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IMPROVING SOME BOUNDS FOR DOMINATING CARTESIAN PRODUCTS

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Abstract

The study of domination in Cartesian products has received its main motivation from attempts to settle a conjecture made by V.G. Vizing in 1968. He conjectured that $\gamma(G)\gamma(H)$ is a lower bound for the domination number of the Cartesian product of any two graphs G and H. Most of the progress on settling this conjecture has been limited to verifying the conjectured lower bound if one of the graphs has a certain structural property.

In addition, a number of authors have established bounds for dominating the Cartesian product of any two graphs. We show how it is possible to improve some of these bounds by imposing conditions on both graphs. For example, we establish a new lower bound for the domination number of $T \,\Box\, T$, when T is a tree, and we improve an upper bound of Vizing in the case when one of the graphs has $k > 1$ dominating sets which cover the vertex set and the other has a dominating set which partitions in a certain way.

Keywords: domination number, Cartesian product, Vizing's conjecture, 2-packing.

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1. Introduction

It is well known that the problem of deciding if a given graph has a dominating set no larger than a given positive integer is NP-complete for the class of arbitrary graphs. However, if the problem is restricted to certain types of graphs such as trees or interval graphs, then polynomial algorithms exist for computing the domination number (denoted by γ in this paper). See Chapter 12 of [8] for a discussion. The special structure of graphs in these restricted classes is exploited to allow for fast computation. In addition, some effort has been given to finding a formula for the domination number of graphs whose structure is simple and well defined. An example of this is the class of complete grid graphs which are Cartesian products of paths. See the second chapter of [8].

For two given graphs G and H the Cartesian product $G \,\square\, H$ is very structured, having many copies of each of G and H as induced subgraphs. It seems natural to try to relate the domination number of this product to the domination numbers of G and H. In 1963 V.G. Vizing ([12]) posed the problem of determining if $\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H)$ for all pairs of graphs G and H. Little progress has been made on this problem, which was made a conjecture by Vizing in [13]. See [1], [11], [6] and [2]. With the exception of the surprising and general result of Clark and Suen ([2]), the progress has been to show the conjectured inequality holds when one of the graphs satisfies some structural condition.

Several authors have proved lower or upper bounds for $\gamma(G \square H)$ in terms of invariants of G and H. The following theorem summarizes some of these.

Theorem 1.1. *Let G and H be arbitrary graphs. Then*

1. [12] $\gamma(G \square H) \leq \min\{\gamma(G)|H|, \gamma(H)|G|\}$, *where $|G|$ denotes the number of vertices of G;*
2. [10] $\gamma(G \,\square\, H) \geq \frac{|H|}{\Delta(H)+1}\gamma(G)$;
3. [11] $\gamma(G \,\square\, H) \geq \max\{\gamma(G)\rho(H), \gamma(H)\rho(G)\}$, *where $\rho(G)$ is the 2-packing number of G;*
4. [3] $\gamma(G \,\square\, H) \geq \min\{|G|, |H|\}$;
5. [2] $\gamma(G \,\square\, H) \geq \frac{1}{2}\gamma(G)\gamma(H)$.

In this paper we do not verify Vizing's conjecture for any new classes of graphs. Rather our approach is to show that some of the bounds for the domination number of a Cartesian product can be improved by restricting the graphs in the product. If T_1 and T_2 are trees, then it is known that $\gamma(T_1 \,\square\, T_2) \geq \gamma(T_1)\gamma(T_2)$ since Vizing's conjecture holds if at least one of the factors is a tree. In Section 4 we establish a more general lower bound when T_1 and T_2 are isomorphic. In Section 3 we do not demand that G and H are the same graph but do require that each has certain properties with regard to different 2-packings, and we establish a lower bound for such graphs. In Section 5 we generalize the upper bound of Vizing from 1963 in the case when the vertex set of one of the graphs can be covered by $k > 1$ dominating sets and the other graph has a dominating set which partitions into k subsets satisfying a certain property.

2. Terminology and Background

All graphs considered in this paper are finite, simple graphs. We follow the definitions and notation of [8]. In particular, for vertex subsets A and B of a graph $G = (V, E)$ we say that A *dominates* B if each vertex of B is in the closed neighborhood of A; that is, each vertex of B is in A or is adjacent to some vertex of A. In case A dominates V we call A a *dominating set* for G. The *domination number* of G is the smallest cardinality, $\gamma(G)$, of a dominating set for G. A subset A of V is called a 2-*packing* of G if the closed neighborhoods of any two distinct vertices of A are disjoint. The 2-*packing number* of G is the maximum cardinality, $\rho(G)$, of a 2-packing of G. Since every dominating set for G has a nonempty intersection with each closed neighborhood, it follows that $\rho(G) \leq \gamma(G)$. We use $|G|$ to denote the order of G. By a *labeling* in G we mean a function $L : X \to \{1, 2, 3, \ldots\}$, where X is allowed to be any subset of V. For ease of illustration we will often write the label of a vertex next to the vertex and then refer to the natural partition of X induced by the labeling. For example, in Figure 1 the vertex set of C_6 is partitioned into three 2-packings, V_1, V_2, V_3 where V_k is the set of vertices labeled k. If $G = (V, E)$ and $H = (W, F)$ are graphs, then the *Cartesian product* of G and H is the graph $G \,\square\, H$, whose vertex set is the (set) Cartesian product $V \times W$. Two vertices (v_1, w_1) and (v_2, w_2) of $G \,\square\, H$ are adjacent if and only if they are equal in one coordinate and adjacent in the other coordinate. Note that we distinguish between the Cartesian product of sets, which is denoted by $\times$, and the Cartesian product of two graphs, which is denoted using the symbol $\square$. It often becomes convenient to consider the subgraph of $G \,\square\, H$ induced by the set of vertices $\{v\} \times W$.

This subgraph is isomorphic to H and is denoted by H_v. Similarly, for a vertex w of H, G_w denotes the subgraph of $G \,\square\, H$ induced by $V \times \{w\}$.

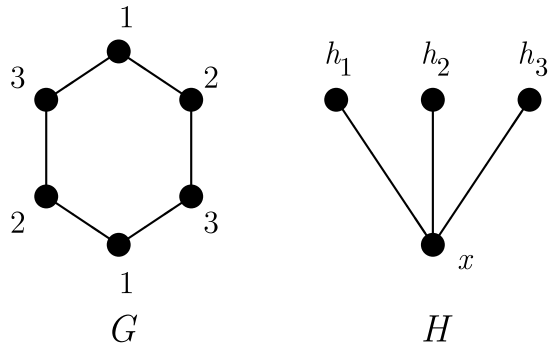

Figure 1. Elementary illustration

3. Lower Bounds

The central idea in our approach to establishing lower bounds for $\gamma(G \square H)$ is to require only part, although a carefully chosen part, of the vertex set of the product to be dominated. Observe that for any vertex u of G, the vertices in the subgraph H_u can be dominated only by those in the set $N_G[u] \times V(H)$. Therefore, if A is any maximum 2-packing of G and D dominates $A \times V(H)$, then for every $a \in A$ it follows that $|D \cap (N_G[a] \times V(H))| \geq \gamma(H)$. The third inequality of Theorem 1.1 is a direct consequence of these observations.

For the above approach, as traditionally applied, to yield a good lower bound, at least one of the two graphs must have a 2-packing that is almost as large as its domination number. In this section we relax that requirement but instead impose conditions which take advantage of the fact that the graph has a number of pairwise disjoint 2-packings. A related condition will also be imposed on the other graph.

Lemma 3.1. *Let $V_1, V_2, \ldots, V_k$ be pairwise disjoint subsets of $V(G)$ such that each is a 2-packing of G and assume H has an independent set of cardinality at least k. Then $\gamma(G \,\square\, H) \geq \sum_{i=1}^{k} |V_i|$.*

Proof. Let D be a subset of $V(G \,\square\, H)$ such that D dominates the set of vertices $W = \cup_{i=1}^{k}(V_i \times \{h_i\})$, where $A = \{h_1, h_2, \ldots, h_k\}$ is an independent set in H. Since W is a 2-packing of $G \,\square\, H$, it follows that no vertex of D can dominate more than one vertex of W. Therefore, $|D| \geq |W|$. Any dominating set for $G \,\square\, H$ must dominate W, and so

$$\gamma(G \,\square\, H) \geq |W| = \sum_{i=1}^{k} |V_i|.$$

■

A simple illustration of Lemma 3.1 is the pair of graphs G and H in Figure 1. Let V_i be the set of vertices of G labeled i. Then $\cup_{i=1}^{k}(V_i \times \{h_i\})$ is a 2-packing of $G \,\square\, H$, so at least six vertices will be required to dominate $G \,\square\, H$. The fact that $V(G) \times \{x\}$ dominates $G \,\square\, H$ shows the domination number of this Cartesian product is exactly six.

Lemma 3.1 may not be helpful in forcing a large lower bound for the domination number of a particular Cartesian product since the 2-packing sets of a graph may all have small cardinality while the domination number is large. The next result generalizes Lemma 3.1 in two ways.

Lemma 3.2. *Let $V_1, V_2, \ldots, V_k$ be pairwise disjoint subsets of $V(G)$ and, for each i, let n_i denote the smallest cardinality of a set W_i that dominates V_i. Let $A_1, A_2, \ldots, A_k$ be a collection of 2-packings of H such that for every $1 \leq i < j \leq k$, if there is a vertex of A_i adjacent to a vertex of A_j, then no vertex of V_i has a neighbor in V_j. Then $\gamma(G \,\square\, H) \geq \sum_{i=1}^{k} |A_i| n_i$.*

Proof. Let D be a subset of $V(G \square H)$ such that D dominates $\cup_{i=1}^{k}(V_i \times A_i)$. For a fixed i, if $x \in V_i$ and u and v are distinct vertices of A_i, then no vertex in D can dominate both (x, u) and (x, v) since A_i is a 2-packing. Therefore, $|A_i| n_i$ vertices of D will be required to dominate $V_i \times A_i$. But for $y \in V_j$ and $w \in A_j$, if $uw \in E(H)$, then $xy \notin E(G)$. Thus no vertex of D can dominate a vertex of $V_i \times A_i$ and a vertex of $V_j \times A_j$. It follows that $\gamma(G \,\square\, H) \geq |D| \geq \sum_{i=1}^{k} |A_i| n_i$. ■

Consider the pair G_1 and H_1 of Figure 2. The graph H_1 has domination number equal to its 2-packing number, and so by the result of Barcalkin and German in [1] it follows that $\gamma(G_1 \,\square\, H_1) \geq \gamma(G_1)\gamma(H_1) = 3 \times 4 = 12$. Let V_1, V_2, V_3, V_4 and A_1, A_2, A_3, A_4 be the sets of vertices of G_1 and H_1, respectively, induced by the given labelings in Figure 2. Then $n_1 = 2, n_2 = 2, n_3 = 1$ and $n_4 = 1$. Using Lemma 3.2 it follows that $\gamma(G_1 \,\square\, H_1) \geq 14$.

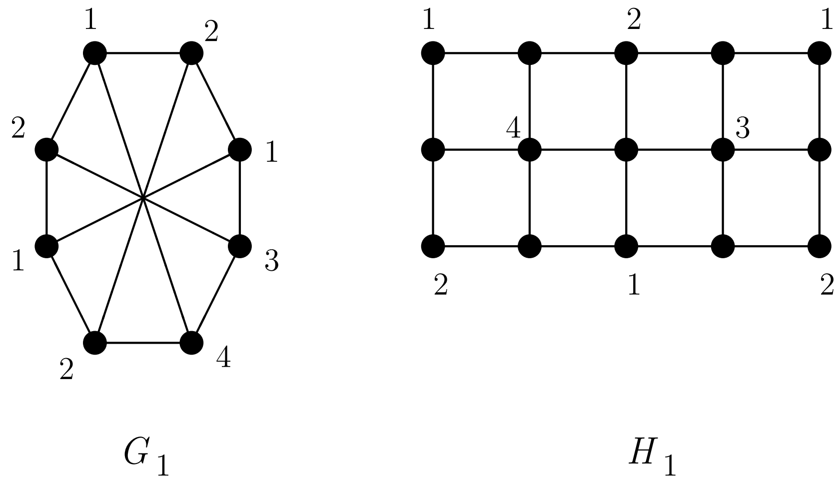

Figure 2. Example for Lemma 3.2

To get a reasonable lower bound using Lemma 3.2 it seems that one of the graphs in a Cartesian product must have some 2-packings of cardinality close to the domination number or must have a number of pairwise disjoint 2-packings. In the general case it is not at all clear how best to apply this lemma. Consider the two graphs G_2 and H_2 in Figure 3. The different labelings of G_2 and H_2 given in the following list illustrate how a range of values can arise as the lower bound derived from the lemma depends on the labelings chosen.

1. Let $V_1 = \{x_1, x_5\}, V_2 = \{x_2, x_7\}, A_1 = \{y_1, y_5, y_9\}, A_2 = \{y_3, y_7\}$. Then $\gamma(G_2 \,\square\, H_2) \geq 10$.
2. Let $V_1 = \{x_1, x_7\}, V_2 = \{x_3, x_6\}, V_3 = \{x_2, x_5\}, A_1 = \{y_1, y_6\}, A_2 = \{y_3, y_8\}, A_3 = \{y_5, y_{10}\}$. Then $\gamma(G_2 \,\square\, H_2) \geq 12$.
3. Let $V_1 = \{x_1, x_7\}, V_2 = \{x_2, x_5\}, A_1 = \{y_1, y_4, y_7, y_{10}\}, A_2 = \{y_2, y_5, y_8\}$. Then $\gamma(G_2 \,\square\, H_2) \geq 14$.
4. Let $V_1 = \{x_1, x_7\}, V_2 = \{x_3, x_6\}, V_3 = \{x_2, x_5\}, V_4 = \{x_4\}, A_1 = \{y_1, y_6\}, A_2 = \{y_3, y_8\}, A_3 = \{y_{10}\}, A_4 = \{y_5\}$. Then $\gamma(G_2 \,\square\, H_2) \geq 11$.

We will now restrict our attention to special classes of graphs and apply Lemma 3.2 to derive lower bounds for the domination number of a Cartesian product. Throughout this discussion we assume one of the graphs H has a collection of 2-packings $A_1, A_2, \ldots, A_k$ whose union A is independent. A simple instance of this is when $\gamma(H) = 1$ and the independence number of H is at least k. Assume $A = \{h_1, h_2, h_3\}$ is independent in H and let $\{x\}$ be a dominating set for H. Let G be the path $P_t : u_1, u_2, \ldots, u_t$. The vertex set of G partitions into three 2-packings, V_1, V_2, V_3, where $V_i = \{u_j | j \equiv i \pmod 3\}$. By Lemma 3.2 it follows that $\gamma(P_t \Box H) \geq t$. Therefore, $\gamma(P_t \Box H) = t$ since the set $V(P_t) \times \{x\}$ dominates $P_t \Box H$.

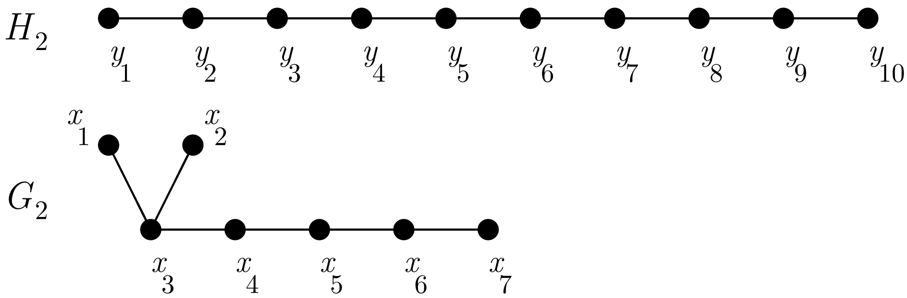

Figure 3. Example showing different labelings

The following result will allow us to produce similar lower bounds for the Cartesian product of a graph H which has 2-packings whose union is independent and a tree having small enough maximum degree.

Lemma 3.3. *Let T be a tree with maximum degree n. The vertex set of T can be partitioned into $n + 1$ sets each of which is a 2-packing.*

Proof. Let $x \in V(T)$ be a vertex of degree n. Root the tree at x and consider its n neighbors $u_1, u_2, \ldots, u_n$. Assign label $n + 1$ to x and i to u_i for $1 \leq i \leq n$. Vertex u_1 has at most $n - 1$ children, so they can be assigned labels from the set $\{2, 3, \ldots, n\}$. Since the children of u_1 are at a distance of three from each of $u_2, u_3, \ldots, u_n$, the subsets of the partial partition of $V(T)$ induced by the labeled vertices are 2-packings. This process can be continued until all vertices of T are labeled. ■

It is clear that if the tree has maximum degree less than n it is still possible, if T has order at least $n + 1$, to label as in the above lemma so that $V(T)$ is partitioned into $n + 1$ sets which are 2-packings. The proof of the next theorem now follows from a direct application of Lemmas 3.2 and 3.3.

Theorem 3.4. *Let H be a graph which has an independent set A which is a union of k pairwise disjoint 2-packings $A_1, A_2, \ldots, A_k$. Let T be a tree of maximum degree at most $k - 1$. Then $\gamma(T \Box H) \geq |T| \min\limits_{1 \leq i \leq k} |A_i|$.*

4. Lower Bound for $\gamma(T \square T)$

As indicated earlier, if T is a tree and H is any graph, then it follows from the result of Barcalkin and German [1] that $\gamma(T \square H) \geq \gamma(T)\gamma(H)$. In [9] Fink, et al, proved that if both G and H have the property that each vertex of degree greater than one has exactly one neighbor of degree one (they call such graphs *generalized combs*), then $\gamma(G \square H) = \gamma(G)\gamma(H)$. Jacobson and Kinch prove in [11] that if T_1 and T_2 are both trees such that $\gamma(T_1 \square T_2) = \gamma(T_1)\gamma(T_2)$, then at least one of them must be a generalized comb. Note that for a tree T the quantity $|T| - 2\gamma(T)$ is strictly positive unless T is a generalized comb, in which case it is zero.

In Corollary 2.2 of [7] Hartnell and Rall show that if T is a tree in which each vertex of degree greater than one has *at least* one neighbor of degree one, then $\gamma(T \square H) \geq \gamma(T)\gamma(H) + (|T| - 2\gamma(T))$ for every graph H of sufficiently large order. We now establish a lower bound which is an improvement over the conjectured lower bound of Vizing for the Cartesian product of any tree with itself.

Theorem 4.1. *If T is any tree, then $\gamma(T \square T) \geq \gamma(T)\gamma(T) + (|T| - 2\gamma(T))$.*

Before giving the proof of Theorem 4.1 consider the following situation which suggests why it might be true. Assume that G is a graph and $A = \{v_1, v_2, \ldots, v_t\}$ is a 2-packing in G. Let R be the vertices that remain when the t closed neighborhoods are removed from G. That is, $R = G - \cup_{i=1}^{t} N[v_i]$. Assume that D is a subset of $V(G \square G)$ which dominates $(A \times A) \cup (R \times R)$. Since $A \times A$ is a 2-packing of $G \square G$, D must contain at least t^2 vertices from $\cup_{i=1}^{t} N[v_i] \times \cup_{i=1}^{t} N[v_i]$, and none of these vertices is adjacent to any vertex of $R \times R$. It is straightforward to see that the set D must then contain at least $|R|$ vertices from $(V(G) - A) \times R$ to dominate $R \times R$. Therefore, $\gamma(G \square G) \geq t^2 + |R|$.

When G is a tree and has a maximum 2-packing, necessarily of order $\gamma(G)$, consisting entirely of leaves, the above bound coincides with that of the theorem.

***Proof of Theorem* 4.1.** Let T be a tree of order n and having domination number and 2-packing number $\gamma(T) = k = \rho(T)$. Choose any maximum 2-packing B of T and color its vertices black. Let $B = \{b_1, b_2, \ldots, b_k\}$. If a black vertex is a leaf, color its only neighbor yellow; otherwise color its neighbors pink. Let Y be the set of yellow vertices, and let P be the set of pink vertices. Color the remaining vertices, if any, red. So $R = V(T)-(B\cup Y\cup P)$ is the set of red vertices. Let D be any subset of $T \,\square\, T$ having the following properties:

1. D dominates all of $B \times B$;
2. For each black vertex b_i that has pink neighbors, say $P_i = P \cap N(b_i) = \{p_1, p_2, \ldots, p_n\}$, D dominates $Q_i = \{(p_1, p_2), (p_2, p_3), \ldots, (p_{n-1}, p_n), (p_n, p_1)\}$;
3. For each connected component C of the subgraph $\langle R\rangle$ induced by R, D dominates all vertices of $C \times C$.

Because the set of black vertices is a 2-packing of T, D must contain $|B|^2 = [\gamma(T)]^2$ vertices to dominate $B \times B$. Assume $b_i \in B$ has a nonempty set $P_i = \{p_1, p_2, \ldots, p_n\}$ of pink neighbors. If $(u, v) \in D$ dominates vertices in both $B \times B$ and $P_i \times P_i$, then either $u = b_i$ or $v = b_i$. That is, such a (u, v) can dominate (b_i, b_i) but no other vertex of $B \times B$. Also, Q_i is a 2-packing (it is actually a 3-packing) in $T \,\square\, T$, and so $Q_i \cup \{(b_i, b_i)\})$ can be dominated by no fewer than $|Q_i| = |P_i|$ members of D.

Consider now a component C of $\langle R\rangle$. Note that the distance in T from any red vertex to a black vertex is at least two, so no vertex of D can simultaneously dominate a vertex of $C \times C$ and a vertex of $B \times B$. If $D \cap (C \times C)$ dominates $C \times C$, then it follows from the fourth inequality of Theorem 1.1 that $|D \cap (C \times C)| \geq |C|$. Note also that no vertex of $C \times C$ dominates a vertex in $P_i \times P_i$, for any i. If a vertex $(r, s) \in C \times C$ is not dominated by $D \cap (C \times C)$, then there must exist in D a vertex d of the form (x, s) or (r, x) where $x \in Y \cup P_j$, for some j. If $x \in P_j$, then d does dominate $(x, x) \in P_j \times P_j$ as well, but does not dominate any vertex of Q_j. If $x \in Y$, then d dominates exactly one required vertex, either (r, r) or (s, s). Therefore,

$$|D| \geq \gamma(T)\gamma(T) + (|T| - 2\gamma(T)). \qquad \blacksquare$$

Since it is not known if Vizing's conjectured bound holds for $G\square G$, a modified Theorem 4.1 with the graph not required to be a tree would provide more evidence in favor of the conjecture. However, such a statement is not true. Although it can easily be shown that $\gamma(G\square G) \geq \gamma(G)\gamma(G)+(|G|-2\gamma(G))$ if $\gamma(G) \leq 2$, the self-complementary graph $G = K_3 \,\square\, K_3$ shows the inequality does not hold in general.

5. Upper Bounds

Nearly all the published results on domination of Cartesian products have been motivated by Vizing's conjecture, and so authors have been interested in lower bounds for the domination number of a Cartesian product. Possible exceptions to this focus are attempts to find the domination number of grid graphs and hypercubes. However, the fundamental challenge in domination theory is to find **small** dominating sets and so it seems natural to establish upper as well as lower bounds.

Consider the upper bound $\gamma(G \Box H) \leq \min\{\gamma(G)|H|, \gamma(H)|G|\}$ of Vizing given in Theorem 1.1. This bound is valid for any pair of graphs G and H and is verified by observing that if D is any dominating set of G, then $D \times V(H)$ dominates $G \Box H$. A similar dominating set for $G \Box H$ can be obtained by interchanging the roles of G and H. Vizing's result then follows.

There are several ways to generalize this upper bound. In what follows we only consider one of two symmetric cases. Instead of using a copy of a minimum dominating set D of G inside G_u, for each vertex u of H, we note that it may be possible to build a smaller dominating set for the Cartesian product if H has large enough maximum degree. Let $x \in V(H)$ be a vertex of degree $\Delta(H)$, and let D be a minimum dominating set of G. The set $(V(G) \times \{x\}) \cup (D \times (V(H) - N[x])$ dominates $G \Box H$. This proves the following theorem.

Theorem 5.1. *For any two graphs G and H,*

$$\begin{aligned}\gamma(G \Box H) \leq\ & \min\{\gamma(G)|H| - (\gamma(G)(\Delta(H)+1) - |G|),\\ & \gamma(H)|G| - (\gamma(H)(\Delta(G)+1) - |H|)\}.\end{aligned}$$

Of course, if H has several vertices of large degree whose neighborhoods are disjoint, or nearly so, then it is possible to modify the above idea to get other upper bounds. The statements of these become too unwieldy to include.

Another way to generalize Vizing's upper bound of $\gamma(G)|H|$ is to build a dominating set for the product graph that uses the domination properties of both graphs. To state this precisely requires several additional definitions. A collection of subsets $\{A_1, A_2, \ldots, A_k\}$ of $V(H)$ is called a *dominating k-cover* of H if each A_i is a dominating set of H and $V(H) = \cup_{i=1}^{k} A_i$. This is a generalization of a *domatic k-partition* of H, in which the subsets are also pairwise disjoint. Note that H has a dominating k-cover for all $k \geq 1$ since each A_i can be taken to be $V(H)$. However, the upper bound given in the next theorem will, in general, be smaller when few vertices are repeated in the cover. A dominating set D of G is called a k *type dominating set* if there is a partition $\{D_1, D_2, \ldots, D_k\}$ of D, called a k *type dominating partition*, such that for every $x \in V(G) - D$ and every $1 \leq i \leq k$, the vertex x has a neighbor in D_i.

The proof of the following theorem follows immediately from the definitions and is omitted. The original upper bound of Vizing is obtained by taking $k = 1$. This will always be possible since $V(H)$ is a dominating set of H, and every dominating set of the graph G is itself a 1 type dominating partition.

Theorem 5.2. *Let k be a positive integer. Assume G is a graph with a k type dominating set and H is any graph. Then*

$$\gamma(G \,\Box\, H) \leq \min \sum_{i=1}^{k} |D_i||A_i|,$$

where the minimum is taken over all dominating k-covers $\{A_1, A_2, \ldots, A_k\}$ of H and all k type dominating partitions $\{D_1, D_2, \ldots, D_k\}$ of a dominating set of G.

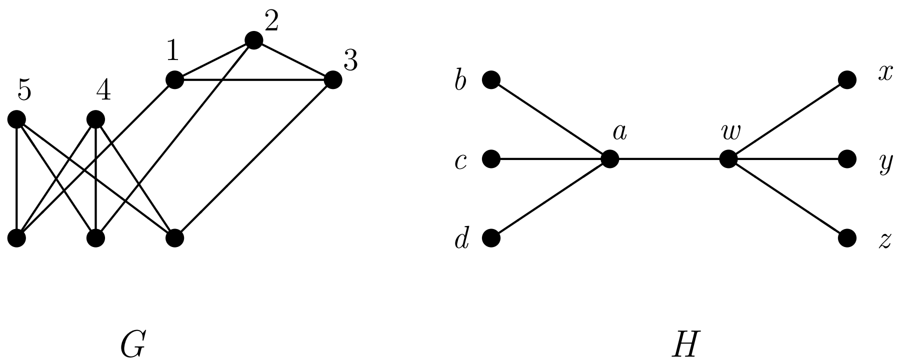

Figure 4. Illustration of the upper bound

As an example of how to apply Theorem 5.2 consider the graphs G and H in Figure 4. Vizing's upper bound for $\gamma(G \,\Box\, H)$ from Theorem 1.1 is 16 and is achieved using the dominating set $V(G) \times \{a, w\}$. Theorem 5.1 does not give any improvement in this case. However, if $D_1 = \{1, 2, 3\}$, $D_2 = \{4\}$ and $D_3 = \{5\}$, then $\{D_1, D_2, D_3\}$ is a 3 type dominating set of G. Let $A_1 = \{a, w\}$, $A_2 = \{a, x, y, z\}$ and $A_3 = \{w, b, c, d\}$. The collection $\{A_1, A_2, A_3\}$ is a dominating 3-cover of H, and so by Theorem 5.2, $\gamma(G \,\square\, H) \leq \sum |D_i||A_i| = 14$.

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