcal{C}_4$ (with b_2, b_3 and their common neighbors in one clique).

Subcase 4/0-2-3. b_5 is \bar{T} -related to exactly one white vertex on T .

By symmetry, either $b_5 \sim w_1$ or $b_5 \sim w_2$.

Let first $b_5 \sim w_1$. If b_1 is not \bar{T} -related to any of b_2, w_2, b_3, w_3, b_4 , then $L(H') \in \mathcal{C}_1$ (with b_2, b_3 and b_4 in one clique). The relations $b_1 \sim b_2$ and $b_1 \sim b_4$ immediately imply traceability. Hence b_1 is \bar{T} -related to w_2, b_3 or w_3 .

If $b_1 \sim b_3$ and, at the same time, $b_1 \sim w_2$ or $b_1 \sim w_3$, then $L(H')$ is traceable, and if $b_1 \sim w_2$ and $b_1 \sim w_3$, then we are in Subcase 4/0-2-1 (where b_1 plays the role of b_5). Hence b_1 is \bar{T} -related to exactly one of b_3, w_2, w_3 .

If $b_1 \sim b_3$, then any additional relation implies $L(H')$ is traceable, and hence we have $L(H') \in \mathcal{C}_4$.

If $b_1 \sim w_2$, then for $b_2 \sim w_3$ we are in Subcase 4/0-2-1 (where b_2 plays the role of b_5) and $L(H') \in \mathcal{C}_6$. Any other additional relation except $b_3 \sim b_4$

implies $L(H')$ is traceable. If $b_3 \sim b_4$, or if there is no additional relation, we have $L(H') \in \mathcal{C}_4$ (with b_3, b_4 in one clique).

If $b_1 \sim w_3$, then any additional relation except for $b_2 \sim b_3$ or $w_2 \sim b_4$ implies $L(H')$ is traceable. For $w_2 \sim b_4$ we have $L(H') \in \mathcal{C}_5$, and if $b_2 \sim b_3$ or if there is no additional relation, then $L(H') \in \mathcal{C}_4$ (with b_2 and b_3 in one clique).

Let now $b_5 \sim w_2$. First observe that there is no \bar{T} -relation containing w_2 since H' is triangle-free and both $w_2 \sim b_1$ and $w_2 \sim b_4$ imply traceability. Secondly, if there is no \bar{T} -relation $x \sim y$ with $x \in \{b_1, w_1, b_2\}$ and $y \in \{b_3, w_3, b_4\}$, then $L(H') \in \mathcal{C}_1$. Since $b_1 \sim b_3$ and $b_1 \sim b_4$ imply traceability, by symmetry, we have $b_2 \sim b_3$, $w_1 \sim b_3$ or $b_1 \sim w_3$. We consider these possibilities separately.

If $b_2 \sim b_3$, then there is no additional \bar{T} -relation containing b_1 (or symmetrically b_4), for otherwise $L(H')$ is traceable. This implies $L(H') \in \mathcal{C}_1$.

If $w_1 \sim b_3$, then similarly $L(H') \in \mathcal{C}_1$, unless there is an additional \bar{T} -relation containing b_1 or b_4 . The only such relations that do not imply traceability are $b_1 \sim w_3$ or $b_3 \sim b_4$, but then $L(H') \in \mathcal{C}_6$ or $L(H') \in \mathcal{C}_4$, respectively.

Finally, if $b_1 \sim w_3$, then we already know there is no further relation, and we have $L(H') \in \mathcal{C}_4$.

Subcase 4/1w.

Recall that in this subcase T visits twice one white vertex. Choose the notation such that $T = b_1 w_1 b_2 w_2 b_3 w_1 b_4$. Clearly, $b_5 \not\sim b_1$, $b_5 \not\sim b_4$ and b_5 cannot be \bar{T} -related to both b_2, b_3 (since in each of these cases $L(H')$ is traceable). Thus, b_5 is \bar{T} -related to at most one black vertex on T . There are two subcases.

Subcase 4/1w-1. b_5 is \bar{T} -related to one black vertex on T .

By symmetry, let $b_5 \sim b_2$. Then $L(H') \in \mathcal{C}_1$ (with w_1, b_2, w_2, b_3 in one clique), since any additional \bar{T} -relation involving any of b_1, b_4, b_5 implies $L(H')$ is traceable.

Subcase 4/1w-2. b_5 is \bar{T} -related only to white vertices.

In this subcase, we distinguish three further possibilities.

Subcase 4/1w-2-1. $b_5 \sim w_1, b_5 \sim w_2$.

Then there is no other relation and $L(H') \in \mathcal{C}_7$.

Subcase 4/1w-2-2. $b_5 \sim w_1, b_5 \not\sim w_2$.

If there is no other relation involving any of b_1, b_2 , then we have $L(H') \in \mathcal{C}_1$ (with b_2, b_3 and their common neighbors in one clique). It is straightforward to check that any further \bar{T} -relation involving b_1 or b_4 gives $L(H') \in \mathcal{C}_8$ (if some of b_1, b_2 is \bar{T} -related to w_2), or traceability of $L(H')$.

Subcase 4/1w-2-3. $b_5 \sim w_2, b_5 \not\sim w_1$.

Then $L(H') \in \mathcal{C}_1$ (with w_1, b_2, w_2, b_3 in one clique) and any \bar{T} -relation between any of b_2, w_2, b_3 and the rest implies traceability of $L(H')$.

Subcase 4/1b.

Choose the notation such that the vertex b_2 is visited twice by T , i.e., $T = b_1(w_1)b_2w_2b_3w_3b_2(w_4)b_4$. Similarly as before, b_5 is \bar{T} -related to at most one black vertex on T , and neither to b_1 nor to b_4 . We distinguish three subcases.

Subcase 4/1b-1. $b_5 \sim b_2, b_5 \not\sim b_3$.

In this case b_2 is the only \bar{T} -relation of b_5 on T (since any other relation implies traceability). Now $L(H') \in \mathcal{C}_2$ and any other \bar{T} -relation between vertices of T gives $L(H') \in \mathcal{C}_1$ or traceability. We distinguish three possibilities.

Subcase 4/1b-2. $b_5 \sim b_3, b_5 \not\sim b_2$.

In this subcase immediately $L(H') \in \mathcal{C}_1$ with $\{b_2, w_2, b_3, w_3\}$ in one clique and any relation joining a vertex from this set to the rest gives traceability.

Subcase 4/1b-3. b_5 is \bar{T} -related only to white vertices.

Then b_5 can have \bar{T} -relations in at most one of the sets $\{w_1, w_4\}, \{w_2, w_3\}$ (otherwise $L(H')$ is traceable). We have two further possibilities.

Subcase 4/1b-3-1. $b_5 \sim w_1$.

For $b_5 \not\sim w_4$ we have $L(H') \in \mathcal{C}_1$, and for $b_5 \sim w_4$ we have $L(H') \in \mathcal{C}_4$ (with b_2, w_2, b_3, w_3 in one clique).

Subcase 4/1b-3-2. $b_5 \sim w_2$.

If $b_5 \not\sim w_3$, then $L(H') \in \mathcal{C}_1$ (with b_2, w_2, b_3, w_3 in one clique), and if $b_5 \sim w_3$, then $L(H') \in \mathcal{C}_8$.

Subcase 3/0.

Let $T = b_1(w_1)b_2(w_2)b_3$.

Subcase 3/0-1. $b_4 \sim b_5$.

If b_4 or b_5 is \bar{T} -related to any vertex on T , then we have a path T' with $\text{bla}(T') \geq 4$, except for the case if $w_{45} = w_{24} = w_{25}$. In this case, $L(H') \in \mathcal{C}_2$.

Subcase 3/0-2. $b_4 \not\sim b_5$.

If b_4 has two relations on T , then the only possibility that does not create a trail T' with $\text{bla}(T') \geq 4$ is $b_4 \sim w_1, b_4 \sim w_2$, but then, for any \bar{T} -relation of b_5 on T we again have a trail T' with $\text{bla}(T') \geq 4$. Hence both b_4 and b_5 have one \bar{T} -relation on T . Then it is straightforward to check that in all nontraceable cases we have $L(H') \in \mathcal{C}_2$ or $L(H') \in \mathcal{C}_1$.

Subcase 2/0.

Let $T = b_1(w_1)b_2$. If any two of b_3, b_4, b_5 are \bar{T} -related, we have a trail T' with $\text{bla}(T') \geq 3$. Hence $b_3 \sim w_1, b_4 \sim w_1, b_5 \sim w_1$, implying $L(H') \in \mathcal{C}_3$. ■

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