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GENERALIZED EDGE-CHROMATIC NUMBERS AND ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary properties of graphs. The generalized edge-chromatic number $\rho'_{\mathcal{Q}}(\mathcal{P})$ is defined as the least integer n such that $\mathcal{P} \subseteq n\mathcal{Q}$. We investigate the generalized edge-chromatic numbers of the properties $\rightarrow H$, $\mathcal{I}_k$, $\mathcal{O}_k$, $\mathcal{W}^*_k$, $\mathcal{S}_k$ and $\mathcal{D}_k$.

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1. Introduction

Following [1] we denote the class of all finite simple graphs by $\mathcal{I}$.

A *property* of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. We say that a graph G *has the property* $\mathcal{P}$ if $G \in \mathcal{P}$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A *homomorphism* of a graph G to a graph H is a mapping of the vertex set $V(G)$ into $V(H)$ such that if $e = \{u, v\} \in E(G)$, then $f(e) = \{f(u), f(v)\} \in E(H)$. Given a graph G and a positive integer k we define $G[k]$ to be the graph with $V(G[k]) = V(G) \times \{1, 2, \ldots, k\}$ and $E(G[k]) = \{(u, l_1)(v, l_2) : uv \in E(G)\}$; $G[k]$ is called a *multiplication* of G. The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G. A *trail* in a graph is a sequence $u_1u_2, u_2u_3, \ldots, u_{k-1}u_k$ of edges, with no edge repeating. If $u_1 \neq u_k$ then the trail is *open*. Since we will only be interested in the length of a trail, we associate a trail T with the set of edges in T.

Example 1. For a positive integer k and a given graph H we define the following well-known properties:

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,

$\mathcal{I}_k = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$,

$\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$,

$\mathcal{W}_k = \{G \in \mathcal{I} :$ each path in G has at most k edges$\}$,

$\mathcal{W}_k^* = \{G \in \mathcal{I} :$ each open trail in G has at most k edges$\}$,

$\mathcal{S}_k = \{G \in \mathcal{I} :$ the maximum degree of G is at most $k\}$,

$\mathcal{D}_k = \{G \in \mathcal{I} : G$ is k-degenerate, i.e., every subgraph of G has a vertex of degree at most $k\}$,

$\to H = \{G \in \mathcal{I} :$ there is a homomorphism from G to $H\}$,

$\mathcal{O}^k = \{G \in \mathcal{I} : G$ is k-colourable$\} = \to K_k$.

Note that for a graph G we have that $G \in \to H$ iff G is a subgraph of a multiplication of H. A property of the form $\to H$ is called a *hom-property*.

Every hereditary property $\mathcal{P}$ is determined by the set of *minimal forbidden subgraphs* $\mathbf{F}(\mathcal{P}) = \{G \in \overline{\mathcal{P}} :$ every proper subgraph of G is in $\mathcal{P}$ $\}$.

If $G = (V, E)$ is a graph and $E' \subseteq E$ then the *subgraph of* G *induced by* E' is the graph (V, E') and is denoted by $G[E']$.

Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ be arbitrary hereditary properties of graphs. An *edge* $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$-*decomposition* of a graph G is a decomposition

$\{E_1, E_2, \ldots, E_n\}$ of $E(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[E_i]$ has the property $\mathcal{Q}_i$. The property $\mathcal{R} = \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \cdots \oplus \mathcal{Q}_n$ is defined as the set of all graphs having an edge $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$-decomposition. It is easy to see that if $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \cdots \oplus \mathcal{Q}_n$ is additive and hereditary too. If $\mathcal{Q}_1 = \mathcal{Q}_2 = \cdots = \mathcal{Q}_n = \mathcal{Q}$, then we write $n\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \cdots \oplus \mathcal{Q}_n$.

The generalized edge-chromatic number $\rho'_{\mathcal{Q}}(G)$ of a graph G is defined as the least integer n such that $G \in n\mathcal{Q}$. For a property $\mathcal{P}$, $\rho'_{\mathcal{Q}}(\mathcal{P})$ is then the least n such that $\mathcal{P} \subseteq n\mathcal{Q}$.

As an example of the non-existence of $\rho'_{\mathcal{Q}}(\mathcal{P})$ we have $\rho'_{\mathcal{S}_1}(\mathcal{D}_1)$ since there exist graphs in $\mathcal{D}_1$ of arbitrary maximum degree. Theorem 1.1 by J. Nešetřil and V. Rödl (see [6]) implies that for some properties $\mathcal{P}$, $\rho'_{\mathcal{Q}}(\mathcal{P})$ exists iff $\rho'_{\mathcal{Q}}(\mathcal{P}) = 1$. Here a graph G is called 3-*chromatic connected* if there is no $S \subseteq V(G)$ such that $G - S$ is disconnected and $G[S]$ is bipartite.

Theorem 1.1 [6]**.** *Let* $\mathbf{F}(\mathcal{P})$ *be a set of* 3*-chromatic connected graphs. Then for every positive integer* k *and graph* $G \in \mathcal{P}$ *there exists a graph* $H \in \mathcal{P}$ *such that for any decomposition* $\{E_1, E_2, \ldots, E_k\}$ *of* $E(H)$ *there is an* $i \in \{1, 2, \ldots, k\}$*, for which* $G \subseteq H[E_i]$. ∎

Corollary 1.2. *If* $\mathbf{F}(\mathcal{P})$ *is a set of* 3*-chromatic connected graphs, then for any hereditary property* $\mathcal{Q}$, $\rho'_{\mathcal{Q}}(\mathcal{P})$ *exists if and only if* $\mathcal{P} \subseteq \mathcal{Q}$. ∎

Proof. Suppose that $\mathcal{P} \not\subseteq \mathcal{Q}$ but $\mathcal{P} \in n\mathcal{Q}$ for some n. Let $G \in \mathcal{P}$ and $G \notin \mathcal{Q}$. By Theorem 1.1 there is an $H \in \mathcal{P}$ such that for every decomposition $\{E_1, E_2, \ldots, E_n\}$ of $E(H)$ there is an $i \in \{1, 2, \ldots, n\}$ for which $G \subseteq H[E_i]$. Let $\{E_1, E_2, \ldots, E_n\}$ be an $n\mathcal{Q}$-decomposition of $E(H)$. Then $G \subseteq H[E_i] \in \mathcal{Q}$ for some i, a contradiction. The converse is trivial. ∎

In particular, for every k and any hereditary property $\mathcal{Q}$ we have that $\rho'_{\mathcal{Q}}(\mathcal{I}_k)$ exists iff $\mathcal{I}_k \subseteq \mathcal{Q}$.

Lemma 1.3. *Let* $\mathcal{P}_1, \mathcal{P}_2$ *and* $\mathcal{Q}$ *be any properties. If* $\mathcal{P}_1 \subseteq \mathcal{P}_2$*, then* $\rho'_{\mathcal{Q}}(\mathcal{P}_1) \leq \rho'_{\mathcal{Q}}(\mathcal{P}_2)$. ∎

Lemma 1.4. *Let* $\mathcal{Q}_1, \mathcal{Q}_2$ *and* $\mathcal{P}$ *be any properties. If* $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$*, then* $\rho'_{\mathcal{Q}_2}(\mathcal{P}) \leq \rho'_{\mathcal{Q}_1}(\mathcal{P})$. ∎

The lattice of (additive) hereditary properties is discussed in [1] — we use the supremum and infimum of properties in our next result without further discussion. A similar result is proved in [5].

Theorem 1.5. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be hereditary properties and $\mathcal{Q}$ an additive hereditary property such that $\rho'_{\mathcal{Q}}(\mathcal{P}_1)$ and $\rho'_{\mathcal{Q}}(\mathcal{P}_2)$ are finite. The following hold:*

(i) $\rho'_{\mathcal{Q}}(\mathcal{P}_1 \cup \mathcal{P}_2) = \rho'_{\mathcal{Q}}(\mathcal{P}_1 \vee \mathcal{P}_2) = \max\{\rho'_{\mathcal{Q}}(\mathcal{P}_1), \rho'_{\mathcal{Q}}(\mathcal{P}_2)\}$.

(ii) $\rho'_{\mathcal{Q}}(\mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\rho'_{\mathcal{Q}}(\mathcal{P}_1), \rho'_{\mathcal{Q}}(\mathcal{P}_2)\}$.

(iii) $\max\{\rho'_{\mathcal{Q}}(\mathcal{P}_1), \rho'_{\mathcal{Q}}(\mathcal{P}_2)\} \leq \rho'_{\mathcal{Q}}(\mathcal{P}_1 \oplus \mathcal{P}_2) \leq \rho'_{\mathcal{Q}}(\mathcal{P}_1) + \rho'_{\mathcal{Q}}(\mathcal{P}_2)$. ■

In the rest of this paper we aim to study the generalized edge-chromatic number $\rho'_{\mathcal{Q}}(\mathcal{P})$ with $\mathcal{Q}$ and $\mathcal{P}$ amongst the properties listed in Example 1.

2. Some Values of $\rho'_{\mathcal{Q}}(\mathcal{P})$

The well-known results of Vizing and Petersen on edge-colourings of graphs imply the following result — see [3] for details.

Theorem 2.1. *Let p and q be any positive integers. Then*

1. $\mathcal{S}_p \oplus \mathcal{S}_q \subseteq \mathcal{S}_{p+q}$.
2. $\mathcal{S}_p \subseteq (p+1)\mathcal{S}_1$.
3. *If p and q are even then $\mathcal{S}_{p+q} = \mathcal{S}_p \oplus \mathcal{S}_q$.*
4. *If q is odd then $\mathcal{S}_{p+q} \not\subseteq \mathcal{S}_p \oplus \mathcal{S}_q$.* ■

Corollary 2.2. *For all positive integers k and n,*

$$\rho'_{\mathcal{S}_n}(\mathcal{S}_k) = \begin{cases} \left\lceil \dfrac{k}{n} \right\rceil, & n \text{ even or } k \leq n, \\ \left\lceil \dfrac{k+1}{n} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. The result is clearly true if $k \leq n$. If n is even then it follows from Part 3 of Theorem 2.1 that $\mathcal{S}_k \subseteq \left\lceil \frac{k}{n} \right\rceil \mathcal{S}_n$ while the lower bound follows by observing that $k > n\left(\left\lceil \frac{k}{n} \right\rceil - 1\right)$ so that $\mathcal{S}_k \not\subseteq \mathcal{S}_{n(\lceil \frac{k}{n} \rceil - 1)} = \left(\left\lceil \frac{k}{n} \right\rceil - 1\right)\mathcal{S}_n$.

Now let n be odd and $k > n$. By Theorem 2.1 we have that $\mathcal{S}_k \subseteq (k+1)\mathcal{S}_1 \subseteq n\left\lceil \frac{k+1}{n} \right\rceil \mathcal{S}_1 \subseteq \left\lceil \frac{k+1}{n} \right\rceil \mathcal{S}_n$. Let $c = \left\lceil \frac{k+1}{n} \right\rceil - 1$. Since $\left\lceil \frac{k+1}{n} \right\rceil \leq \frac{k+1}{n} + \frac{n-1}{n}$ it follows that $k \geq nc$. If $c = 1$ then, since $k > n$, $\rho'_{\mathcal{S}_n}(\mathcal{S}_k) \geq 2 = c+1 = \left\lceil \frac{k+1}{n} \right\rceil$, so assume that $c \geq 2$. Now $\mathcal{S}_k \supseteq \mathcal{S}_{cn} = \mathcal{S}_{(c-1)n+n} \not\subseteq \mathcal{S}_{(c-1)n} \oplus \mathcal{S}_n \supseteq (c-1)\mathcal{S}_n \oplus \mathcal{S}_n \supseteq c\mathcal{S}_n$ so that $\rho'_{\mathcal{S}_n}(\mathcal{S}_k) \geq c+1$. ■

Our next result states that, in some cases, the determination of the generalized edge-chromatic number $\rho'_{\mathcal{Q}}(\to H)$ reduces to the determination of $\rho'_{\mathcal{Q}}(H)$.

Theorem 2.3. *For any additive hereditary property $\mathcal{Q}$ which is closed under multiplications and any graph H, $\rho'_{\mathcal{Q}}(\to H) = \rho'_{\mathcal{Q}}(H)$.*

Proof. Since $H \in \to H$ we have $\rho'_{\mathcal{Q}}(\to H) \geq \rho'_{\mathcal{Q}}(H)$. Now suppose that $H \in m\mathcal{Q}$ and let $(E_1, E_2, \ldots, E_m)$ be an $m\mathcal{Q}$-decomposition of $E(H)$. If $G \in \to H$ then G is a subgraph of a multiplication of H. Let, for every $i \in \{1, 2, \ldots, m\}$, $E'_i = \{(u, l_1)(v, l_2) : uv \in E_i\}$. Then $G[E'_i]$ is a subgraph of a multiplication of $H[E_i]$ for every i and, since $\mathcal{Q}$ is closed under multiplications and hereditary, $G[E'_i] \in \mathcal{Q}$. Therefore $(E'_1, E'_2, \ldots, E'_m)$ is an $m\mathcal{Q}$-decomposition of $E(G)$, hence $\rho'_{\mathcal{Q}}(\to H) \leq \rho'_{\mathcal{Q}}(H)$. ■

Theorem 2.4. *For all positive integers $n \geq 2$ and k, if $\mathcal{P}$ satisfies $\mathcal{O}_{k-1} \subseteq \mathcal{P} \subseteq \mathcal{O}^k$, then $\rho'_{\mathcal{O}^n}(\mathcal{P}) = \lceil \log_n k \rceil$.*

Proof. It is well known that $\mathcal{O}^{ab} = \mathcal{O}^a \oplus \mathcal{O}^b$ (see e.g. [3]). This implies that $\mathcal{O}^k \subseteq \mathcal{O}^{n^{\lceil \log_n k \rceil}} = \lceil \log_n k \rceil \, \mathcal{O}^n$ hence $\rho'_{\mathcal{O}^n}(\mathcal{O}^k) \leq \lceil \log_n k \rceil$.

Since $n^{\lceil \log_n k \rceil - 1} < n^{\log_n k} = k$ it follows that $K_k \notin \mathcal{O}^{n^{\lceil \log_n k \rceil - 1}} = (\lceil \log_n k \rceil - 1)\mathcal{O}^n$. Therefore $\mathcal{O}_{k-1} \not\subseteq (\lceil \log_n k \rceil - 1)\mathcal{O}^n$ and thus $\rho'_{\mathcal{O}^n}(\mathcal{O}_{k-1}) \geq \lceil \log_n k \rceil$. Therefore, by Lemma 1.3 it follows that $\rho'_{\mathcal{O}^n}(\mathcal{P}) = \lceil \log_n k \rceil$. ■

For our next result we define $\rho_\chi(\mathcal{P})$ to be the least k such that $\mathcal{P} \subseteq \mathcal{O}^k$ and $\chi^*(\mathcal{P})$ to be the greatest k such that $\mathcal{O}^k \subseteq \mathcal{P}$.

Corollary 2.5. *For any additive hereditary properties $\mathcal{Q}$, $\mathcal{P} \neq \mathcal{I}$ for which $\rho_\chi(\mathcal{P})$ and $\rho_\chi(\mathcal{Q})$ exist, $\left\lceil \log_{\rho_\chi(\mathcal{Q})} \chi^*(\mathcal{P}) \right\rceil \leq \rho'_{\mathcal{Q}}(\mathcal{P}) \leq \left\lceil \log_{\chi^*(\mathcal{Q})} \rho_\chi(\mathcal{P}) \right\rceil$.*

Proof. Since $\mathcal{O}^{\chi^*(\mathcal{Q})} \subseteq \mathcal{Q}$ and $\mathcal{P} \subseteq \mathcal{O}^{\rho_\chi(\mathcal{P})}$ we have by Lemma 1.3, Lemma 1.4 and Theorem 2.4 that $\left\lceil \log_{\chi^*(\mathcal{Q})} \rho_\chi(\mathcal{P}) \right\rceil \geq \rho'_{\mathcal{Q}}(\mathcal{P})$. Similarly, since $\mathcal{Q} \subseteq \mathcal{O}^{\rho_\chi(\mathcal{Q})}$ and $\mathcal{O}^{\chi^*(\mathcal{P})} \subseteq \mathcal{P}$ we have that $\rho'_{\mathcal{Q}}(\mathcal{P}) \geq \left\lceil \log_{\rho_\chi(\mathcal{Q})} \chi^*(\mathcal{P}) \right\rceil$. ■

Since, for any graph H, $\rho_\chi(\to H) = \chi(H)$ and $\chi^*(\to H) = \omega(H)$ we have the following corollary.

Corollary 2.6. *For all graphs G and H,*

$$\left\lceil \log_{\chi(G)} \omega(H) \right\rceil \leq \rho'_{\to G}(\to H) \leq \left\lceil \log_{\omega(G)} \chi(H) \right\rceil. \qquad ■$$

3. Some Results on $\mathcal{D}_k$

The next result is stated in [2].

Theorem 3.1. *For all positive integers a and b, we have* $\mathcal{D}_{a+b} \subseteq \mathcal{D}_a \oplus \mathcal{D}_b$. ∎

From this theorem it follows that, for all positive integers c and n, $\mathcal{D}_{cn} \subseteq c\mathcal{D}_n$. We now show that this cannot be improved, even if we restrict the graphs to be bipartite.

Theorem 3.2. *For all positive integers c and n,* $\mathcal{D}_{cn+1} \cap \mathcal{O}^2 \nsubseteq c\mathcal{D}_n$.

Proof. Let $t = (n+1)c^{cn+1}$. Clearly, $G = K_{cn+1,t} \in \mathcal{D}_{cn+1} \cap \mathcal{O}^2$. We show that $G \notin c\mathcal{D}_n$: Suppose, to the contrary, that $\{E_1, E_2, \ldots, E_c\}$ is a $c\mathcal{D}_n$-decomposition of $E(G)$. Let $V_1 = \{v_1, v_2, \ldots, v_{cn+1}\}$ be the partite set of order $cn+1$ and V_2 the partite set of order t. Consider the edges incident with v_1. At least t/c of them must be in the same colour class, hence there is a subset U_1 of V_2 with $|U_1| = t/c$ such that all edges in $G[U_1 \cup V_1]$ incident with v_1 have the same colour. Similarly, there is a subset U_2 of U_1 with $|U_2| = t/c^2$ such that all edges in $G[U_2 \cup V_1]$ incident with v_2 have the same colour (not necessarily the same as for v_1). Continuing in this way we obtain a subset U of V_2 with $|U| = n+1$ such that, for every $v \in V_1$, all edges of $G[U \cup V_1]$ incident with v have the same colour.

Since there are c colours it follows that for some $i \in \{1, 2, \ldots, c\}$ we have that $K_{n+1,n+1} \subseteq G[E_i]$. This is a contradiction, since $K_{n+1,n+1} \notin \mathcal{D}_n$. Thus $K_{cn+1,t} \notin c\mathcal{D}_n$. ∎

Theorem 3.3. *For all positive integers k and n, we have that*

$$\rho'_{\mathcal{D}_n}(\mathcal{D}_k) = \left\lceil \frac{k}{n} \right\rceil.$$

Proof. It follows from Theorem 3.1, by induction on c, that $\mathcal{D}_{cn} \subseteq c\mathcal{D}_n$ for all c and n. Now let k and n be positive integers and let $c = \left\lceil \frac{k}{n} \right\rceil$. Then $k \le cn$ hence $\mathcal{D}_k \subseteq \mathcal{D}_{cn} \subseteq c\mathcal{D}_n$ and the upper bound follows.

For the lower bound, since $k \ge (c-1)n + 1$ we have that $\mathcal{D}_k \supseteq \mathcal{D}_{(c-1)n+1} \nsubseteq (c-1)\mathcal{D}_n$ by Theorem 3.2. ∎

We know that if $pq > a + b$, then $\mathcal{D}_{a+b} \subseteq \mathcal{O}^{a+b+1} \subseteq \mathcal{O}^{pq} = \mathcal{O}^p \oplus \mathcal{O}^q$ and $\mathcal{D}_{a+b} \subseteq \mathcal{D}_a \oplus \mathcal{D}_b$. Our next result shows that for graphs in $\mathcal{D}_{a+b}$ we can find simultaneous $(\mathcal{O}^p, \mathcal{O}^q)$- and $(\mathcal{D}_a, \mathcal{D}_b)$-partitions. First a set-theoretic lemma.

Lemma 3.4. *Let a, b, p and q be positive integers such that $a \geq b$, $2 \leq q \leq b+1$ and $pq > a+b$. If X is a set with $a+b$ elements and $\{U_1, U_2, \dots, U_p\}$ and $\{V_1, V_2, \dots, V_q\}$ are partitions of X then there exists a partition $\{A, B\}$ of X and i and j such that $|A| = a$, $A \cap U_i = \emptyset$ and $B \cap V_j = \emptyset$.*

Proof. It is sufficient (and necessary) to find i and j such that $U_i \cap V_j = \emptyset$, $|U_i| \leq b$ and $|V_j| \leq a$. Let k be the number of U_i's such that $|U_i| > b$ and m the number of V_j's such that $|V_j| > a$. We will show that $(p-k)(q-m) > c = |X \setminus (\bigcup\{U_i : |U_i| > b\} \cup \bigcup\{V_j : |V_j| > a\})|$. It then follows that among the sets of the required size there is a disjoint pair (there are $(p-k)(q-m)$ ways to choose a pair (U_i, V_j) of sets of the required size. Since the U_i's are pairwise disjoint and the V_j's are pairwise disjoint it would follow that $c \geq (p-k)(q-m)$ if all such pairs have nonempty intersection). Note that $m \leq 1$ since $a \geq b$ and that $c \leq \min\{a+b-k(b+1), a+b-m(a+1)\}$. Also, $k < p$, for otherwise we get $a+b = |X| \geq p(b+1) \geq pq$. We have three cases to consider.

(1) $m = 0$: In this case we have $(p-k)q = pq - kq \geq a+b+1-k(b+1) > c$.

(2) $m = 1$ and $k \leq \frac{a+1}{b+1}$: We want to show that $(p-k)(q-1) > b-1$ since $c \leq b-1$. If $q = b+1$ this is clearly true, hence we assume that $q \leq b$. We have

$$\begin{aligned}
\frac{b-1}{q-1} + kq - a &\leq \frac{a+1}{b+1}q - a + \frac{b-1}{q-1} \\
&= a\Big(\frac{q}{b+1} - 1\Big) + \frac{b-1}{q-1} + \frac{q}{b+1} \\
&\leq b\Big(\frac{q}{b+1} - 1\Big) + \frac{b-1}{q-1} + \frac{q}{b+1} \qquad \text{since } a \geq b \text{ and } q \leq b \\
&= b\Big(\frac{1}{q-1} - 1\Big) + q - \frac{1}{q-1} \\
&\leq q\Big(\frac{1}{q-1} - 1\Big) + q - \frac{1}{q-1} \\
&= 1
\end{aligned}$$

Suppose now that $(p-k)(q-1) \leq b-1$. Then we have $pq \leq \frac{b-1}{q-1}q + kq = b - 1 + \frac{b-1}{q-1} + kq - a + a \leq a + b$, a contradiction.

(3) $m = 1$ and $k > \frac{a+1}{b+1}$: Again we may assume that $q \leq b$. We show that $(p-k)(q-1) > a+b-k(b+1) \geq c$. We have

$$\begin{aligned}
-k(b+1) + \frac{a+b-k(b+1)}{q-1} + kq & \\
&= \frac{a+b}{q-1} + k\Big(q-(b+1)-\frac{b+1}{q-1}\Big) \\
&\leq \frac{a+b}{q-1} + \frac{a+1}{b+1}\Big(q-(b+1)-\frac{b+1}{q-1}\Big) \qquad \text{since } q \leq b \\
&= a\Big(\frac{q}{b+1}-1\Big) + \frac{q}{b+1} + \frac{b-q}{q-1} \\
&\leq b\Big(\frac{q}{b+1}-1\Big) + \frac{q}{b+1} + \frac{b-q}{q-1} \\
&= (q-b)\Big(1-\frac{1}{q-1}\Big) \\
&\leq 0
\end{aligned}$$

Suppose now that $(p-k)(q-1) \leq a+b-k(b+1)$. Then we have $pq \leq q\frac{a+b-k(b+1)}{q-1} + kq = a+b-k(b+1) + \frac{a+b-k(b+1)}{q-1} + kq \leq a+b$. ■

Theorem 3.5. *Let a, b, p and q be positive integers such that $a \geq b$, $2 \leq q \leq b+1$ and $pq > a+b$. Then $\mathcal{D}_{a+b} \subseteq (\mathcal{D}_a \cap \mathcal{O}^p) \oplus (\mathcal{D}_b \cap \mathcal{O}^q)$.*

Proof. Let G be a counterexample of minimum order and let v be a vertex of G of degree at most $a+b$. Then $G-v$ has a $(\mathcal{D}_a \cap \mathcal{O}^p, \mathcal{D}_b \cap \mathcal{O}^q)$-decomposition and Lemma 3.4 is exactly what we need to extend this decomposition to G for a contradiction. ■

These results now put us in a position to refine Theorem 3.3.

Theorem 3.6. *For all positive integers k, n and $p \geq 2$ we have that:*

$$\begin{aligned}
\rho'_{\mathcal{D}_n \cap \mathcal{O}^p}(\mathcal{D}_k) &= \left\lceil \log_p(k+1) \right\rceil, \ \textit{if } k \leq n, \\
&= \left\lceil \frac{k}{n} \right\rceil, \ \textit{if } k > n \textit{ and } p^2 > 2n, \\
&\leq \left\lceil \log_p(n+1) \right\rceil + \left\lceil \frac{k}{n} \right\rceil - 1, \ \textit{otherwise.}
\end{aligned}$$

Proof. Firstly we note that from Theorem 3.5 it follows that $\mathcal{D}_{cn} \subseteq \mathcal{D}_{(c-1)n} \oplus (\mathcal{D}_n \cap \mathcal{O}^2) \subseteq \mathcal{D}_{(c-1)n} \oplus (\mathcal{D}_n \cap \mathcal{O}^p)$ for all $c \geq 2$ and therefore $\mathcal{D}_{cn} \subseteq \mathcal{D}_{2n} \oplus (c-2)(\mathcal{D}_n \cap \mathcal{O}^p)$.

Suppose that $k \leq n$. Then $\rho'_{\mathcal{D}_n \cap \mathcal{O}^p}(\mathcal{D}_k) = \rho'_{\mathcal{O}^p}(\mathcal{D}_k) = \left\lceil \log_p(k+1) \right\rceil$ by Theorem 2.4.

Now suppose that $k > n$ and $p^2 > 2n$. Then $\mathcal{D}_{cn} \subseteq \mathcal{D}_{2n} \oplus (c-2)(\mathcal{D}_n \cap \mathcal{O}^p) \subseteq c(\mathcal{D}_n \cap \mathcal{O}^p)$, using Theorem 3.5 and the fact that $p^2 > 2n$. Now $\mathcal{D}_k \subseteq \mathcal{D}_{\lceil \frac{k}{n} \rceil n} \subseteq \left\lceil \frac{k}{n} \right\rceil (\mathcal{D}_n \cap \mathcal{O}^p)$ giving the upper bound. The lower bound follows from Theorem 3.3 and Lemma 1.4.

Suppose that $p^2 \leq 2n$. From $\mathcal{D}_{cn} \subseteq \mathcal{D}_{2n} \oplus (c-2)(\mathcal{D}_n \cap \mathcal{O}^p)$ we get that $\mathcal{D}_{cn} \subseteq \mathcal{D}_n \oplus (c-1)(\mathcal{D}_n \cap \mathcal{O}^p)$. Moreover, by Theorem 2.4 we have that $\mathcal{D}_n \subseteq \mathcal{O}^{n+1} \subseteq \left\lceil \log_p(n+1) \right\rceil (\mathcal{D}_n \cap \mathcal{O}^p)$. Therefore $\mathcal{D}_k \subseteq \mathcal{D}_{\lceil \frac{k}{n} \rceil n} \subseteq \mathcal{D}_n \oplus \left(\left\lceil \frac{k}{n} \right\rceil - 1 \right) (\mathcal{D}_n \cap \mathcal{O}^p) \subseteq \left(\left\lceil \log_p(n+1) \right\rceil + \left\lceil \frac{k}{n} \right\rceil - 1 \right) (\mathcal{D}_n \cap \mathcal{O}^p)$ giving the desired bound. ■

4. Results on $\mathcal{W}_k^*$ and $\mathcal{W}_k$

It has been conjectured (see e.g. [4]) that the generalized vertex-chromatic number $\rho_{\mathcal{W}_n}(\mathcal{W}_k)$ equals $\left\lceil \frac{k+1}{n+1} \right\rceil$. We now consider the similar problems of determining $\rho'_{\mathcal{W}_n^*}(\mathcal{W}_k^*)$ and $\rho'_{\mathcal{W}_n}(\mathcal{W}_k)$.

We will say that two trails in a graph *intersect* if they have a common edge.

Theorem 4.1. *For $a \geq 9$ and $b \geq 1$ we have $\mathcal{W}^*_{\lceil \frac{2a-6}{3} \rceil + b} \subseteq \mathcal{W}_a^* \oplus \mathcal{W}_b^*$.*

Proof. Consider any graph G in $\mathcal{W}^*_{\lceil \frac{2a-6}{3} \rceil + b}$. Take E_1 to be a maximal subset of $E(G)$ such that $G[E_1]$ is in $\mathcal{W}_a^*$. Let $E_2 = E(G) - E_1$. Suppose that there is an open trail T in $G[E_2]$ of length $b+1$ and let e_1 and e_2 denote the end-edges of T. Since E_1 is maximal in $\mathcal{W}_a^*$ it follows that there is an open trail T_1 of length $a+1$ in $G[E_1 \cup \{e_1\}]$ and an open trail T_2 of length $a+1$ in $G[E_1 \cup \{e_2\}]$. Let T_{11} and T_{12} denote the trails on either side of e_1 such that $T_{11} \cup \{e_1\} \cup T_{12} = T_1$. Similarly, let $T_{21} \cup \{e_2\} \cup T_{22} = T_2$. Now suppose, without loss of generality, that $x = |E(T_{11})| \leq y = |E(T_{12})|$, so that $x + y = a$.

It is easily seen that if $y \geq \left\lfloor \frac{2a}{3} \right\rfloor + 1$, then by taking the trail $T_{12} \cup T$ or $T_{12} \cup (T - e_1)$, as the case may be, we get a trail of length at least $\left\lfloor \frac{2a}{3} \right\rfloor + 1 + b$ and therefore an open trail of length at least $\left\lfloor \frac{2a}{3} \right\rfloor + 1 + b - 1 \geq \frac{2a-2}{3} + b > \frac{2a-4}{3} + b \geq \left\lceil \frac{2a-6}{3} \right\rceil + b$ in G, a contradiction. Therefore

$\lceil \frac{a}{2} \rceil \leq y \leq \left\lfloor \frac{2a}{3} \right\rfloor$. Moreover, each T_{ij}, $i, j \in \{1, 2\}$ has length at least $\lfloor \frac{a}{3} \rfloor$, since $x = a - y \geq a - \left\lfloor \frac{2a}{3} \right\rfloor \geq a - \frac{2a}{3} = \frac{a}{3} \geq \lfloor \frac{a}{3} \rfloor$.

Note that T_{11} and T_{12} are neccessarily edge disjoint as are T_{21} and T_{22}. T_{12} must intersect T_{21} and T_{22}, otherwise we get an open trail of length at least $\lceil \frac{a}{2} \rceil + b - 2 + \lfloor \frac{a}{3} \rfloor \geq \frac{a}{2} + \frac{a-2}{3} + b - 2 = \frac{5a-16}{6} + b > \left\lceil \frac{2a-6}{3} \right\rceil + b$ in G; containing T_{12}, $T - e_1 - e_2$ and T_{21} or T_{22}.

In the following, when we say that T_{21} intersects T_{12} *first* we mean that there is a trail starting from an end-vertex of e_2, following T_{21} and ending with an edge of T_{12}, containing no edge of T_{11}. Similarly for T_{22} intersecting T_{12} first or T_{2i} intersecting T_{11} first. Note that since T_{11} and T_{12} are disjoint and T_{12} intersects T_{21} and T_{22}, we must have that T_{2i}, $i \in \{1, 2\}$ intersects one of T_{11} and T_{12} first.

Suppose that both T_{21} and T_{22} intersect T_{12} first. Then we obtain an open trail of length at least $x+b-1+\lceil \frac{y}{2} \rceil \geq a-y+\frac{y}{2}+b-1 \geq a-\frac{1}{2}y-1+b \geq a-\frac{1}{2}\left\lfloor \frac{2a}{3} \right\rfloor -1+b \geq a-\frac{1}{2}(\frac{2a}{3})-1+b = \frac{2a-3}{3}+b > \left\lceil \frac{2a-6}{3} \right\rceil +b$ in G; containing T_{11}, $T - e_1$ and at least a half of T_{12}.

Now, suppose that T_{21} or T_{22} intersects T_{11} first, say T_{21}. Then we obtain an open trail of length at least $y+\lceil \frac{x}{2} \rceil +b-2 = y+\left\lceil \frac{1}{2}(a-y) \right\rceil +b-2 \geq y+\frac{a-y}{2}+b-2 \geq \frac{a}{2}+\frac{1}{2}\lceil \frac{a}{2} \rceil +b-2 \geq \frac{3a}{4}+b-2 > \left\lceil \frac{2a-6}{3} \right\rceil +b$ in G; containing T_{12}, $T - e_1 - e_2$ and at least a half of T_{11}. ∎

We remark that a similar result has been proved for vertex partitions and $\mathcal{W}_k$ in [5].

Theorem 4.2. *For all positive integers k and $n \geq 9$, $\rho'_{\mathcal{W}_n^*}(\mathcal{W}_k^*) \leq \left\lceil \frac{3k}{2n-6} \right\rceil$.*

Proof. From Theorem 4.1 it follows by induction on c that $\mathcal{W}^*_{c\lceil \frac{2n-6}{3} \rceil} \subseteq c\mathcal{W}_n^*$ for all positive integers c and n. Now, with $c = \left\lceil \frac{3k}{2n-6} \right\rceil$ we have that $\mathcal{W}_k^* \subseteq \mathcal{W}^*_{c\lceil \frac{2n-6}{3} \rceil} \subseteq c\mathcal{W}_n^*$. ∎

Theorem 4.3. *For all positive integers k and $n \geq 2$, $\left\lfloor \frac{k-2}{n-1} \right\rfloor + 1 \leq \rho'_{\mathcal{W}_n}(\mathcal{W}_k) \leq 2k$.*

Proof. We first show that $\mathcal{W}_{2ac+2} \not\subseteq c\mathcal{W}_{2a+1}$ for every positive integer c: Clearly, $G = K_{ac+1,t} \in \mathcal{W}_{2ac+2}$ for every t. Let t be large and suppose that $G \in c\mathcal{W}_{2a+1}$. Let $\{E_1, E_2, \ldots, E_c\}$ be a corresponding decomposition of $E(G)$. As in the proof of Theorem 3.2 we get, if t is large enough, for some $i \in \{1, 2, \ldots, c\}$ that $K_{a+1,a+2} \subseteq G[E_i]$, a contradiction.

Now let $a = \frac{n-1}{2}$ and $c = \left\lfloor \frac{k-2}{n-1} \right\rfloor$. Since $k \geq 2ac + 2$ we have $\mathcal{W}_k \supseteq \mathcal{W}_{2ac+2} \not\subseteq c\mathcal{W}_n$. Therefore $\rho'_{\mathcal{W}_n}(\mathcal{W}_k) \geq c + 1$.

For the upper bound we have $\mathcal{W}_k \subseteq \mathcal{D}_k \subseteq k\mathcal{D}_1 \subseteq 2k\mathcal{W}_2 \subseteq 2k\mathcal{W}_n$ from Theorem 3.3 and the well-known fact that every tree has a $2(\mathcal{W}_2 \cap \mathcal{D}_1)$ edge decomposition. ■

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