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ON CYCLICALLY EMBEDDABLE (n,n)-GRAPHS

Agnieszka Görlich
Monika Pilśniak
and
Mariusz Woźniak*

Faculty of Applied Mathematics AGH
Department of Discrete Mathematics
al. Mickiewicza 30, 30–059 Kraków, Poland

e-mail: {forys,pilsniak,mwozniak}@uci.agh.edu.pl

Abstract

An *embedding* of a simple graph G into its complement $\overline{G}$ is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In this note we consider the embeddable (n,n)-graphs. We prove that with few exceptions the corresponding permutation may be chosen as cyclic one.

Keywords: packing of graphs, cyclic permutation.

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1. Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs G of order $n = |V(G)|$ and size $|E(G)|$. All graphs will be assumed to have neither loops nor multiple edges. If a graph G has order n and size m, we say that G is an (n,m)-graph.

Assume now that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph is the union of k (≥ 2) disjoint copies of a graph H, then we write $G = kH$.

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An *embedding* of G (in its complement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In other words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G (of order n) into a complete graph K_n. If, additionally, an embedding of G is a cyclic permutation we say that G is *cyclically embeddable* (CE for short).

In the paper we continue the study of families of CE graphs of [10] and [11]. It will be helpful to formulate some results proved in [10, 11] as a theorem.

Theorem 1. *The following graphs are cyclically embeddable:*
1. $(n, n-2)$*-graphs,*
2. *non-star trees,*
3. *cycles* C_i *for* $i \geq 6$,
4. *unicyclic graphs* (*connected* (n, n)*-graphs*) *except for graphs that are not embeddable at all* (*see Figure* 1), *and five graphs given in Figure* 2.

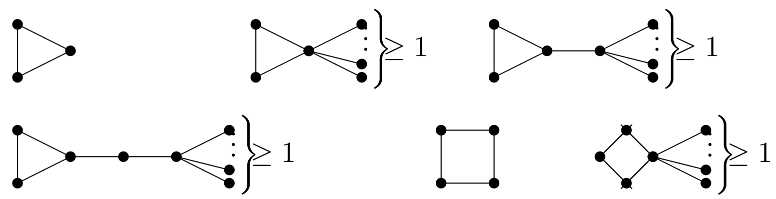


Figure 1. Non-embeddable unicyclical graphs

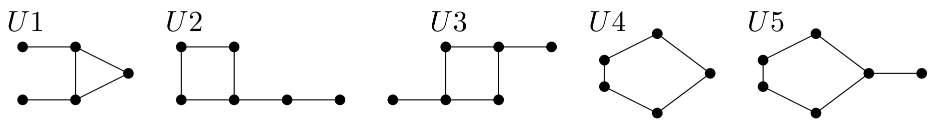


Figure 2. Five embeddable unicyclical graphs which are not CE

Consider now the family of $(n, n-1)$-graphs. The following theorem, originally proved in [4] and independently in [7], completely characterizes those graphs with n vertices and $n-1$ edges that are embeddable.

Theorem 2. *Let G be a graph of order n. If $|E(G)| \leq n-1$ then either G is embeddable or G is isomorphic to one of the following graphs: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ with $n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$ (see Figure 3).* ■

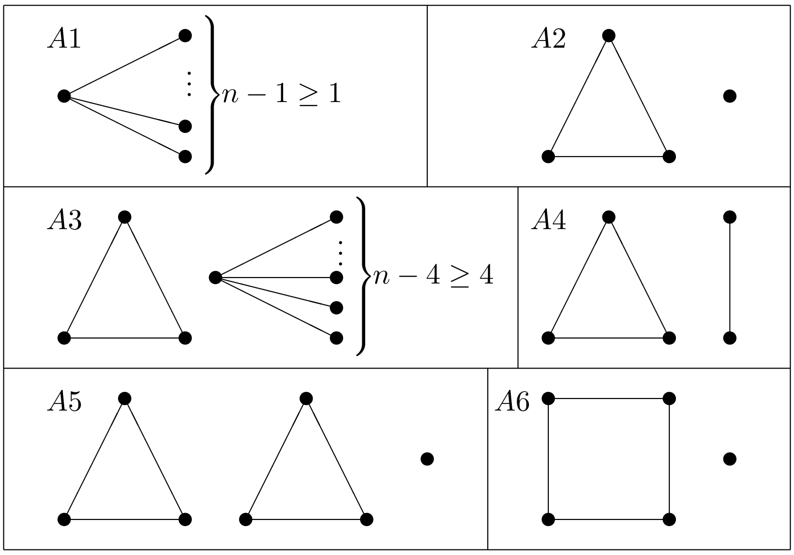


Figure 3. Non-embeddable $(n, n-1)$ graphs

Note that the graphs $K_{1,2} \cup K_3$ and $K_{1,3} \cup K_3$ are embeddable but cannot be embedded without fixed vertices. It is interesting to note that all other $(n, n-1)$-graphs that are contained in their complements can be embedded without fixed vertices. More precisely, we have the following theorem mentioned first in [8].

Theorem 3. *Let G be a graph of order n with $|E(G)| \leq n-1$ and such that*

a) *G is not an exceptional graph of Theorem 2,*

b) *$G \neq K_{1,2} \cup K_3$ and $G \neq K_{1,3} \cup K_3$.*

Then there exists a fixed-point-free embedding of G. ■

Somewhat unexpectedly, with only one exceptional graph more we have considerably stronger result proved in [11].

Theorem 4. *Let G be a graph of order n with $|E(G)| \leq n-1$ and such that*
a) *G is not an exceptional graph of Theorem 2,*
b) *$G \neq K_{1,2} \cup K_3$ and $G \neq K_{1,3} \cup K_3$,*
c) *$G \neq K_1 \cup C_5$.*

Then there exists a cyclic embedding of G. ■

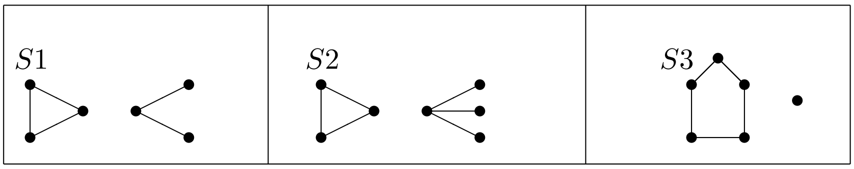


Figure 4. Embeddable $(n, n-1)$ graphs which are not CE

In this paper we shall consider the case where G is a graph on n vertices with n edges. The more general result known on embeddings of (n,n)-graphs is the following theorem proved in [5].

Theorem 5. *Let G be a graph of order n. If $|E(G)| = n$ then either G is embeddable or G is isomorphic to one of the graphs of Figure 5.* ■

We shall consider the cyclic embedding of (n,n)-graphs. First, we notice that for $n \leq 4$ the number of edges in the complete graph K_n is less then $2n$. In Section 3 in Lemma 15 we notice that neither $(5,5)$-graph is CE too. Therefore we consider all (n,n)-graphs for $n \geq 6$. We prove that only five embeddable graphs are not CE.

Theorem 6. *Let $G = (V,E)$ be an embeddable (n,n)-graph $(n \geq 6)$. Then either G is cyclically embeddable or G is isomorphic to:*
A) *one of the unicyclic graphs $U2, U3, U5$ given in Figure 2;*
B) *one of five graphs $F1, F2, F3, F4, F5$ of Figure 6.*

The general references for these and other packing problems are in the papers of B. Bollobás, H.P. Yap and M. Woźniak (see [1], [12] and [13], and [9] respectively).

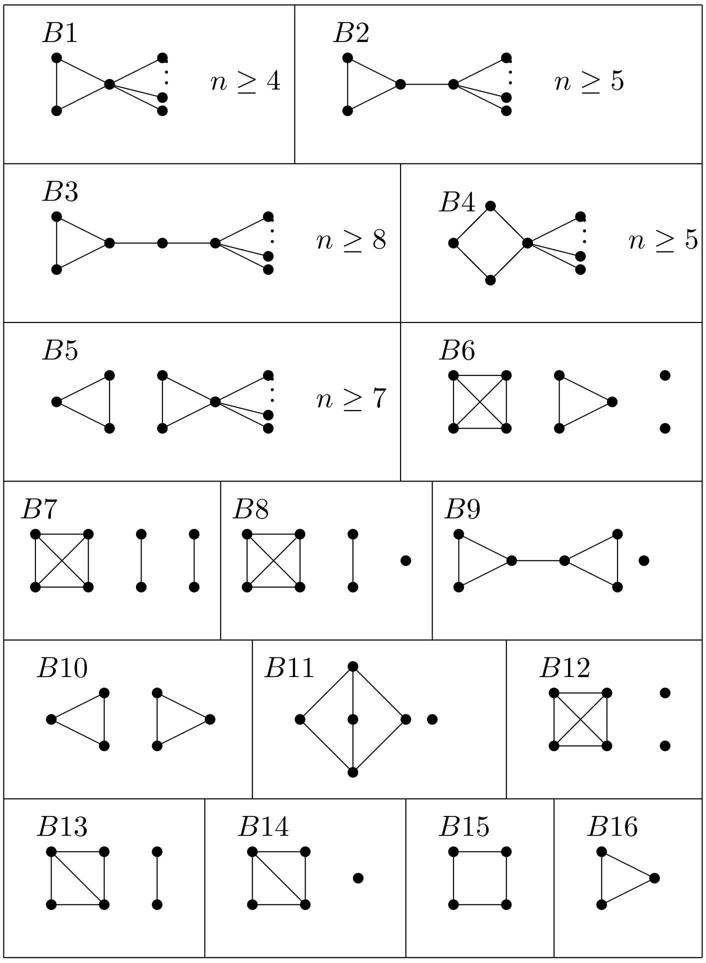


Figure 5. Non-embeddable (n, n) graphs

The rest of the paper is organized as follows: in Section 2 we recall some results and we prove some lemmas, which will be helpful in the proof of Theorem 6. In Section 3 we show that the graphs Fi, $i \in \{1, 2, 3, 4, 5\}$ are not cyclically embeddable and in Section 4 we given the proof of Theorem 6.

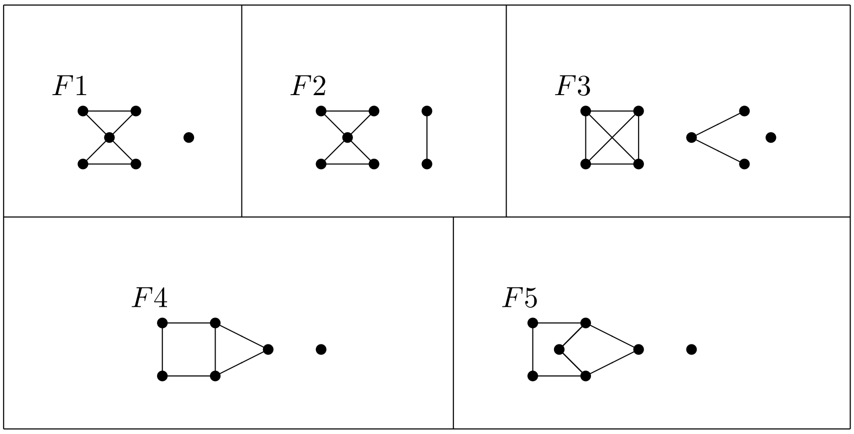


Figure 6. Embeddable (n, n) graphs which are not CE

2. Some Lemmas

We shall need some additional definitions in order to formulate the results. Let G and H be two rooted graphs at u and x, respectively. The graph of order $|V(G)| + |V(H)| - 1$ obtained from G and H by identifying u with x will be called the *touch* of G and H and will be denoted by $G \cdot H$. A similar operation consisting in the identification of a couple of vertices of G, say (u_1, u_2) with a couple of vertices of H, say (x_1, x_2) will be called the 2-*touch* of G and H and will be denoted by $G : H$. The graph $G : H$ is of order $|V(G)| + |V(H)| - 2$. By definition, the edge say u_1u_2 belongs to $E(G : H)$ if $u_1u_2 \in E(G)$ or $x_1x_2 \in E(H)$.

Let σ be a cyclic permutation defined on $V(G)$. Let assume the vertices of G define a polygon. σ is defined as a clockwise rotation of these vertices. For $u \in V(G)$, we denote the vertex $\sigma(u)$ by u^+ and $\sigma^{-1}(u)$ by u^-. Let u, v are the vertices of G and σ is its cyclic permutation. If between u and v are $k - 1$ and $n - 1 - k$ vertices (for $k > 1$) and $k - 1 \leq n - k - 1$, then the edge uv is said to be *of length* k (with respect to σ).

The easy proofs of the following lemmas can be found in [10].

Lemma 7. *Let G be a graph obtained from the graph H by removing a pendent vertex. If G is* CE *then H is* CE. ■

Lemma 8. *Let H be a graph with at least one isolated vertex v and let $G = H - \{v, x\}$ be a graph obtained from the graph H by removing v and another vertex x. If G has an isolated vertex and is* CE *then H is* CE. ■

Lemma 9. *Let G and H be two* CE *graphs. Then $G \cup H$ is* CE. ■

Lemma 10. *Let G and H be two* CE *graphs rooted at u and x, respectively. Then the graph $G \cdot H$ is* CE. ■

Remark. A similar result holds also if "cyclically embeddable" is replaced by "embeddable" (see [6]).

Lemma 11. *Let G and H be two* CE *graphs such that the vertices v, u of G and x, y of H are consecutive with respect to the cyclic embeddings of G and H, respectively. Suppose that: the edges uu^+ and xx^- as well as the edges yy^+ and vv^- are not simultaneously present.*

Then the graph $G : H$ obtained by identifying u with x and v with y is CE. ■

We shall need also some new lemmas.

Lemma 12. *Let G be a* CE *graph and σ its cyclic packing. Let $x \in V(G)$ be a vertex of degree two and let $y, y' \in V(G)$ be the neighbours of x. Let G' be a graph obtained from G by inserting new vertex u on the edge xy, i.e., $V' = V \cup \{u\}$, $E' = E \setminus \{xy\} \cup \{ux, uy\}$.*

Then the graph G' is CE.

Proof. We distinguish two cases. Without loss of generality we may assume that x is between y' and y with respect to the orientation given by σ.

Case 1. Let $y \neq x^+$ and $y' \neq x^-$ with respect to σ. We define σ' as follows: $\sigma'(x) = u$, $\sigma'(u) = x^+$, and $\sigma'(a) = \sigma(a)$ for other vertices of G'. It is easy to see that σ' is a cyclic embedding of G'.

Case 2. Let $y = x^+$ with respect to σ. Then, we define σ' as follows: $\sigma'(x^-) = u$, $\sigma'(u) = x$, and $\sigma'(a) = \sigma(a)$ for remaining vertices of G'. As above, it is easy to see that σ' is a cyclic embedding of G' except for the case where $y' = x^{++}$ with respect to σ. In this case we define σ' as follows: $\sigma'(y) = u$, $\sigma'(u) = y'$, and $\sigma'(a) = \sigma(a)$ for all remaining vertices of G'.

The same reasoning is true if $y = x^-$. ■

Lemma 13. *If $G = (V, E)$ is a* CE *graph and has an isolated vertex then the graph $G \cup K_3$ is* CE.

Proof. We obtain a cyclic packing of the graph $G \cup K_3$ by 2-touch of G and the graph $2K_1 \cup K_3$. The result follows from Lemma 11. ∎

Lemma 14. *Let $G = (V, E)$ be a* CE *graph and σ its cyclic packing. If there exists a vertex $x \in V$ such that $d(x) + d(x^+) \leq n - 2$ then the graph $G \cup K_3$ is* CE.

Proof. First, note that we can choose y and y^+ such that neither $y \notin \{x, x^+\}$ nor $y^+ \notin \{x, x^+\}$ and neither the edge xy nor x^+y^+ is in E. For, each edge $xz \in E$ effects $y \neq z$ and each edge $x^+z \in E$ effects $y \neq z^-$, so these vertices z are blocked by x or x^+. If edges of x, x^+ together block at least $n-1$ vertices of G then we cannot find any y and y^+ such that an edge xy does not exist neither an edge x^+y^+. This situation is not possibility by our assumption of $d(x) + d(x^+) \leq n - 2$.

Now, we can define σ' by adding two vertices v and w of K_3 between y and y^+ and a vertex u of K_3 between x and x^+. Let σ' be a packing of $G \cup K_3$ then $\sigma'(x) = u$, $\sigma'(u) = x^+$, $\sigma'(y) = v$, $\sigma'(v) = w$, $\sigma'(w) = y^+$ and it is easy to see that σ' is cyclically embeddable. ∎

3. Exceptional Graphs

In this section we prepare for the proof of Theorem 6. We start with case $n = 5$.

Lemma 15. *Let $G = (V, E)$ be an embeddable graph and $|V| = |E| = 5$. Then the graph G is not* CE.

Proof. Let σ be a cyclic embedding of G. If G is CE then it has at most four edges: two of length one and two of length two. ∎

Now, we are going to prove that the graphs Fi from Theorem 6 are not cyclically embeddable.

Consider first the graph $F1$. Let u be the vertex of degree four and v the isolated vertex of $F1$. It is easy to see that each packing permutation of the graph $F1$ contains the transposition (u, v), so any packing permutation is not cyclic.

Suppose that σ is a cycling embedding for the graph $F2$. Denote by u the vertex of degree four and by a, b the vertices of degree one. It is easy to see that u cannot be sent on a vertex of degree two. This implies that u cannot be an image (by σ) of a vertex of degree two (since σ^{-1} is also an embedding). Without loss of generality, we suppose that $\sigma(u) = a$. If $\sigma(a) = u$ then σ would contain a transposition. Thus we have $\sigma(b) = u$. Note that $ua^+ \in E$, since u is adjacent to all other vertices. But then, $\sigma(ba) = ua^+$ which contradicts the fact that σ is an embedding.

Let σ be an embedding of the graph $F3$. The set of the images of the vertices of K_4 have to contain: an isolated vertex, two vertices of degree one and one vertex of $V(K_4)$. It is easy to see that in this case, the vertex of degree two has to be mapped on itself. Thus σ has a fixed point and is not cyclic.

We know from Theorem 4 there does not exist a cyclic embedding for the graph $C_5 \cup K_1$ i.e., $S3$, which has one edge less than the graph $F4$. So there does not exist a cyclic embedding of $F4$.

Now, we shall show that the graph $F5$ is not cyclically embeddable. Let the vertices of $F5$ be as in the Figure 7. First we consider a packing of a graph $C_4 \cup 3K_1$. There exist four cyclic embedding of this graph (see Figure 8), so we distinguish four cases. Let σ be an embedding of $F5$.

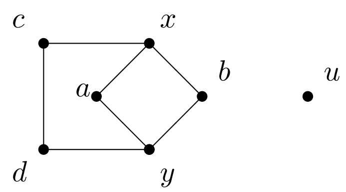


Figure 7. A graph $F5$

Case A. Without loss a generality we may suppose that the vertices are such as in Figure 8. Notice $\sigma(a)$ cannot be neither a vertex c, because $\sigma(cx) = \sigma(xb)$, nor a vertex d, because $\sigma(yd) = \sigma(ax)$. So $\sigma(a) = u$. It is not possibility $\sigma^{-1}(y) = d$, because then $\sigma(dy) = \sigma(ya)$. Therefore $\sigma(b) = d$ and $\sigma(d) = c$, but $\sigma(xc) = \sigma(by)$, so there does not exist cyclic packing of $F5$ in this case.

Analogously we can consider the remaining cases.

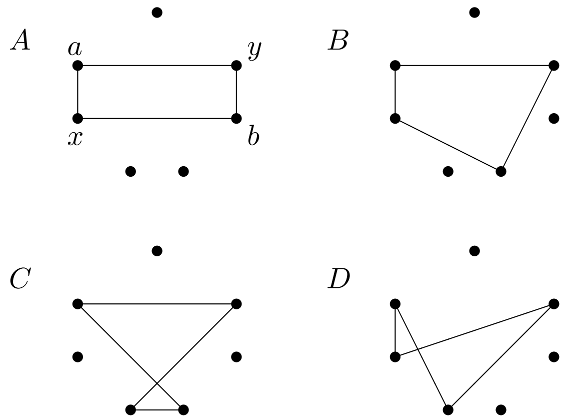


Figure 8. Four different cyclical embeddings of the graphs $C_4 \cup 3K_1$

4. Proof of Theorem 6

We use induction on order of graphs and we show that the graphs Fi are the only exceptions. First, we consider $G = (V, E)$ a $(6, 6)$-graph, which is embeddable and is not unicyclic exception. Let σ be a cyclic packing of G. If graph G is CE, it has at most three edges of length one, three edges of length two and one edge of length three. It is easy to see that each graph G is either a cycle C_6, thus is cyclically embeddable by Theorem 1 or G is a subgraph of graph from Figure 9, which is CE.

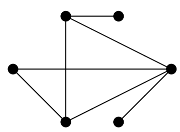


Figure 9. Cyclically embeddable graph

Now, let $n \geq 7$ and assume that our result is true for all $n' < n$. Consider a graph G of order and size equal to n, which is embeddable and is not an exceptional graph (neither one of the graphs $U2, U3, U5$ nor one of the graphs $F1, F2, F3, F4, F5$). We distinguish four main cases.

Case 1. G has a pendent vertex x.
Let $G' = G \setminus \{x\}$. Then G' is an $(n-1, n-1)$-graph. If G' is CE then by Lemma 7 G is CE. If G' is not embeddable then it is either one of graphs Bi, $i \in \{1, \ldots, 14\}$ ($B15, B16$ are (n, n)-graphs with $n < 5$) or G' is one of exceptions Fi, $i \in \{1, \ldots, 5\}$. Let y be a vertex of G'. Now $G = G' \cup \{x\}$ is a graph obtained from Bi or Fi by adding a vertex x with an edge xy.

Note, if G is a graph obtained from Bi, $i \in \{1, 2, 3, 4\}$ then G is CE, because it is an unicyclic graph and is not an exceptional graph of Theorem 1.

If G is obtained from B5 then, by assumption, the vertex y can be only the vertex a, b or c. All these graphs are CE (see Figure 10).

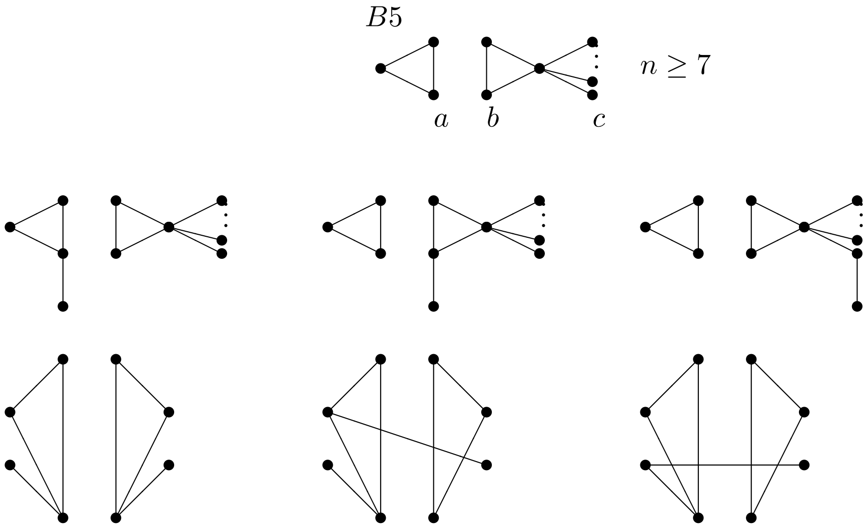


Figure 10. Cyclical embedding of graphs obtained from $B5$

In the next constructions we repeat following reasoning. If the vertex y of the graph G is degree four and G has isolated vertex u, we get a $(n, n-2)$-graph $G'' = G \setminus \{y, u\}$, which has an isolated vertex x. Then by Theorem 1 and Lemma 8 G is CE. If the vertex y of the graph G is degree three and isolated vertex u, we get a $(n, n-1)$-graph $G'' = G \setminus \{y, u\}$, which has an isolated vertex x. Then if G'' is not an exceptional graph of Theorem 4, by Lemma 8 G is CE. So we consider only this cases of graph G, in which it is not possible this reasoning.

We consider the graph G obtained from $B6$. If y is the vertex of K_4 or K_3, we use reasoning like above. If y is an isolated vertex then we obtain a graph $K_4 \cup K_3 \cup K_2 \cup K_1$, which has a cyclic packing (see Figure 11).

Figure 11. Cyclical embedding of the graphs $K_4 \cup K_3 \cup K_2 \cup K_1$ and $K_4 \cup P_3 \cup K_2$

Let G be obtained from $B7$. If y is the vertex of K_4, we get a graph G'' from G by removing two pendent vertices different from x. We get an $(n, n-2)$-graph G''' from G'' by removing an isolated vertex and a vertex y. Then by Lemma 7 and Lemma 8 G is CE. If y is a pendent vertex of $B7$ then we obtain the graph G isomorphic to $K_4 \cup P_3 \cup K_2$, which has a cyclic packing (see Figure 11).

Let G be obtained from $B8$. If y is a pendent vertex of $B8$ then we obtain the graph $F5$, which is not CE. If y is an isolated vertex of $B8$ then we obtain the graph $B7$, which is not embeddable.

We consider the case of G obtained from $B9$. If y is an isolated vertex of $B9$ then we obtain the graph $L1$, which is CE (see Figure 12).

Figure 12. Graph $L1$ and its cyclical embedding

If G is obtained from $B10$ then G is the graph $B5$ and is not embeddable.

Let us consider the case where G is obtained from $B11$. If y is a vertex of $B11$ of degree two then we obtain the graph $L2$, which is CE and if y is an isolated vertex of $B11$ then we obtain the graph $L3$, which is CE, too (see Figure 13).

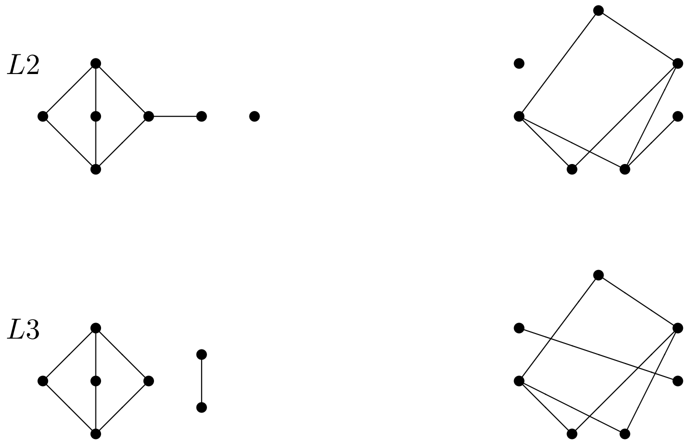


Figure 13. Graphs $L2$ and $L3$ and their cyclical embeddings

Let G be obtained from $B12$. If y is an isolated vertex of $B12$ then we obtain a graph $B8$, which is not embeddable.

Let G be a graph obtained from $B13$ and y be a vertex of degree three of $B13$. We get a graph G'' from G by removing a pendent vertex different from x and we get a $(n, n-2)$-graph G''' from G'' by removing an isolated vertex and a vertex y of degree four. Then by Lemma 7 and Lemma 8 G is CE. If y is a pendent vertex of $B13$ then we obtain the graph $L4$ and if y is a vertex of valency two then we obtain a graph $L5$. The graphs $L4$ and $L5$ are CE as it is showed in Figure 14.

If G is obtained from $B14$ by adding a pendent vertex x then G is a $(6, 6)$-graph. Then G is either the subgraph of Figure 9, which is CE or the graph $B13$, which is not embeddable.

Now, let G be a graph obtained from Fi, $i \in \{1, \ldots, 5\}$ by adding a new vertex x with an edge xy. As above, if y is a vertex of degree three or four we apply reasoning like by the graphs Bi.

We consider the case of G obtained from $F1$. If y is an isolated vertex of $F1$ then we obtain the graph $F2$, which is not CE. If y is an other vertex $F1$ then we repeat reasoning as above.

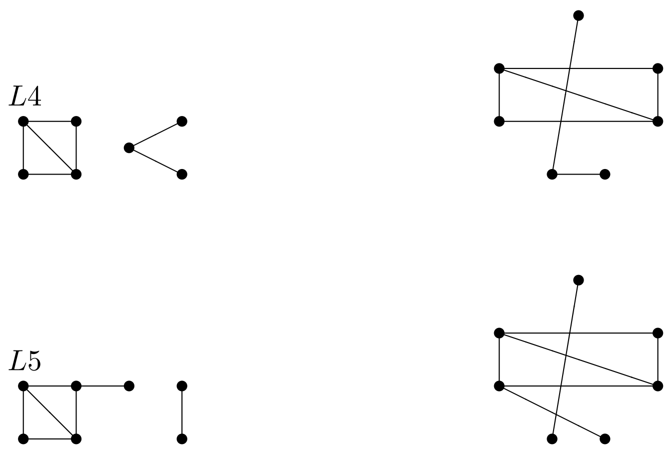


Figure 14. Graphs $L4$ and $L5$ and their cyclical embeddings

We consider the case of G obtained from $F2$. If y is a vertex of degree two $F2$, then by removing a pendent vertex different from x we get the graph $L6$ which is CE (see Figure 15). The same reasoning can by applied if y is a vertex of degree one $F2$. Then we obtain the graph $L7$, which is CE (see Figure 15).

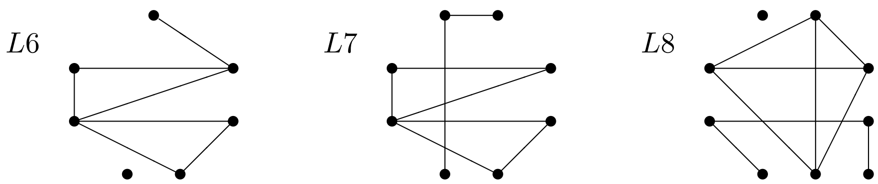
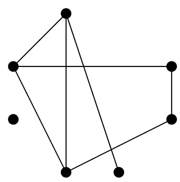


Figure 15. Cyclical embeddings of graphs $L6, L7, L8$

We consider the case of G obtained from $F3$. If y is an isolated vertex $F3$ then we obtain the graph $K_4 \cup P_3 \cup K_2$, which is CE, (see Figure 15). And if y is a vertex of degree one $F3$ then the graph $L8$, which is CE, too (see Figure 15).

We consider the case of G obtained from $F4$. If y is a vertex of degree two $F4$, which has two neighbours of degree three, we obtain the graph $L9$, which is CE (see Figure 16). If y is an isolated vertex $F4$ then we obtain the graph $L10$, which is CE (see Figure 16).

$L9$



$L10$

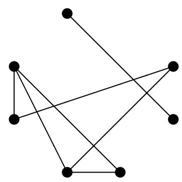
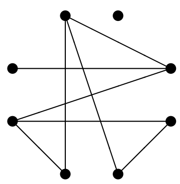


Figure 16. Cyclical embeddings of the graphs $L9, L10$

We consider the case of G obtained from $F5$. If y is a vertex of degree two $F5$, which has two neighbours of degree three, we obtain the graph $L11$, which is CE (see Figure 17). If y is an isolated vertex $F5$ then we obtain the graph $L12$, which is CE (see Figure 17).

$L11$



$L12$

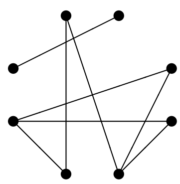


Figure 17. Cyclical embeddings of the graphs $L11, L12$

Case 2. G has exactly one isolated vertex and it does not have any pendent vertex.
It is easy to see that either G has one vertex of valency four or two vertices of valency three. So, we consider two subcases.

Subcase a. Let x be a vertex of degree four in G and G has two connected components. We can say that x is a joint vertex of two cycles in G.

The graph $F1$ is this smallest $(6, 6)$-graph, which obtained from two K_3 by join in a vertex x and it is not CE. Next graph is $Z1$ and it is CE (see Figure 18). Every larger graphs with two connected components can be obtained by a construction of Lemma 12.

Figure 18. Graph $Z1$ and its cyclic embedding

Other graphs with one vertex of valency four, one isolated vertex and without any pendent vertices are obtained by adding one or more cycle components to $F1$ or to graph obtained from $F1$ by a construction of Lemma 12. Thereby we obtain a cyclic embedding of the graph $W1$ (see Figure 19). Each larger graph with three connected components we can obtained by a construction of Lemma 12 and every graph, which has more connected components we can obtained by Lemma 13 and Lemma 12. The assumptions of this lemmas are satisfy in our case.

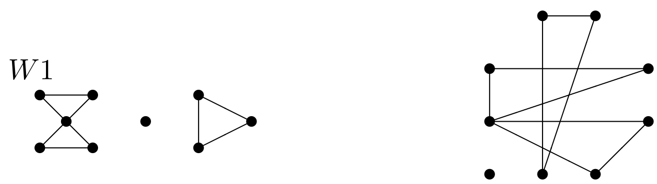


Figure 19. Graph $W1$ and its cyclical embedding

Subcase b. Let G have two vertices x and y of valency three. Observe that x and y are joined by exactly three or one paths, so we consider two subcases: b_1 and b_2.

According to subcase b_1 we consider only embeddable graphs, therefore $F5, Z2$ and $Z3$ are the smallest graphs satisfying our conditions. We consider additionally $(8,8)$-graphs $Z4$ and $Z5$ (see Figure 20), because $F5$ is not CE. All graphs Zi, $i \in \{2,3,4,5\}$ are CE as it is showed in Figure 20. Using a construction of Lemma 12 we can obtain all graphs satisfying our conditions and having two connected components.

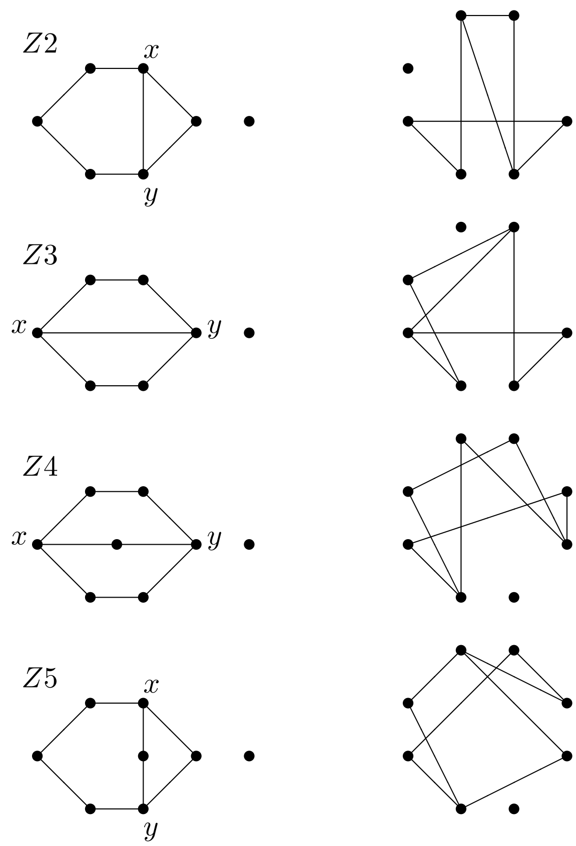


Figure 20. Graphs $Z2, Z3, Z4, Z5$ and their cyclical embeddings

$W2$ and $W3$ are the smallest graphs, which have three connected components. There are constructed by adding C_3 to $B14$ and $B11$, and they are CE (see Figure 21). Each larger graph with three connected components can be obtained by a construction of Lemma 12 and each graph, which has more connected components can be obtained by Lemma 13 and Lemma 12.

In subcase b_2 the graphs $Z6$ and $Z7$ are the smallest, which satisfy our conditions and have two connected components. They are CE (see Figure 22). The addition of C_3 to $B9$, i.e., a graph $W4$ satisfies our conditions too, but has three connected components and is CE (see Figure 22). Now similarly to the previous cases, Lemma 13 and Lemma 12 we can used to obtain every larger graph and its are CE.

Figure 21. Cyclical embeddings of graphs $W2, W3$

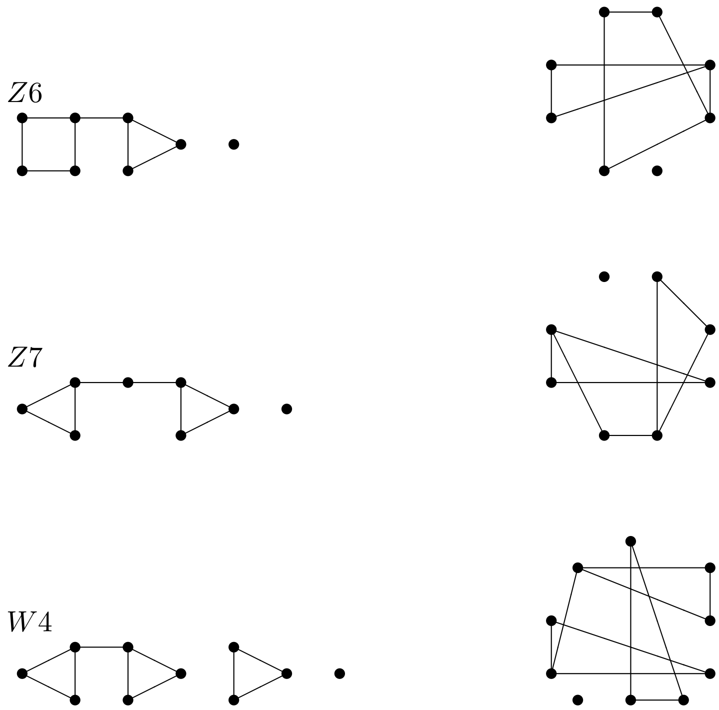


Figure 22. Graphs $Z6, Z7, W4$ and their cyclical embeddings

Case 3. G has at least two isolated vertices, say u and v and it does not have any pendent vertex.

It is easy to see that either G has at least one vertex of valency greater or equal to four or at least two vertices of valency three. We consider two subcases.

Subcase a. Let G have at least one vertex of valency at least four, say x. Consider a graph G' obtained from the graph G by removing the vertices u and x. Then G' has $n-2$ vertices and at most $n-4$ edges. Therefore G' is CE and by Lemma 8 G is CE, too.

Subcase b. Let G have at least two vertices x and y of valency three. Consider a graph $G' = G \setminus \{x, v\}$. Then G' has $n-2$ vertices and $n-3$ edges. By induction, either G' is cyclically embeddable and then G is CE by Lemma 7 or graph G' is not embeddable, i.e., is one of the graphs of Figure 3, or it is one of exceptions of Figure 4. Graph G is embeddable by assumption, therefore it is obtained from $A5$, $A6$ or $S3$ by adding a vertex x of degree three and an isolated vertex v.

If G is constructed from $A5$ then new edges join the vertex x with two vertices of first C_3 and with one vertex of second C_3. If G is obtained from $A6$ then new edges join the vertex x with three vertices of C_4. In the C_5 we can select three vertices by two ways, either all three are adjacent or only two are adjacent. So from $S3$ (see Figure 4) we can obtain two graphs G. All these four graphs G are CE by Lemma 8, because another choice of the vertex of degree three, for example y, leads to a CE $(n-2, n-3)$-graph.

Case 4. G has only vertices of valency two.
If graph G is a cycle then by Theorem 1 (recall that $n \geq 6$) it is CE. Let G be the union of cycles. $C_3 \cup C_4$ is the smallest graph, which has two connected components and $3C_3$ is the smallest graph, which has three connected components and both are CE as it is showed in Figure 23. Every larger graph with two connected components can be obtained by construction of Lemma 12 and every graph, which has more connected components can be obtained by Lemma 14 and Lemma 12.

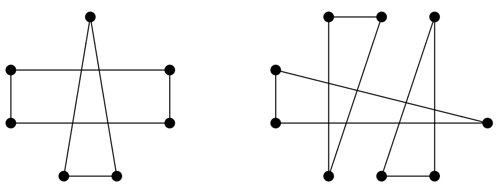


Figure 23. Cyclical embedding of graphs $C_3 \cup C_4$ and $3C_3$

Thus, by induction, the proof is complete. ■

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