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ON THE PACKING OF TWO COPIES OF A CATERPILLAR IN ITS THIRD POWER

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Abstract

H. Kheddouci, J.F. Saclé and M. Woźniak conjectured in 2000 that if a tree T is not a star, then there is an edge-disjoint placement of T into its third power.

In this paper, we prove the conjecture for caterpillars.

Keywords: packing, placement, permutation, power of tree, caterpillar.

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1. Introduction

Suppose $G_1, \ldots, G_k$ are graphs of order n. We say that there is a *packing* of $G_1, \ldots, G_k$ (into the complete graph K_n) if there exist injections $\alpha_i : V(G_i) \to V(K_n), i = 1, \ldots, k$, such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the map $\alpha_i^* : E(G_i) \to E(K_n)$ is the one induced by α_i.

A packing of k copies of a graph G will be called a k-*placement* of G. A 2-placement of G (in its complement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

The following theorem was proved, independently, in [1], [3] and [7].

Theorem 1. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 2$ then G is contained in its complement.* ■

This result has been improved in many ways. The main references of this paper and of other packing problems are to be found in the last chapter of Bollobás' book [1], the 4th Chapter of Yap's book [11] and the survey paper [10].

In this paper we shall consider the case in which G is a tree on n vertices. The example of the star S_n shows that Theorem 1 cannot be improved by raising the size of G even in the case when G is a tree. However, in that case, we have the following result:

Theorem 2. *Let T be a tree of order n, $T \neq S_n$. Then T is contained in its own complement.* ■

Theorem 2 was first proved by Straight (unpublished, cf. [4]). Besides, this result has been improved in many ways. For instance, the packing of two trees was considered in [4] and the 3-placement of a tree in [8].

Another possibility of improving Theorem 2 is to consider some additional information about embeddings *i.e.*, packing permutations.
An example of such a result is the following theorem contained as a lemma in [9] (cf. also [2]).

Theorem 3. *Let T be a non-star tree of order n, with $n > 3$. Then there exists a 2-placement σ of T such that for every $x \in V(T)$, $\mathrm{dist}_T(x, \sigma(x)) \leq 3$.* ■

This theorem immediately implies the following

Corollary 4. *Let T be a non-star tree of order n, with $n > 3$. Then there exists an embedding σ of T such that $\sigma(T) \subset T^7$.*

Since T^7 is, in general, a proper subgraph of K_n, the last corollary can be considered as an improvement of Theorem 2.

Kheddouci, Saclé and Woźniak considered the problem of the 2-placement of a tree T into T^p such that p is as small as possible. In [6], they proved the following result.

Theorem 5. *Let T be a non-star tree of order n, with $n > 3$. Then there exists a 2-placement σ of T such that $\sigma(T) \subset T^4$.*

And they posed the following conjecture.

Conjecture 6. *Let T be a non-star tree of order n, with $n > 3$. Then there exists a 2-placement σ of T such that $\sigma(T) \subset T^3$.*

Kheddouci in [5] studied the packing of some trees into their third power, namely a path of length at least 3, a star-path-star and any non-star tree without vertices of degree 2.

In this note, we prove the following result.

Theorem 7. *Any non-star caterpillar admits a 2-placement into its third power.*

We shall need some additional definitions and notations in order to formulate our results.

A tree is called a *caterpillar* if by removing all end-vertices it becomes a path. The path obtained by removing all end-vertices of a caterpillar is called the *main path* of a caterpillar. The end-vertices of a main path are called the *nodes*. In particular, if a caterpillar is a star then the main path is reduced to a single node. A caterpillar $\mathcal{C}$ is said *complete* if any vertex of its main path is adjacent to at least one end-vertex in $\mathcal{C}$. Let x be a vertex of a caterpillar $\mathcal{C}$. The components of $\mathcal{C} - x$ are called *neighbor caterpillars* of x. If y is any neighbor of x in $\mathcal{C}$, we denote by $\mathcal{C}_y$ the neighbor caterpillar of x which contains y. Consequently, if we delete an edge $e = xy$ of $\mathcal{C}$, we obtain two components of $\mathcal{C} - e$, respectively the neighbor caterpillar $\mathcal{C}_x$ of y and the neighbor tree $\mathcal{C}_y$ of x. Consider two distinct caterpillars $\mathcal{C}'$ and $\mathcal{C}''$, together with the path $(x_1, x_2, \ldots, x_p)$. We denote $\mathcal{C} = \mathcal{C}' \cdot (x_1, x_2, \ldots, x_p) \cdot \mathcal{C}''$ the caterpillar obtained by identifying x_1 with a node of the first caterpillar, and x_p with a node of the second caterpillar. By doing this, the first caterpillar becomes the neighbor caterpillar $\mathcal{C}_{x_1}$ of x_2, and the second caterpillar becomes the neighbor caterpillar $\mathcal{C}_{x_p}$ of x_{p-1}. A special case of this construction is examplified by the star-path-star $S' \cdot (x_1, \ldots, x_p) \cdot S''$, where S' and S'' are two stars with centers respectively x_1 and x_p.

For a 2-placement σ of a graph G and for each edge e of G at least one end-vertex of e is not fixed. Note that in the packing of some trees into their third power some vertices must be fixed for any permutation. It is easy to verify, for instance, that for the path $P_7 = (x_1, x_2, \ldots, x_7)$, any permutation of packing of P_7 into P_7^3 must keep x_4 as fixed vertex.

Here we define a permutation which uses fixed vertices in the packing of a caterpillar. Let $\mathcal{C}$ be a non-star caterpillar and L be its main path. Let e be any node of L. Let y be the neighbor of e on L and x be the neighbor of y

on L, $x \neq e$. Let f be an end-vertex neighbor of e in $\mathcal{C}$. A permutation σ on $V(\mathcal{C})$ is said to be $(\mathcal{C}, e)$-*good* iff the following three conditions are satisfied:

1. σ is a 2-placement of $\mathcal{C}$,
2. $\sigma(\mathcal{C}) \subset \mathcal{C}^3$,
3. $\mathrm{dist}_{\mathcal{C}}(e, \sigma(e)) = 1$ and $\mathrm{dist}_{\mathcal{C}}(f, \sigma(f)) \in \{0, 2\}$, or σ on the vertices x, y, e and f is given by the cyclic permutation (x, y, f, e).

The caterpillar itself is said to be e-*good* if there exists a $(\mathcal{C}, e)$-good permutation. By using this terminology we shall prove the following version of Theorem 7.

Theorem 8. *Let $\mathcal{C}$ be a non-star caterpillar. Let $(e_1, e_2, \ldots, e_n)$ be a main path of $\mathcal{C}$, with $n \geq 3$. Then $\mathcal{C}$ is e_n-good.*

The proof of Theorem 8 is divided into two parts. We start with a sequence of lemmas (Section 2) that we use in the main part of the proof given in Section 3.

2. Lemmas

Let us recall the following lemma on a 2-placement of a path given in [5].

Lemma 9. *Let $P = (x_1, x_2, \ldots, x_n)$ be a path with $n \geq 4$. There exists a 2-placement σ of P into P^3 such that* $\mathrm{dist}(x_1, \sigma(x_1)) = 1$ *and* $\mathrm{dist}(x_n, \sigma(x_n)) \in \{0, 1\}$.

Proof. The proof is by induction on n. For $n = 4, 5, 6$ and 7, one can see that the result holds (see Lemma 9 in [5]).

Suppose that the result holds for all $n' < n$ and $n \geq 8$. Let $x_j x_{j+1}$ be an edge of P such that $4 \leq j \leq n - 4$. The neighbor paths P_{x_j} and $P_{x_{j+1}}$ of $P - x_j x_{j+1}$ are respectively of order j and $n - j$. By the induction hypothesis, There exists a 2- placement $\sigma_{P_{x_j}}$ of P_{x_j} into $P^3_{x_j}$ such that $\mathrm{dist}(x_1, \sigma_{P_{x_j}}(x_1)) = 1$ and $\mathrm{dist}(x_j, \sigma_{P_{x_j}}(x_j)) \in \{0, 1\}$, and there exists a 2-placement $\sigma_{P_{x_{j+1}}}$ of $P_{x_{j+1}}$ into $P^3_{x_{j+1}}$ such that $\mathrm{dist}(x_{j+1}, \sigma_{P_{x_{j+1}}}(x_{j+1})) = 1$ and $\mathrm{dist}(x_n, \sigma_{P_{x_{j+1}}}(x_n)) \in \{0, 1\}$. Then the composition of these two permutations (not commutative) gives a permutation $\sigma = \sigma_{P_{x_j}} \circ \sigma_{P_{x_{j+1}}}$ on P such that $\mathrm{dist}_P(x_1, \sigma(x_1)) = 1$, $\mathrm{dist}_P(\sigma(x_j), \sigma(x_{j+1})) \leq 3$ and $\mathrm{dist}_P(x_n, \sigma(x_n)) \in \{0, 1\}$. ■

From now, the whitened vertices (if they exist) in each figure are fixed (for instance, see Figure 1). In the following lemma we give a 2-placement of a non-star complete caterpillar:

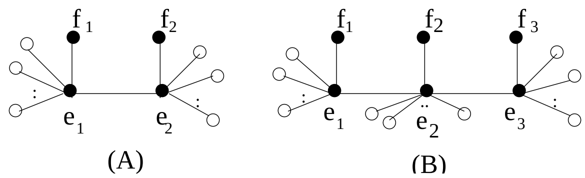

Figure 1. σ_K on darkened vertices is (A) (e_2, f_2, e_1, f_1). (B) $(e_3, f_3, e_2, f_2, e_1, f_1)$.

Lemma 10. *Let K be a non-star complete caterpillar. Let e_1 and e_n be the nodes of the main path of K and f_1 be an end-vertex neighbor of e_1 in K. Then there exists a (K, e_n)-good permutation σ_K with* $\text{dist}(e_1, \sigma_K(e_1)) = \text{dist}(e_n, \sigma_K(e_n)) = 1$ *and* $\text{dist}(f_1, \sigma_K(f_1)) \in \{2, 3\}$.

Proof. Let $(e_1, e_2, \ldots, e_n)$ be the main path of K. For $1 \leq i \leq n$, let f_i be an end-vertex neighbor of e_i. The proof is by induction on n, with $n \geq 2$. If $n = 2$ or 3, the lemma holds (see Figure 1). Suppose that $n \geq 4$ and the lemma holds for all $n' < n$. By removing the edge $e_j e_{j+1}$, with $2 \leq j \leq n-2$, we obtain two non-star complete sub-caterpillars K_{e_j} and $K_{\ e_{j+1}}$. By the induction hypothesis, there exists a (K_{e_j}, e_j)-good permutation $\sigma_{K_{e_j}}$ with $\text{dist}_{K_{e_j}}(e_1, \sigma_{K_{e_j}}(e_1)) = \text{dist}_{K_{e_j}}(e_j, \sigma_{K_{e_j}}(e_j)) = 1$ and $\text{dist}_{K_{e_j}}(f_1, \sigma_{K_{e_j}}(f_1)) \in \{2, 3\}$. Moreover, there exists a $(K_{e_{j+1}}, e_n)$-good permutation $\sigma_{K_{e_{j+1}}}$ with $\text{dist}_{K_{e_{j+1}}}(e_{j+1}, \sigma_{K_{e_{j+1}}}(e_{j+1})) = \text{dist}_{K_{e_{j+1}}}(e_n, \sigma_{K_{e_{j+1}}}(e_n)) = 1$. So the composition of these two permutations on K gives a (K, e_n)-good permutation σ_K such that $\text{dist}_K(\sigma_{K_{e_j}}(e_j), \sigma_{K_{e_{j+1}}}(e_{j+1})) = 3$, $\text{dist}_K(e_1, \sigma_K(e_1)) = \text{dist}_K(e_n, \sigma_K(e_n)) = 1$ and $\text{dist}_K(f_1, \sigma_K(f_1)) \in \{2, 3\}$. ■

Lemma 11. *Let $\mathcal{C}'$ be an e_1-good caterpillar with a node e_1, $P = (e_1, e_2, \ldots, e_q)$ a path with $q \geq 3$ and let K be a non-star complete caterpillar with nodes e_q and e. Then there exits a $(\mathcal{C}' \cdot P \cdot K, e)$-good permutation.*

Proof. Let $\mathcal{C} = \mathcal{C}' \cdot P \cdot K$. Let $\sigma_{\mathcal{C}}$ be a $(\mathcal{C}', e_1)$-good permutation. By Lemma 10, there exists a (K, e)-good permutation σ_K with $\text{dist}_K(e_q, \sigma(e_q)) = 1$. We construct a $(\mathcal{C}, e)$-good permutation $\sigma_{\mathcal{C}}$. We shall consider two cases depending on $\sigma_{\mathcal{C}'}$. For the particular cases of Figure 2, if nothing is said about a vertex v of $\mathcal{C}'$ (resp. K), then $\sigma_{\mathcal{C}}(v) = \sigma_{\mathcal{C}'}(v)$ (resp. $\sigma_{\mathcal{C}}(v) = \sigma_K(v)$).

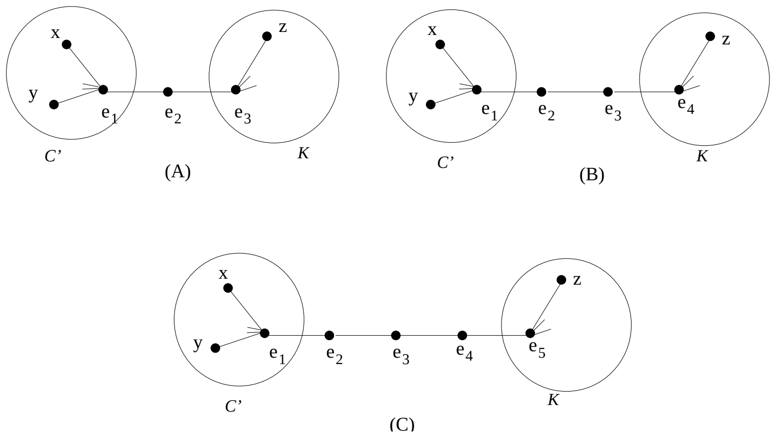

Figure 2. (A) e_2 is fixed by $\sigma_{\mathcal{C}}$, (B) $\sigma_{\mathcal{C}}(e_2) = y$, $\sigma_{\mathcal{C}}(f_1) = e_2$ and e_3 is fixed, (C) $\sigma_{\mathcal{C}}(e_2) = y$, $\sigma_{\mathcal{C}}(f_1) = e_3$, $\sigma_{\mathcal{C}}(e_3) = e_2$.

Case 1. $\text{dist}_{\mathcal{C}'}(e_1, \sigma_{\mathcal{C}'}(e_1)) = 1$. So, by definition, there exists an end-vertex f_1 neighbor of e_1 such that $\text{dist}(f_1, \sigma_{\mathcal{C}'}(f_1)) \in \{0, 2\}$. For $3 \leq q \leq 5$, let $x = \sigma_{\mathcal{C}'}(e_1)$, $y = \sigma_{\mathcal{C}'}(f_1)$ and $z = \sigma_K(e_q)$. We obtain $\sigma_{\mathcal{C}}$ as it is shown in Figure 2. If $q \geq 6$ then, by Lemma 9, there exists a 2-placement $\sigma_{P'}$ of the path $P' = (e_2, \ldots, e_{q-1})$ into P'^3 such that $\text{dist}_{P'}(e_2, \sigma_{P'}(e_2)) = 1$ and $\text{dist}_{P'}(e_{q-1}, \sigma_{P'}(e_{q-1})) \in \{0, 1\}$. So $\sigma_{\mathcal{C}}$ is given by $\sigma_{\mathcal{C}'}$, $\sigma_{P'}$ and σ_K on respectively $\mathcal{C}'$, P' and K. Observe that we have $\text{dist}_{\mathcal{C}}(\sigma(e_1), \sigma(e_2))$ $\text{dist}_{\mathcal{C}}$ $(\sigma_{\mathcal{C}'}(e_1), \sigma_{P'}(e_2)) = 3$, $\text{dist}_{\mathcal{C}}(\sigma(e_{q-1}), \sigma(e_q)) = \text{dist}_{\mathcal{C}}(\sigma_{P'}(e_{q-1}), \sigma_K(e_q)) \in \{2, 3\}$. Moreover, as $\sigma_{\mathcal{C}} = \sigma_K$ on vertices e and its neighbors, then $\sigma_{\mathcal{C}}$ is a $(\mathcal{C}, e)$-good permutation.

Case 2. $\text{dist}_{\mathcal{C}'}(e_1, \sigma_{\mathcal{C}'}(e_1)) = 2$. Let u, v, e_1 and f_1 be the vertices satisfying the cycle (u, v, f_1, e_1) given by the $(\mathcal{C}', e_1)$-good permutation $\sigma_{\mathcal{C}'}$. For $3 \leq q \leq 6$ we obtain $\sigma_{\mathcal{C}}$ as it is shown in Figure 3. If $q \geq 7$ then, by Lemma 9, there exists a 2-placement $\sigma_{P'}$ of the path $P' = (e_3, \ldots, e_{q-1})$ into P'^3 such that $\text{dist}_{P'}(e_3, \sigma_{P'}(e_3)) = 1$ and $\text{dist}_{P'}(e_{q-1}, \sigma_{P'}(e_{q-1})) \in \{0, 1\}$. So the permutation $\sigma_{\mathcal{C}}$ is given by $\sigma_{\mathcal{C}}(e_2) = e_2$ and for each vertex x of $\mathcal{C}'$, P' or K, we have $\sigma_{\mathcal{C}}(x)$ is equal respectively to $\sigma_{\mathcal{C}'}(x)$, $\sigma_{P'}(x)$ or $\sigma_K(x)$. We obtain $\text{dist}_{\mathcal{C}}(\sigma(e_1), \sigma(e_2)) = 3$, $\text{dist}_{\mathcal{C}}(\sigma(e_2), \sigma(e_3)) = 2$ and thus, as in the previous case, $\text{dist}_{\mathcal{C}}(\sigma(e_{q-1}), \sigma(e_q)) \in \{2, 3\}$, then $\sigma_{\mathcal{C}}$ is a $(\mathcal{C}, e)$-good permutation. ■

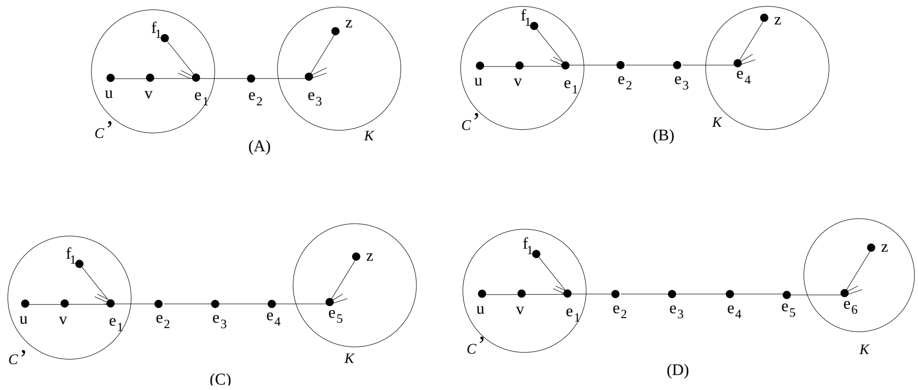

Figure 3. (A) e_2 is fixed by $\sigma_{\mathcal{C}}$, (B) $(u, v, f_1, e_2, e_1)(e_3)$, (C) $(u, v, e_2)(e_1, f_1, e_3)(e_4)$, and e_4 is fixed and (D) $(u, v, f_1)(e_1, e_2, e_4, e_3)(e_5)$.

Lemma 12. *Let $\mathcal{C}'$ be a e'-good caterpillar with e' a node of $\mathcal{C}'$, P be a path of length at least 2 and S be a star of center e and size at least 1. Then $\mathcal{C}' \cdot P \cdot S$ is e-good.*

Proof. Let $\mathcal{C} = \mathcal{C}' \cdot P \cdot S$. Let $P = (e', e_1, e_2, \ldots, e_n, e)$ be a path where e' is a node of $\mathcal{C}'$ and $n \geq 1$. Let f be an end-vertex of S. Let $\sigma_{\mathcal{C}'}$ be a $(\mathcal{C}', e')$-good permutation. By definition, the vertex e' of the caterpillar $\mathcal{C}'$ can be moved at distance 1 or 2 by $\sigma_{\mathcal{C}'}$. So, we discuss each displacement of e' given by $\sigma_{\mathcal{C}'}$.

Case 1. $\text{dist}(e', \sigma'_{\mathcal{C}}(e')) = 1$. So there exists an end-vertex f' in $\mathcal{C}'$ adjacent to e' such that $\text{dist}(f', \sigma'_{\mathcal{C}}(f')) \in \{0, 2\}$.

If $1 \leq n \leq 2$ then, let $y = \sigma_{\mathcal{C}'}(f')$. The permutations are given in Figure 4. Hence we may suppose that $n \geq 3$. Let $P' = P - e'$. By Lemma 9, there exists a 2-placement $\sigma_{P'}$ of P' into its third power such that $\text{dist}(e, \sigma_{P'}(e)) = 1$ and $\text{dist}(e_1, \sigma_{P'}(e_1)) \in \{0, 1\}$. Therefore, the distance on $\mathcal{C}$ between $\sigma_{\mathcal{C}'}(e')$ and $\sigma_{P'}(e_1)$ is at most 3. Then the composition of these two permutations $(\sigma_{\mathcal{C}'}, \sigma_{P'})$ and the fact of keeping all end-vertices neighbors of e fixed give a permutation $\sigma_{\mathcal{C}}$ of $\mathcal{C}$ into its third power.

Case 2. $\text{dist}(e', \sigma_{\mathcal{C}'}(e')) = 2$. So, by definition, there exists an end-vertex f' adjacent to e' such that $\sigma_{\mathcal{C}'}(f') = e'$.

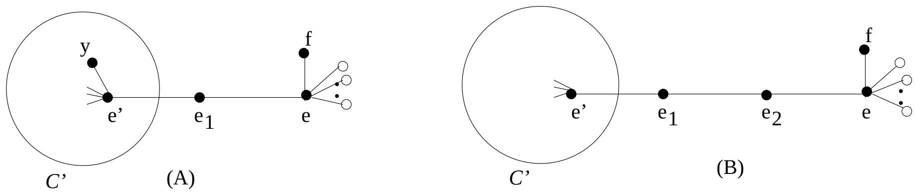

Figure 4. The extension of $\sigma_{\mathcal{C}'}$ to $\mathcal{C}$ is given by: (A) $\sigma_{\mathcal{C}}(e) = e_1$, $\sigma_{\mathcal{C}}(e_1) = y$, $\sigma_{\mathcal{C}}(f') = e$ and $\sigma_{\mathcal{C}}(f) = f$, (B) (e_1, e_2, f, e).

If $1 \leq n \leq 3$ then, let $x = \sigma_{\mathcal{C}'}(e')$. The permutations are given in Figure 5. If $n \geq 4$, we remove the edge e_1e_2. So we obtain the caterpillars $\mathcal{C}_{e_1} = \mathcal{C}' + e'e_1$ and $\mathcal{C}_{e_2}$. Let σ' be a permutation given on $\mathcal{C}_{e_1}$ such that $\sigma'(v) = \sigma_{\mathcal{C}'}(v)$, for each vertex v of $\mathcal{C}' - f'$, $\sigma'(f') = e_1$ and $\sigma'(e_1) = e'$. Let $P' = P - \{e', e_1\}$. By Lemma 9, there exists a 2-placement $\sigma_{P'}$ of P' into its third power such that $\text{dist}(e, \sigma_{P'}(e)) = 1$ and $\text{dist}(e_2, \sigma_{P'}(e_2)) \in \{0, 1\}$. Therefore, the distance on $\mathcal{C}$ between $\sigma'(e_1)$ and $\sigma_{P'}(e_2)$ is at most 3. Then the composition of these two permutations and the fact to keep all end-vertices neighbors of e fixed give a permutation $\sigma_{\mathcal{C}}$ of $\mathcal{C}$ into its third power.

Finally, it is easy to see (in the two cases) that the vertices e_{n-1}, e_n, e and f by $\sigma_{\mathcal{C}}$ verify the property (3) of the definition of a e-good permutation. ■

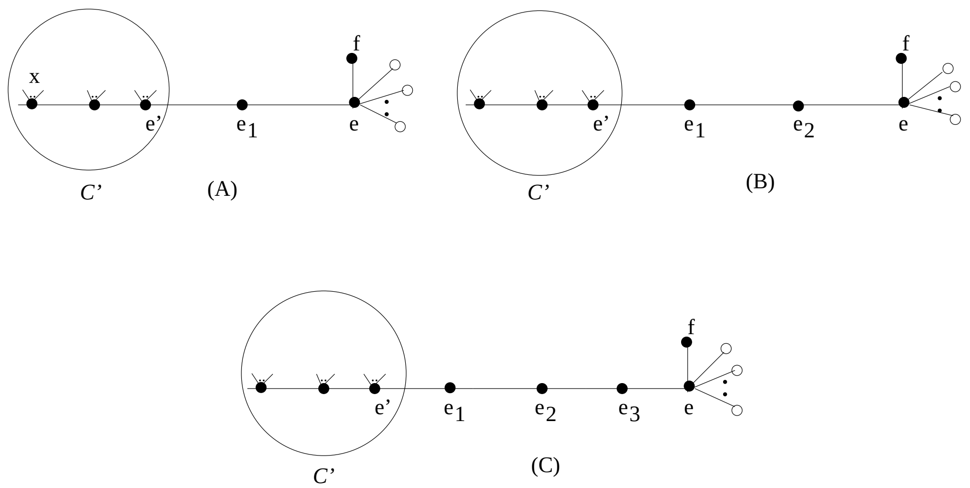

Figure 5. The extension of $\sigma_{\mathcal{C}'}$ to $\mathcal{C}$ is given by: (A) (e', e_1, f, e) and $\sigma_{\mathcal{C}}(f') = x$, (B) $(e', e_1, e, e_2)(f)$ and $\sigma_{\mathcal{C}}(f') = x$, (C) $(e_1)(e_2, e, f, e_3)$.

Lemma 13. *Let $P = (e_1, e_2, \ldots, e_n)$ be a path of length of at least 2, S be a star of center e_1 and of a size of at least 1 and K be a complete caterpillar. Let e_n and e be nodes of the main path of K. Then $(S \cdot P \cdot K)$ is e-good.*

Proof. Let $\mathcal{C}$=$S \cdot P \cdot K$. Let f (resp. f') be an end-vertex neighbor of e_1 (resp. e_n) in S (resp. K). We study two cases.

Case 1. K is a star ($e = e_n$).
If $3 \leq n \leq 6$, the e-good permutations of $\mathcal{C}$ into $\mathcal{C}^3$ are given in Figure 6. If $n \geq 7$, let $P' = (e_1, e_2, \ldots, e_{n-3})$. By Lemma 9, there exists a 2-placement $\sigma_{P'}$ of P' into its third power such that $\text{dist}(e_1, \sigma_{P'}(e_1)) = 1$ and $\text{dist}(e_{n-3}, \sigma_{P'}(e_{n-3})) \in \{0, 1\}$. Let $P'' = (e_{n-2}, e_{n-1}, e_n, f')$. Put $\sigma_{P''} = (e_{n-2}, e_{n-1}, f', e_n)$. Observe that $\text{dist}_{\mathcal{C}}(\sigma_{P'}(e_{n-3}), \sigma_{P''}(e_{n-2})) \leq 3$. So we obtain a $(\mathcal{C}, e)$-good permutation $\sigma_{\mathcal{C}}$ on $\mathcal{C}$ which is given by $\sigma_{P'}$ on P', $\sigma_{P''}$ on P'' and keeps the end-vertices of $K - f'$ and S fixed.

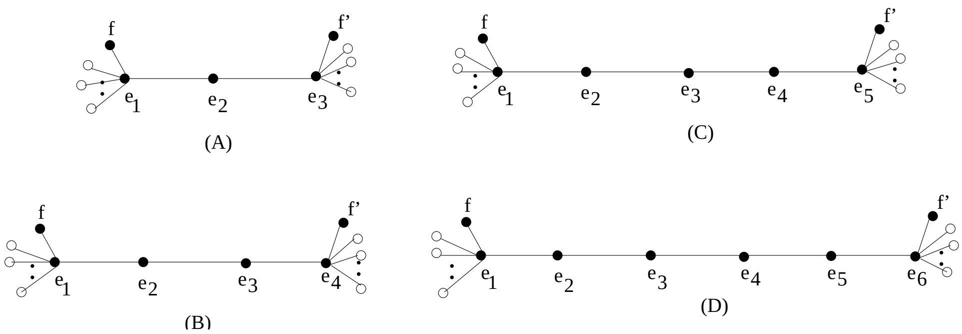

Figure 6. A permutation $\sigma_{\mathcal{C}}$ of darkened vertices is given by: (A) $(f, e_1, e_3, e_2)(f')$, (B) $(f)(e_1, e_2, e_4, e_3)(f')$, (C) $(f)(e_1, e_2, e_5, e_4)(e_3)(f')$, (D) $(f, e_1, e_4)(e_2, e_6, e_5)$ $(e_3)(f')$.

Case 2. K is not a star.
By Lemma 10, there exists a (K, e)-good permutation σ_K on K such that $\text{dist}(e_n, \sigma_K(e_n)) = 1$ and $\text{dist}(f', \sigma_K(f')) \in \{2, 3\}$. If $n = 3$, we obtain the permutation $\sigma_{\mathcal{C}}$ by putting $\sigma_{\mathcal{C}}(f') = e_1$, $\sigma_{\mathcal{C}}(e_1) = e_2$, $\sigma_{\mathcal{C}}(e_2) = \sigma_K(f')$ and $\sigma_{\mathcal{C}}(x) = \sigma_K(x)$ for all vertices of $K - \{e_3, f'\}$ and $\sigma_{\mathcal{C}}(y) = y$ for each end-vertex y of S. In case $n = 4$, we obtain a $(\mathcal{C}, e)$-good permutation $\sigma_{\mathcal{C}}$ by putting $\sigma_{\mathcal{C}}(f) = e_1$, $\sigma_{\mathcal{C}}(e_1) = e_3$, $\sigma_{\mathcal{C}}(e_2) = f$, $\sigma_{\mathcal{C}}(e_3) = e_2$ and $\sigma_{\mathcal{C}}(x) = \sigma_K(x)$ for each x of K and $\sigma_{\mathcal{C}}(y) = y$ for each y in $S - \{f, e_1\}$. For $n \geq 5$, let $P' = P - e_n$. There exists a permutation $\sigma_{P'}$ such that $\text{dist}(e_1, \sigma_{P'}(e_1)) = 1$,

$\text{dist}(e_{n-1}, \sigma_{P'}(e_{n-1})) \in \{0, 1\}$. The e-good permutation $\sigma_{\mathcal{C}}$ is given by σ_K on vertices of K, by $\sigma_{P'}$ on vertices of P' and by putting $\sigma_{\mathcal{C}}(x) = x$ for each end-vertex x of S. Observe that $\text{dist}_{\mathcal{C}}(\sigma_{P'}(e_{n-1}), \sigma_K(e_n)) \leq 3$. So $\mathcal{C}$ is e-good. ∎

3. Proof of Theorem 8

Let $L = (e_1, e_2, \ldots, e_n)$ be the main path of $\mathcal{C}$, where e_1 and e_n are the nodes of $\mathcal{C}$. Observe that if $\mathcal{C}$ is a complete caterpillar, then by Lemma 10, $\mathcal{C}$ is e_n-good and the proof is finished. Otherwise, suppose that $\mathcal{C}$ is not complete. The proof is done by induction on n, with $n \geq 3$. If $n = 3$, $\mathcal{C}$ is a star-path-star and it is e_n-good by Lemma 13. Suppose that any sub-caterpillar $\mathcal{C}'$ is $e_{n'}$-good for each $3 \leq n' < n$. Let e_j be the vertex of degree 2 in $\mathcal{C}$ as close to e_n as possible along L. So $2 \leq j \leq n-1$. Remove from $\mathcal{C}$ the edge e_je_{j+1}, then we obtain the neighbor sub-caterpillars $\mathcal{C}_{e_j}$ and $\mathcal{C}_{e_{j+1}}$. It is easy to see that $\mathcal{C}_{e_{j+1}}$ is a complete sub-caterpillar (in particular, it can be a star). Let e_i be the last vertex of degree 2 in $\mathcal{C}$ along $\bar{L}$ (from e_n to e_1) such that $2 \leq i \leq j \leq n-1$. Remove from $\mathcal{C}$ the edge $e_{i-1}e_i$. Then we obtain the neighbor sub-caterpillars $\mathcal{C}_{e_{i-1}}$ and $\mathcal{C}_{e_i}$. Denote $\mathcal{C}_{e_{i-1}}$ by $\mathcal{C}'$ and $\mathcal{C}_{e_{j+1}}$ by $\mathcal{C}''$. Observe that $\mathcal{C} = \mathcal{C}' \cdot P \cdot \mathcal{C}''$, with $P = (e_{i-1}, e_i, \ldots, e_{j+1})$. In order to give the $(\mathcal{C}, e_n)$-good permutation, we study two cases. First suppose that $\mathcal{C}'$ is not a star. By hypothesis of induction $\mathcal{C}'$ is e_{i-1}-good. If $\mathcal{C}''$ is not a star then by Lemma 11 there exists a $(\mathcal{C}, e_n)$-good permutation, else ($\mathcal{C}''$ is a star) by Lemma 12 there exists a $(\mathcal{C}, e_n)$-good permutation. Now if $\mathcal{C}'$ is a star then by Lemma 13, there exists a $(\mathcal{C}, e_n)$-good permutation. ∎

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