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HAJÓS' THEOREM FOR LIST COLORINGS OF HYPERGRAPHS*

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Abstract

A well-known theorem of Hajós claims that every graph with chromathic number greater than k can be constructed from disjoint copies of the complete graph K_{k+1} by repeated application of three simple operations. This classical result has been extended in 1978 to colorings of hypergraphs by C. Benzaken and in 1996 to list-colorings of graphs by S. Gravier. In this note, we capture both variations to extend Hajós' theorem to list-colorings of hypergraphs.

Keywords: list-coloring, Hajós' construction, hypergraph.

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1. Introduction

In 1961, Hajós [5] gave a construction of the graphs that are not k-colorable. The construction uses the following simple operations:

(1) Add a new vertex or edge.

(2) Let G_1, G_2 be two vertex-disjoint graphs, and a_1b_1 and a_2b_2 be edges in G_1 and G_2, respectively. Make a graph G from $G_1 \cup G_2$ by deleting the edge a_ib_i from G_i (for $i = 1, 2$), identifying a_1 and a_2 (the resulting vertex is called a_1a_2), and adding a new edge b_1b_2 (see Figure 1).

(3) Identify two non-adjacent vertices.

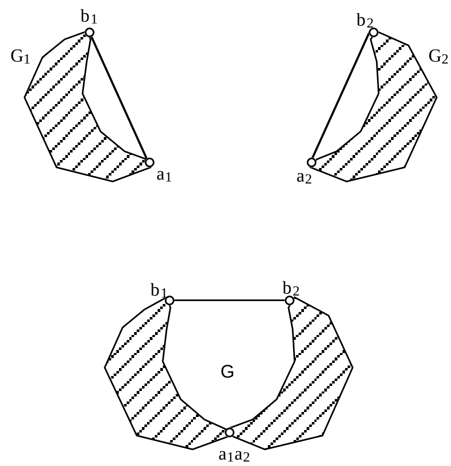

Figure 1. Operation (2)

Theorem 1.1 (Hajós)**.** *Every non-k-colorable graph can be constructed by operations* (1) − (3) *from disjoint copies of the complete graph* K_{k+1}.

This classical result has been extended to colorings of hypergraphs by Benzaken [1, 2] and to list-colorings of graphs by Gravier [4]. In this note we capture both variations to extend Hajós' theorem to list-colorings of hypergraphs. However, Zhu [8] gave an analogue of Hajós' theorem for the circular chromathic number. Recently, the classical result was extended by Mohar [6] in three slightly different ways to colorings and circular colorings of edge-weighted graphs (enhancing the channel assignment problem as well). Moreover, it is mentioned in [6] that one of these extensions sheds some new light on the fact that today no nontrivial application of Hajós' theorem is known.

2. Hajós' Theorem for List Colorings of Hypergraphs

In a hypergraph $\mathcal{H}$, the set of vertices and the set of hyperedges are denoted by $V(\mathcal{H})$ and $E(\mathcal{H})$, respectively. Given a hypergraph $\mathcal{H}$, a *k-coloring* of the vertices of $\mathcal{H}$ is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that for every hyperedge e of $\mathcal{H}$ there exist two vertices $x, y \in e$ with $c(x) \neq c(y)$, or shortly $|c(e)| \geq 2$. A hypergraph $\mathcal{H}$ is *k-colorable* if it admits a k-coloring, and the *chromatic number* of $\mathcal{H}$ is the smallest integer k such that $\mathcal{H}$ is k-colorable.

Vizing [7] and independently Erdős, Rubin, and Taylor [3] introduced the concept of list colorings. This concept can be naturally extended to hypergraphs in the following way. Suppose that each vertex v is assigned a list $L(v)$ of possible colors; we then want to find a vertex-coloring c such that $c(v) \in L(v)$ for all $v \in V(\mathcal{H})$. In the case where such a coloring exists we will say that the hypergraph $\mathcal{H}$ is *L-colorable*; we may also say that c is an *L-coloring* of $\mathcal{H}$. Given an integer k, the hypergraph $\mathcal{H}$ is called *k-choosable* if it is L-colorable for every assignment L that satisfies $|L(v)| \geq k$ for all $v \in V(\mathcal{H})$. Finally, the *choice number* or *list-chromatic number* $\chi_l(\mathcal{H})$ of $\mathcal{H}$ is the smallest k such that $\mathcal{H}$ is k-choosable.

Concerning the problem of coloring the hypergraphs, without loss of generality, we can restrict ourselves to hypergraphs with the Sperner property, i.e., no hyperedge contains (as a subset) another hyperedge in a hypergraph.

Indeed, if we have a coloring c of a hypergraph $\mathcal{H}$ and e, f are hyperedges of $\mathcal{H}$ with $e \subseteq f$, then condition $|c(e)| \geq 2$ implies that $|c(f)| \geq 2$. In all of our constructions given below, by deleting the superfluous hyperedges of the newly constructed hypergraph, we may assume that it has the Sperner property.

In order to obtain Hajós' theorem for list colorings of hypergraphs, we will use the following operations:

(H1) Add a new hyperedge (possibly with new vertices) or a new isolated vertex in a hypergraph $\mathcal{H}$. The new hypergraph obtained by adding a new hyperedge e is denoted by $\mathcal{H} \vee e$.

(H2) Let $\mathcal{H}_1, \mathcal{H}_2$ be two vertex-disjoint hypergraphs, and e_1 and e_2 be hyperedges in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Also, let $a_1 \in e_1$ and $a_2 \in e_2$. Make a new hypergraph $\mathcal{H}$ from $\mathcal{H}_1 \cup \mathcal{H}_2$ by deleting the edge e_i from $\mathcal{H}_i$ (for $i = 1, 2$), identifying a_1 and a_2 (the resulting vertex is called a_1a_2), and adding a new hyperedge $e_1 \setminus \{a_1\} \cup e_2 \setminus \{a_2\} \cup \{a_1a_2\}$.

(H3) If $\mathcal{H}$ is not L-colorable for some assignment L with $|L(x)| \geq k$ for each $x \in V(\mathcal{H})$, then identify two vertices u and v of $\mathcal{H}$ with $L(u) = L(v)$ into a new vertex uv. After this, if there are two hyperedges e, e' of $\mathcal{H}$ with $e' \subseteq e$ then remove e.

Notice that if $\mathcal{H}_1$ and $\mathcal{H}_2$ have the Sperner property then the hypergraph obtained from $\mathcal{H}_1$ and $\mathcal{H}_2$ by operation (H2) also has the Sperner property. Regarding the operation (H3), every hyperedge e which contains u or v is replaced by $e \setminus \{u, v\} \cup \{uv\}$. Moreover remark that one could apply operation (H3) to two adjacent vertices u, v. The second step of (H3) guarantees to preserve Sperner Property.

Theorem 2.1. *A hypergraph $\mathcal{H}$ can be constructed by operations* (H1) – (H3) *from disjoint copies of any bipartite graph with choice number equal to* $k + 1$ *if and only if* $\chi_l(\mathcal{H}) \geq k + 1 \geq 2$.

Proof. Note that introducing a new vertex or a new hyperedge in a given hypergraph does not decrease the choice number. The same holds if we identify two (non-)adjacent vertices under the assumption of operation (H3).

Next we will show that the class of non-k-choosable hypergraphs is closed under operation (H2). We use the same notation as in its description. For $i = 1, 2$, since $\mathcal{H}_i$ is not k-choosable, there exists an assignment L_i with $|L_i(v)| = k$ for all $v \in V(\mathcal{H}_i)$ and such that $\mathcal{H}_i$ is not L_i-colorable. We may assume that $L_1(a_1) = L_2(a_2)$ by a suitable permutation of the colors. Now, we create a list assignment L of $V(\mathcal{H})$ by setting $L(v) = L_i(v)$ for $v \in V(\mathcal{H}_i)$. We claim that $\mathcal{H}$ is not L-colorable. Indeed, suppose that there is an L-coloring c of $\mathcal{H}$. Then, $|c(e_1 \cup e_2)| \geq 2$. Moreover $c(a_1a_2) = c(a_1) = c(a_2)$, which implies that either $|c(e_1)| \geq 2$ or $|c(e_2)| \geq 2$. Therefore, c is either an L_1-coloring of $\mathcal{H}_1$ or an L_2-coloring of $\mathcal{H}_2$, a contradiction. Since $|L(v)| = k$ for all $v \in V(\mathcal{H})$, this shows that $\mathcal{H}$ is not k-choosable.

Thus, the only if part of the theorem is established. To prove the if part, we will prove first that every non-k-choosable hypergraph can be obtained by (H1) – (H3) starting with (hyper)graphs from the family of complete multipartite graphs with choice number $k + 1$.

So, assume that this is false and that there exists a counterexample. By operation (H1), we may assume that there is such a counterexample $\mathcal{H}$ having Sperner Property. Then, there exists an assignment L with $|L(v)| = k$ for all $v \in V(\mathcal{H})$ such that $\mathcal{H}$ is not L-colorable.

If $\chi_l(\mathcal{H}) = \infty$, then it contains a hyperedge with precisely one vertex. In that case, starting with K_{k+1} use (H3) to construct a hypergraph with a single vertex and a single hyperedge, and afterwards use (H1) to obtain $\mathcal{H}$. Now, we may assume that the choice number of $\mathcal{H}$ is finite.

Define a relation $\preceq$ on the hypergraphs whose set of vertices is $V(\mathcal{H})$, in the following way:

$$\mathcal{H}_a \preceq \mathcal{H}_b \quad \text{if and only if} \quad \forall e_a \in E(\mathcal{H}_a)\ \exists e_b \in E(\mathcal{H}_b) \quad \text{such that} \quad e_b \subseteq e_a.$$

Obviously, $\preceq$ is a transitive and reflexive relation. By the Sperner property, it follows that this relation is also antisymmetric. So, it is a partial ordering. We say that $\mathcal{H}_b$ is *greater* than $\mathcal{H}_a$ (with respect to the relation $\preceq$). Note that if $\mathcal{H}_a$ is non-k-choosable, then $\mathcal{H}_b$ is also non-k-choosable.

According to the partial order $\preceq$, we may assume that $\mathcal{H}$ is as great as possible hypergraph regarding $\preceq$ (which is still not constructible). Thus, every greater hypergraph than $\mathcal{H}$ is constructible.

In what follows, we will prove that for any independent sets I_1, I_2 of $\mathcal{H}$ with non-empty intersection, the set $I_1 \cup I_2$ is also independent in $\mathcal{H}$. (Recall that a set is independent if it contains no hyperedge as a subset.) Consider the hypergraphs $\mathcal{H} \vee I_1$ and $\mathcal{H} \vee I_2$. Since I_1, I_2 are independent, we infer that $\mathcal{H} \preceq \mathcal{H} \vee I_i$ and $\mathcal{H} \neq \mathcal{H} \vee I_i$ for $i = 1, 2$. So, it follows that these two hypergraphs can be constructed from complete multipartite graphs by operations (H1) – (H3).

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two vertex-disjoint copies of $\mathcal{H} \vee I_1$ and $\mathcal{H} \vee I_2$, respectively. For every vertex x from $\mathcal{H}$, we denote by x_1 and x_2 its counterparts in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
Let $a \in I_1 \cap I_2$. Now, using the same notation as in (H2) with I_1, I_2 playing the roles of e_1, e_2, and a playing the role of a_1 in $\mathcal{H}_1$ and a_2 in $\mathcal{H}_2$, we construct a new hypergraph $\mathcal{H}^*$. Define an assignment L^* on $\mathcal{H}^*$ by setting $L^*(v_i) = L(v)$ for each $v \in V(\mathcal{H})$ and each $i = 1, 2$. Observe that $\mathcal{H}^*$ is not L^*-colorable. Finally, using the operation (H3), identify vertices x_1, x_2 from $\mathcal{H}^*$ for each vertex x of $\mathcal{H}$. Since (H3) preserves the Sperner property, we have that the obtained hypergraph is isomorphic to $\mathcal{H}$ if and only if the set $I_1 \cup I_2$ is not independent. Therefore, if $I_1 \cup I_2$ is not an independent set, we obtain a construction of $\mathcal{H}$, which is a contradiction. So, the property for independent sets is established.

From this property, it easily follows that the relation $\sim$ on vertices of $\mathcal{H}$ defined as

$$a \sim b \quad \text{if and only if} \quad \{a\} \cup \{b\} \text{ is independent set,}$$

is an equivalence relation. In particular this means that $\mathcal{H}$ is a complete multipartite graph, which is a contradiction.

In [4], it was proven that using only rules (H1) and (H3) applied on graphs, from any complete bipartite graph with choice number $k+1$, we can construct every non-k-choosable multipartite graph. To achieve the proof of the theorem, it is sufficient to observe that similarly using only rules (H1) and (H3), from any bipartite graph with choice number $k+1$, we can construct every non-k-choosable bipartite complete graph. ■

Theorem 2.1 shows that, for a fixed k, any minimal graph (for the subgraph relation) in the class of non-k-choosable bipartite graphs forms a basis for the non-k-choosability of hypergraphs.

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