Discussiones Mathematicae
Graph Theory 23 (2003) 159–162

PERFECT CONNECTED-DOMINANT GRAPHS

Igor Edmundovich Zverovich

RUTCOR, Rutgers University
640 Bartholomew Rd., Piscataway, NJ 08854 USA

e-mail: igor@rutgers.rutcor.edu

Abstract

If D is a dominating set and the induced subgraph $G(D)$ is connected, then D is a *connected* dominating set. The minimum size of a connected dominating set in G is called *connected domination number* $\gamma_c(G)$ of G. A graph G is called a *perfect connected-dominant* graph if $\gamma(H) = \gamma_c(H)$ for each connected induced subgraph H of G.

We prove that a graph is a perfect connected-dominant graph if and only if it contains no induced path P_5 and induced cycle C_5.

Keywords: Connected domination, perfect connected-dominant graph.

2000 Mathematics Subject Classification: 05C69.

All graphs will be finite and undirected, without loops or multiple edges. Let $G = (V, E)$ be a graph. As usual, $N(u)$ denotes the neighborhood of a vertex $u \in V$; $N[u] = \{u\} \cup N(u)$. For a set $D \subseteq V$ we put $N[D] = \bigcup_{u \in D} N[u]$. We say that a set D *dominates* a set X if $X \subseteq N[D]$. If D dominates V then D is a dominating set of G. A *minimum* dominating set of G has the minimum cardinality among all dominating sets of G. The *domination number* $\gamma(G)$ of G is the cardinality of a minimum dominating set of G.

The subgraph of G induced by a set $X \subseteq V(G)$ is denoted by $G(X)$. If D is a dominating set and $G(D)$ is a connected subgraph, then D is called a *connected* dominating set. Accordingly, the minimum size of a connected

---

Supported by the Office of Naval Research (Grant N0001492F1375), NSF (Grant DMS-9806389), INTAS and the Belarus Government (Project INTAS-BELARUS 97-0093).

dominating set in G is called *connected domination number* $\gamma_c(G)$ of G. Clearly,

$$\gamma(G) \leq \gamma_c(G)$$

for any connected graph G.

Definition 1. A graph G is called a *perfect connected-dominant* graph if $\gamma(H) = \gamma_c(H)$ for each connected induced subgraph H of G.

Theorem 1. *A graph G is a perfect connected-dominant graph if and only if G contains no induced path P_5 and induced cycle C_5.*

Proof. Necessity is clear, since both P_5 and C_5 are connected, $\gamma(P_5) = \gamma(C_5) = 2$ and $\gamma_c(P_5) = \gamma_c(C_5) = 3$.

Sufficiency. Suppose that the statement is not true and let G be a minimal counterexample, i.e., G is a connected graph without induced P_5 and C_5, but $\gamma(G) < \gamma_c(G)$.

We choose a minimum dominating set D of G such that $H = G(D)$ has the minimal number of connected components among all minimum dominating sets of G. Since $\gamma(G) < \gamma_c(G)$, H is a disconnected subgraph. Let us fix two connected components K and L of H.

By connectivity of G, there is a shortest path $P = (u_1, u_2, \ldots, u_t)$ such that $u_1 \in K$ and $u_t \in L$. ∎

Claim 1. $t = 3$.

Proof. Clearly, $t \geq 3$. Since P_5 is not an induced subgraph of G, $t \leq 4$. Thus, $t \in \{3, 4\}$.

Suppose that $t = 4$. First we show that

$$D' = (D \backslash \{u_1, u_4\}) \cup \{u_2, u_3\}$$

is a dominating set of G. If it is not so, then there is a vertex v such that D' does not dominate v. But D is a dominating set of G. Hence v is adjacent to at least one of u_1, u_4 (since $D \backslash D' = \{u_1, u_4\}$). Then $\{u_1, u_2, u_3, u_4, v\}$ induces either P_5 or C_5, a contradiction.

Thus, D' is a minimum dominating set of G. By the choice of D, the number of components in $G(D')$ is not less than the number of components in $G(D)$. It follows that the set $(K \backslash \{u_1\}) \cup (L \backslash \{u_4\}) \cup \{u_2, u_3\}$ induces a subgraph F with at least two components. Let M be a component of F

which does not contain u_2 and u_3. We may assume that $M \subseteq K$. By connectivity of K, there is a vertex $w \in M$ such that u_1 and w are adjacent.

Then $\{w, u_1, u_2, u_3, u_4\}$ induces P_5, a contradiction. ■

Let us denote $D_i = (D \backslash \{u_i\}) \cup \{u_2\}$, $i \in \{1, 3\}$.

Claim 2. At least one of D_1, D_3 is a dominating set of G.

Proof. Suppose that both D_1 and D_3 are not dominating sets of G. Then there are vertices v_i $(i \in \{1, 3\})$ such that D_i does not dominate v_i. Since D_i is a dominating set, v_i is adjacent to u_i, $i \in \{1, 3\}$. We obtain that $\{v_1, u_1, u_2, u_3, v_3\}$ induces either P_5 or C_5, a contradiction. ■

By Claim 2 and using symmetry, we may assume that D_1 is a dominating set of G. Since $|D_1| = |D|$, D_1 is a minimum dominating set of G. By the choice of D, there is a component $N \subseteq K$ of $G(D_1)$. By connectivity of K, there is a vertex $w \in N$ which is adjacent to u_1.

Claim 3. The set $D' = (D_1 \backslash \{w\}) \cup \{u_1\}$ is a minimum dominating set of G.

Proof. If it is not true, there is a vertex y which is not dominated by D'. Clearly, y is adjacent to w. Then $\{y, w, u_1, u_2, u_3\}$ induces P_5, a contradiction. ■

Claim 4. $G(D')$ has less components than $G(D)$.

Proof. Otherwise $G(D')$ contains a component $P \subseteq K$ such that $u_1 \notin P$. By connectivity of K, there is a vertex $z \in P$ which is adjacent to w. Then $\{z, w, u_1, u_2, u_3\}$ induces P_5, a contradiction. ■

Claim 3 and Claim 4 produce the final contradiction. ■

References

[1] S. Arumugam and J.J. Paulraj, *On graphs with equal domination and connected domination numbers*, Discrete Math. **206** (1999) 45–49.

[2] K. Arvind and R.C. Pandu, *Connected domination and Steiner set on weighted permutation graphs*, Inform. Process. Lett. **41** (1992) 215–220.

[3] H. Balakrishnan, A. Rajaram and R.C. Pandu, *Connected domination and Steiner set on asteroidal triple-free graphs*, Lecture Notes Math. **709** (1993) 131–141.

[4] C. Bo and B. Liu, *Some inequalities about connected domination number*, Discrete Math. **159** (1996) 241–245.

[5] J.E. Dunbar, J.W. Grossman, J.H. Hattingh, S.T. Hedetniemi and A.A. McRae, *On weakly connected domination in graphs*, Discrete Math. **167/168** (1997) 261–269.

[6] D.V. Korobitsyn, *On the complexity of determining the domination number in monogenic classes of graphs*, Discrete Math. **2** (1990) 90–96.

[7] J.J. Paulrau and S. Arumugam, *On connected cutfree domination in graphs*, Indian J. Pure Appl. Math. **23** (1992) 643–647.

[8] L. Sun, *Some results on connected domination of graphs*, Math. Appl. **5** (1992) 29–34.

[9] E.S. Wolk, *A note on 'The comparability graph of a tree'*, Proc. Amer. Math. Soc. **16** (1966) 17–20.

[10] E.S. Wolk, *The comparability graph of a tree*, Proc. Amer. Math. Soc. **13** (1962) 789–795.

Received 16 August 2001