GENERALIZED EDGE-CHROMATIC NUMBERS AND ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. Let \( P \) and \( Q \) be hereditary properties of graphs. The generalized edge-chromatic number \( \rho'_Q(P) \) is defined as the least integer \( n \) such that \( P \subseteq nQ \). We investigate the generalized edge-chromatic numbers of the properties \( \rightarrow, I_k, O_k, W_k^{*}, S_k \) and \( D_k \).

Keywords: property of graphs, additive, hereditary, generalized edge-chromatic number.

2000 Mathematics Subject Classification: 05C15.

1This research forms part of the author’s PhD studies at the Rand Afrikaans University.
1. Introduction

Following [1] we denote the class of all finite simple graphs by $\mathcal{I}$.

A property of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. We say that a graph $G$ has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A property $\mathcal{P}$ is called hereditary if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A homomorphism of a graph $G$ to a graph $H$ is a mapping of the vertex set $V(G)$ into $V(H)$ such that if $e = \{u, v\} \in E(G)$, then $f(e) = \{f(u), f(v)\} \in E(H)$.

Given a graph $G$ and a positive integer $k$ we define $G[k]$ to be the graph with $V(G[k]) = V(G) \times \{1, 2, \ldots, k\}$ and $E(G[k]) = \{(u, l_1)(v, l_2) : uv \in E(G)\}$; $G[k]$ is called a multiplication of $G$. The clique number $\omega(G)$ of a graph $G$ is the maximum order of a complete subgraph of $G$. A trail in a graph is a sequence $u_1u_2, u_2u_3, \ldots, u_{k-1}u_k$ of edges, with no edge repeating. If $u_1 \neq u_k$ then the trail is open. Since we will only be interested in the length of a trail, we associate a trail $T$ with the set of edges in $T$.

Example 1. For a positive integer $k$ and a given graph $H$ we define the following well-known properties:

- $\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,
- $\mathcal{I}_k = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$,
- $\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of $G$ has at most $k + 1$ vertices$\}$,
- $\mathcal{W}_k = \{G \in \mathcal{I} :$ each path in $G$ has at most $k$ edges$\}$,
- $\mathcal{W}_k^e = \{G \in \mathcal{I} :$ each open trail in $G$ has at most $k$ edges$\}$,
- $\mathcal{S}_k = \{G \in \mathcal{I} :$ the maximum degree of $G$ is at most $k$\},
- $\mathcal{D}_k = \{G \in \mathcal{I} : G$ is $k$-degenerate, i.e., every subgraph of $G$ has a vertex of degree at most $k\}$,
- $\rightarrow H = \{G \in \mathcal{I} : G$ has a homomorphism from $G$ to $H\}$,
- $\mathcal{O}^k = \{G \in \mathcal{I} : G$ is $k$-colourable$\} = \rightarrow K_k$.

Note that for a graph $G$ we have that $G \in \rightarrow H$ iff $G$ is a subgraph of a multiplication of $H$. A property of the form $\rightarrow H$ is called a hom-property.

Every hereditary property $\mathcal{P}$ is determined by the set of minimal forbidden subgraphs $\mathcal{F}(\mathcal{P}) = \{G \in \overline{\mathcal{P}} :$ every proper subgraph of $G$ is in $\mathcal{P}\}$.

If $G = (V, E)$ is a graph and $E' \subseteq E$ then the subgraph of $G$ induced by $E'$ is the graph $(V, E')$ and is denoted by $G[E']$.

Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ be arbitrary hereditary properties of graphs. An edge $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$-decomposition of a graph $G$ is a decomposition
\{E_1, E_2, \ldots, E_n\} of E(G) such that for each \(i = 1, 2, \ldots, n\) the induced subgraph \(G[E_i]\) has the property \(Q_i\). The property \(R = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n\) is defined as the set of all graphs having an edge \((Q_1, Q_2, \ldots, Q_n)\)-decomposition. It is easy to see that if \(Q_1, Q_2, \ldots, Q_n\) are additive and hereditary, then \(R = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n\) is additive and hereditary too. If \(Q_1 = Q_2 = \cdots = Q_n = Q\), then we write \(nQ = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n\).

The generalized edge-chromatic number \(\rho'_Q(G)\) of a graph \(G\) is defined as the least integer \(n\) such that \(G \in nQ\). For a property \(P\), \(\rho'_Q(P)\) is then the least \(n\) such that \(P \subseteq nQ\).

As an example of the non-existence of \(\rho'_Q(P)\) we have \(\rho'_S(\mathcal{D}_1)\) since there exist graphs in \(\mathcal{D}_1\) of arbitrary maximum degree. Theorem 1.1 by J. Nešetril and V. Rödl (see [6]) implies that for some properties \(P\), \(\rho'_Q(P)\) exists iff \(\rho'_Q(P) = 1\). Here a graph \(G\) is called 3-chromatic connected if there is no \(S \subseteq V(G)\) such that \(G - S\) is disconnected and \(G[S]\) is bipartite.

**Theorem 1.1** [6]. Let \(F(P)\) be a set of 3-chromatic connected graphs. Then for every positive integer \(k\) and graph \(G \in P\) there exists a graph \(H \in P\) such that for any decomposition \(\{E_1, E_2, \ldots, E_k\}\) of \(E(H)\) there is an \(i \in \{1, 2, \ldots, k\}\), for which \(G \subseteq H[E_i]\).

**Corollary 1.2.** If \(F(P)\) is a set of 3-chromatic connected graphs, then for any hereditary property \(Q\), \(\rho'_Q(P)\) exists if and only if \(P \subseteq Q\).

**Proof.** Suppose that \(P \not\subseteq Q\) but \(P \in nQ\) for some \(n\). Let \(G \in P\) and \(G \not\in Q\). By Theorem 1.1 there is an \(H \in P\) such that for every decomposition \(\{E_1, E_2, \ldots, E_n\}\) of \(E(H)\) there is an \(i \in \{1, 2, \ldots, n\}\) for which \(G \subseteq H[E_i]\). Let \(\{E_1, E_2, \ldots, E_n\}\) be an \(nQ\)-decomposition of \(E(H)\). Then \(G \subseteq H[E_i]\) \(\in Q\) for some \(i\), a contradiction. The converse is trivial.

In particular, for every \(k\) and any hereditary property \(Q\) we have that \(\rho'_Q(I_k)\) exists iff \(I_k \subseteq Q\).

**Lemma 1.3.** Let \(P_1, P_2\) and \(Q\) be any properties. If \(P_1 \subseteq P_2\), then \(\rho'_Q(P_1) \leq \rho'_Q(P_2)\).

**Lemma 1.4.** Let \(Q_1, Q_2\) and \(P\) be any properties. If \(Q_1 \subseteq Q_2\), then \(\rho'_{Q_2}(P) \leq \rho'_{Q_1}(P)\).

The lattice of (additive) hereditary properties is discussed in [1] — we use the supremum and infimum of properties in our next result without further discussion. A similar result is proved in [5].
Theorem 1.5. Let $P_1$ and $P_2$ be hereditary properties and $Q$ an additive hereditary property such that $\rho'_Q(P_1)$ and $\rho'_Q(P_2)$ are finite. The following hold:

(i) $\rho'_Q(P_1 \cup P_2) = \rho'_Q(P_1 \lor P_2) = \max\{\rho'_Q(P_1), \rho'_Q(P_2)\}$.

(ii) $\rho'_Q(P_1 \cap P_2) \leq \min\{\rho'_Q(P_1), \rho'_Q(P_2)\}$.

(iii) $\max\{\rho'_Q(P_1), \rho'_Q(P_2)\} \leq \rho'_Q(P_1 \oplus P_2) \leq \rho'_Q(P_1) + \rho'_Q(P_2)$.

In the rest of this paper we aim to study the generalized edge-chromatic number $\rho'_Q(P)$ with $Q$ and $P$ amongst the properties listed in Example 1.

2. Some Values of $\rho'_Q(P)$

The well-known results of Vizing and Petersen on edge-colourings of graphs imply the following result — see [3] for details.

Theorem 2.1. Let $p$ and $q$ be any positive integers. Then

1. $S_p \oplus S_q \subseteq S_{p+q}$.

2. $S_p \subseteq (p+1)S_1$.

3. If $p$ and $q$ are even then $S_{p+q} = S_p \oplus S_q$.

4. If $q$ is odd then $S_{p+q} \nsubseteq S_p \oplus S_q$.

Corollary 2.2. For all positive integers $k$ and $n$,

$$\rho'_{S_n}(S_k) = \begin{cases} \left\lfloor \frac{k}{n} \right\rfloor, & n \text{ even or } k \leq n, \\ \left\lfloor \frac{k+1}{n} \right\rfloor, & \text{otherwise.} \end{cases}$$

Proof. The result is clearly true if $k \leq n$. If $n$ is even then it follows from Part 3 of Theorem 2.1 that $S_k \subseteq \left\lfloor \frac{k}{n} \right\rfloor S_n$ while the lower bound follows by observing that $k > n \left(\left\lfloor \frac{k}{n} \right\rfloor - 1\right)$ so that $S_k \nsubseteq S_n\left(\left\lfloor \frac{k}{n} \right\rfloor - 1\right) = \left(\left\lfloor \frac{k}{n} \right\rfloor - 1\right)S_n$.

Now let $n$ be odd and $k > n$. By Theorem 2.1 we have that $S_k \subseteq (k+1)S_1 \subseteq n \left\lfloor \frac{k+1}{n} \right\rfloor S_1 \subseteq \left\lfloor \frac{k+1}{n} \right\rfloor S_n$. Let $c = \left\lfloor \frac{k+1}{n} \right\rfloor - 1$. Since $\left\lfloor \frac{k+1}{n} \right\rfloor \leq \frac{k+1}{n} + \frac{n-1}{n}$ it follows that $k \geq nc$. If $c = 1$ then, since $k > n$, $\rho'_{S_n}(S_k) \geq 2 = c+1 = \left\lfloor \frac{k+1}{n} \right\rfloor$, so assume that $c \geq 2$. Now $S_k \supseteq S_{cn} = S_{(c-1)n+n} \nsubseteq S_{(c-1)n} \oplus S_n \supseteq (c-1)S_n \oplus S_n \supseteq cS_n$ so that $\rho'_{S_n}(S_k) \geq c+1$. 


Our next result states that, in some cases, the determination of the generalized edge-chromatic number $\rho'_Q(\rightarrow H)$ reduces to the determination of $\rho'_Q(H)$.

**Theorem 2.3.** For any additive hereditary property $Q$ which is closed under multiplications and any graph $H$, $\rho'_Q(\rightarrow H) = \rho'_Q(H)$.

**Proof.** Since $H \in \rightarrow H$ we have $\rho'_Q(\rightarrow H) \geq \rho'_Q(H)$. Now suppose that $H \in mQ$ and let $(E_1, E_2, \ldots, E_m)$ be an $mQ$-decomposition of $E(H)$. If $G \in \rightarrow H$ then $G$ is a subgraph of a multiplication of $H$. Let, for every $i \in \{1, 2, \ldots, m\}$, $E'_i = \{(u, l_1)(v, l_2) : uv \in E_i\}$. Then $G[E'_i]$ is a subgraph of a multiplication of $H[E_i]$ for every $i$ and, since $Q$ is closed under multiplications and hereditary, $G[E'_i] \in Q$. Therefore $(E'_1, E'_2, \ldots, E'_m)$ is an $mQ$-decomposition of $E(G)$, hence $\rho'_Q(\rightarrow H) \leq \rho'_Q(H)$.

**Theorem 2.4.** For all positive integers $n \geq 2$ and $k$, if $\mathcal{P}$ satisfies $O_{k-1} \subseteq \mathcal{P} \subseteq O^k$, then $\rho'_{O_n}(\mathcal{P}) = \lfloor \log n \rfloor$.

**Proof.** It is well known that $O^{ab} = O^a \oplus O^b$ (see e.g. [3]). This implies that $O^k \subseteq O^{\lfloor \log n \rfloor k} = \lfloor \log n \rfloor O^n$ hence $\rho'_{O_n}(O^k) \leq \lfloor \log n \rfloor$.

Since $n^{\lfloor \log n \rfloor k - 1} < n^{\log n k} = k$ it follows that $K_k \notin O^{\lfloor \log n \rfloor k - 1} = (\lfloor \log n \rfloor - 1)O^n$. Therefore $O_{k-1} \subseteq (\lfloor \log n \rfloor - 1)O^n$ and thus $\rho'_{O_n}(O_{k-1}) \geq \lfloor \log n \rfloor$. Therefore, by Lemma 1.3 it follows that $\rho'_{O_n}(\mathcal{P}) = \lfloor \log n \rfloor$.

For our next result we define $\rho_\chi(\mathcal{P})$ to be the least $k$ such that $\mathcal{P} \subseteq O^k$ and $\chi^*(\mathcal{P}) = k$ such that $O^k \subseteq \mathcal{P}$.

**Corollary 2.5.** For any additive hereditary properties $Q$, $\mathcal{P} \neq \mathcal{I}$ for which $\rho_\chi(\mathcal{P})$ and $\rho_\chi(\mathcal{Q})$ exist, $\left| \log_{\rho_\chi(\mathcal{Q})} \chi^*(\mathcal{P}) \right| \leq \rho'_Q(\mathcal{P}) \leq \left| \log_{\rho_\chi(\mathcal{Q})} \rho_\chi(\mathcal{P}) \right|$.

**Proof.** Since $O^{\chi^*(\mathcal{Q})} \subseteq \mathcal{Q}$ and $\mathcal{P} \subseteq O_{\rho_\chi(\mathcal{P})}$ we have by Lemma 1.3, Lemma 1.4 and Theorem 2.4 that $\left| \log_{\chi^*(\mathcal{Q})} \rho_\chi(\mathcal{P}) \right| \geq \rho'_Q(\mathcal{P})$. Similarly, since $\mathcal{Q} \subseteq O_{\rho_\chi(\mathcal{Q})}$ and $O^{\chi^*(\mathcal{P})} \subseteq \mathcal{P}$ we have that $\rho'_Q(\mathcal{P}) \geq \left| \log_{\rho_\chi(\mathcal{Q})} \chi^*(\mathcal{P}) \right|$.

Since, for any graph $H$, $\rho_\chi(\rightarrow H) = \chi(H)$ and $\chi^*(\rightarrow H) = \omega(H)$ we have the following corollary.

**Corollary 2.6.** For all graphs $G$ and $H$,

$$\left| \log_{\chi(G)} \omega(H) \right| \leq \rho'_{\rightarrow G}(\rightarrow H) \leq \left| \log_{\omega(G)} \chi(H) \right|.$$
3. Some Results on $\mathcal{D}_k$

The next result is stated in [2].

**Theorem 3.1.** For all positive integers $a$ and $b$, we have $\mathcal{D}_{a+b} \subseteq \mathcal{D}_a \oplus \mathcal{D}_b$. □

From this theorem it follows that, for all positive integers $c$ and $n$, $\mathcal{D}_{cn} \subseteq c\mathcal{D}_n$. We now show that this cannot be improved, even if we restrict the graphs to be bipartite.

**Theorem 3.2.** For all positive integers $c$ and $n$, $\mathcal{D}_{cn+1} \cap \mathcal{O}^2 \nsubseteq c\mathcal{D}_n$.

**Proof.** Let $t = (n+1)c^{n+1}$. Clearly, $G = K_{cn+1,t} \in \mathcal{D}_{cn+1} \cap \mathcal{O}^2$. We show that $G \notin c\mathcal{D}_n$: Suppose, to the contrary, that $\{E_1, E_2, \ldots, E_c\}$ is a $c\mathcal{D}_n$-decomposition of $E(G)$. Let $V_1 = \{v_1, v_2, \ldots, v_{cn+1}\}$ be the partite set of order $cn + 1$ and $V_2$ the partite set of order $t$. Consider the edges incident with $v_1$. At least $t/c$ of them must be in the same colour class, hence there is a subset $U_1$ of $V_2$ with $|U_1| = t/c$ such that all edges in $G[U_1 \cup V_1]$ incident with $v_1$ have the same colour. Similarly, there is a subset $U_2$ of $U_1$ with $|U_2| = t/c^2$ such that all edges in $G[U_2 \cup V_1]$ incident with $v_2$ have the same colour (not necessarily the same as for $v_1$). Continuing in this way we obtain a subset $U$ of $V_2$ with $|U| = n + 1$ such that, for every $v \in V_1$, all edges of $G[U \cup V_1]$ incident with $v$ have the same colour.

Since there are $c$ colours it follows that for some $i \in \{1, 2, \ldots, c\}$ we have that $K_{n+1,n+1} \subseteq G[E_i]$. This is a contradiction, since $K_{n+1,n+1} \notin \mathcal{D}_n$. Thus $K_{cn+1,t} \notin c\mathcal{D}_n$. □

**Theorem 3.3.** For all positive integers $k$ and $n$, we have that

$$\rho'_\mathcal{D}_n(\mathcal{D}_k) = \left\lceil \frac{k}{n} \right\rceil.$$

**Proof.** It follows from Theorem 3.1, by induction on $c$, that $\mathcal{D}_{cn} \subseteq c\mathcal{D}_n$ for all $c$ and $n$. Now let $k$ and $n$ be positive integers and let $c = \left\lceil \frac{k}{n} \right\rceil$. Then $k \leq cn$ hence $\mathcal{D}_k \subseteq \mathcal{D}_{cn} \subseteq c\mathcal{D}_n$ and the upper bound follows.

For the lower bound, since $k \geq (c-1)n + 1$ we have that $\mathcal{D}_k \supseteq \mathcal{D}_{(c-1)n+1} \nsubseteq (c-1)\mathcal{D}_n$ by Theorem 3.2. □

We know that if $pq > a + b$, then $\mathcal{D}_{a+b} \subseteq \mathcal{O}^{a+b+1} \subseteq \mathcal{O}^{pq} = \mathcal{O}^p \oplus \mathcal{O}^q$ and $\mathcal{D}_{a+b} \subseteq \mathcal{D}_a \oplus \mathcal{D}_b$. Our next result shows that for graphs in $\mathcal{D}_{a+b}$ we can find simultaneous $(\mathcal{O}^p, \mathcal{O}^q)$- and $(\mathcal{D}_a, \mathcal{D}_b)$-partitions. First a set-theoretic lemma.
Lemma 3.4. Let $a$, $b$, $p$ and $q$ be positive integers such that $a \geq b$, $2 \leq q \leq b + 1$ and $pq > a + b$. If $X$ is a set with $a + b$ elements and $\{U_1, U_2, \ldots, U_p\}$ and $\{V_1, V_2, \ldots, V_q\}$ are partitions of $X$ then there exists a partition $\{A, B\}$ of $X$ and $i$ and $j$ such that $|A| = a$, $A \cap U_i = \emptyset$ and $B \cap V_j = \emptyset$.

Proof. It is sufficient (and necessary) to find $i$ and $j$ such that $U_i \cap V_j = \emptyset$, $|U_i| \leq b$ and $|V_j| \leq a$. Let $k$ be the number of $U_i$’s such that $|U_i| > b$ and $m$ the number of $V_j$’s such that $|V_j| > a$. We will show that $(p - k)(q - m) > c = |X \setminus (\bigcup \{U_i : |U_i| > b\} \cup \bigcup \{V_j : |V_j| > a\})|$. It then follows that among the sets of the required size there is a disjoint pair (there are $(p - k)(q - m)$ ways to choose a pair $(U_i, V_j)$ of sets of the required size. Since the $U_i$’s are pairwise disjoint and the $V_j$’s are pairwise disjoint it would follow that $c \geq (p - k)(q - m)$ if all such pairs have nonempty intersection). Note that $m \leq 1$ since $a \geq b$ and that $c \leq \min\{a + b - k(b + 1), a + b - m(a + 1)\}$. Also, $k < p$, for otherwise we get $a + b = |X| \geq p(b + 1) \geq pq$. We have three cases to consider.

(1) $m = 0$: In this case we have $(p - k)q = pq - kq \geq a + b + 1 - k(b + 1) > c$.

(2) $m = 1$ and $k \leq \frac{q + 1}{b + 1}$: We want to show that $(p - k)(q - 1) > b - 1$ since $c \leq b - 1$. If $q = b + 1$ this is clearly true, hence we assume that $q \leq b$. We have

$$
\frac{b - 1}{q - 1} + kq - a \leq \frac{a + 1}{b + 1}q - a + \frac{b - 1}{q - 1} = a\left(\frac{q}{b + 1} - 1\right) + \frac{b - 1}{q - 1} + \frac{q}{b + 1} \\
\leq b\left(\frac{q}{b + 1} - 1\right) + \frac{b - 1}{q - 1} + \frac{q}{b + 1} \quad \text{since } a \geq b \text{ and } q \leq b \\
= b\left(\frac{1}{q - 1} - 1\right) + q - \frac{1}{q - 1} \\
\leq q\left(\frac{1}{q - 1} - 1\right) + q - \frac{1}{q - 1} = 1
$$

Suppose now that $(p - k)(q - 1) \leq b - 1$. Then we have $pq \leq \frac{b - 1}{q - 1}q + kq = b - 1 + \frac{b - 1}{q - 1} + kq - a + a \leq a + b$, a contradiction.

(3) $m = 1$ and $k > \frac{q + 1}{b + 1}$: Again we may assume that $q \leq b$. We show that $(p - k)(q - 1) > a + b - k(b + 1) \geq c$. We have
\[ -k(b + 1) + \frac{a + b - k(b + 1)}{q - 1} + kq \]
\[ = \frac{a + b}{q - 1} + k\left(q - (b + 1) - \frac{b + 1}{q - 1}\right) \]
\[ \leq \frac{a + b + 1}{q - 1} + \frac{a + 1}{b + 1}\left(q - (b + 1) - \frac{b + 1}{q - 1}\right) \quad \text{since } q \leq b \]
\[ = a\left(\frac{q}{b + 1} - 1\right) + \frac{q}{b + 1} + \frac{b - q}{q - 1} \]
\[ \leq b\left(\frac{q}{b + 1} - 1\right) + \frac{q}{b + 1} + \frac{b - q}{q - 1} \]
\[ = (q-b)\left(1-\frac{1}{q-1}\right) \]
\[ \leq 0 \]

Suppose now that \((p - k)(q - 1) \leq a + b - k(b + 1)\). Then we have
\[ pq \leq \frac{a + b - k(b + 1)}{q - 1} + kq = a + b - k(b + 1) + \frac{a + b - k(b + 1)}{q - 1} + kq \leq a + b. \]

**Theorem 3.5.** Let \(a, b, p, q\) be positive integers such that \(a \geq b\), \(2 \leq q \leq b + 1\) and \(pq > a + b\). Then \(D_{a + b} \subseteq (D_a \cap O^p) \oplus (D_b \cap O^q)\).

**Proof.** Let \(G\) be a counterexample of minimum order and let \(v\) be a vertex of \(G\) of degree at most \(a + b\). Then \(G - v\) has a \((D_a \cap O^p, D_b \cap O^q)\)-decomposition and Lemma 3.4 is exactly what we need to extend this decomposition to \(G\) for a contradiction.

These results now put us in a position to refine Theorem 3.3.

**Theorem 3.6.** For all positive integers \(k, n\) and \(p \geq 2\) we have that:
\[ \rho'_{D_n \cap O^p}(D_k) = \left\lfloor \log_p(k + 1) \right\rfloor, \quad \text{if } k \leq n, \]
\[ = \left\lfloor \frac{k}{n} \right\rfloor, \quad \text{if } k > n \text{ and } p^2 > 2n, \]
\[ \leq \left\lfloor \log_p(n + 1) \right\rfloor + \left\lfloor \frac{k}{n} \right\rfloor - 1, \quad \text{otherwise}. \]

**Proof.** Firstly we note that from Theorem 3.5 it follows that \(D_{cn} \subseteq D_{(c-1)n} \oplus (D_n \cap O^2) \subseteq D_{(c-1)n} \oplus (D_n \cap O^p)\) for all \(c \geq 2\) and therefore \(D_{cn} \subseteq D_{2n} \oplus (c - 2)(D_n \cap O^p)\).
Suppose that \( k \leq n \). Then \( \rho_{D_n \cap O_p}(D_k) = \rho'_{O_p}(D_k) = \left\lceil \log_p(k + 1) \right\rceil \) by Theorem 2.4.

Now suppose that \( k > n \) and \( p^2 > 2n \). Then \( D_{cn} \subseteq D_{2n} \oplus (c - 2)(D_n \cap O_p) \subseteq c(D_n \cap O_p) \), using Theorem 3.5 and the fact that \( p^2 > 2n \). Now \( D_k \subseteq D_{\left\lceil \frac{k}{n} \right\rceil n} \subseteq \left\lceil \frac{k}{n} \right\rceil (D_n \cap O_p) \) giving the upper bound. The lower bound follows from Theorem 3.3 and Lemma 1.4.

Suppose that \( p^2 \leq 2n \). From \( D_{cn} \subseteq D_{2n} \oplus (c - 2)(D_n \cap O_p) \) we get that \( D_{cn} \subseteq D_n \oplus (c - 1)(D_n \cap O_p) \). Moreover, by Theorem 2.4 we have that \( D_n \subseteq O^{n+1} \subseteq \left\lceil \log_p(n + 1) \right\rceil (D_n \cap O_p) \). Therefore \( D_k \subseteq D_{\left\lceil \frac{k}{n} \right\rceil n} \subseteq D_n \oplus \left( \left\lceil \frac{k}{n} \right\rceil - 1 \right)(D_n \cap O_p) \subseteq \left( \left\lceil \log_p(n + 1) \right\rceil + \left\lceil \frac{k}{n} \right\rceil - 1 \right)(D_n \cap O_p) \) giving the desired bound.

\[ \square \]

### 4. Results on \( W_k^a \) and \( W_k^b \)

It has been conjectured (see e.g. [4]) that the generalized vertex-chromatic number \( \rho_{W_n}(W_k) \) equals \( \left\lceil \frac{k+1}{n+1} \right\rceil \). We now consider the similar problems of determining \( \rho_{W_n}(W_k^a) \) and \( \rho_{W_n}(W_k^b) \).

We will say that two trails in a graph intersect if they have a common edge.

**Theorem 4.1.** For \( a \geq 9 \) and \( b \geq 1 \) we have \( W_{\left\lceil \frac{2a-6}{3} \right\rceil + b}^a \subseteq W_a^a \oplus W_b^b \).

**Proof.** Consider any graph \( G \) in \( W_{\left\lceil \frac{2a-6}{3} \right\rceil + b}^a \). Take \( E_1 \) to be a maximal subset of \( E(G) \) such that \( G[E_1] \) is in \( W_a^a \). Let \( E_2 = E(G) - E_1 \). Suppose that there is an open trail \( T \) in \( G[E_2] \) of length \( b+1 \) and let \( e_1 \) and \( e_2 \) denote the end-edges of \( T \). Since \( E_1 \) is maximal in \( W_a^a \) it follows that there is an open trail \( T_1 \) of length \( a+1 \) in \( G[E_1 \cup \{ e_1 \}] \) and an open trail \( T_2 \) of length \( a+1 \) in \( G[E_1 \cup \{ e_2 \}] \). Let \( T_{11} \) and \( T_{12} \) denote the trails on either side of \( e_1 \) such that \( T_{11} \cup \{ e_1 \} \cup T_{12} = T_1 \). Similarly, let \( T_{21} \cup \{ e_2 \} \cup T_{22} = T_2 \). Now suppose, without loss of generality, that \( x = |E(T_{11})| \leq y = |E(T_{12})| \), so that \( x + y = a \).

It is easily seen that if \( y \geq \left\lceil \frac{2a}{3} \right\rceil + 1 \), then by taking the trail \( T_{12} \cup T \) or \( T_{12} \cup (T - e_1) \), as the case may be, we get a trail of length at least \( \left\lceil \frac{2a}{3} \right\rceil + 1 + b \) and therefore an open trail of length at least \( \left\lceil \frac{2a}{3} \right\rceil + 1 + b - 1 \geq \frac{2a-2}{3} + b > \frac{2a-4}{3} + b \geq \left\lceil \frac{2a-6}{3} \right\rceil + b \) in \( G \), a contradiction. Therefore
\[\left\lceil \frac{a}{3} \right\rceil \leq y \leq \left\lceil \frac{2a}{3} \right\rceil.\] Moreover, each \(T_{ij}, i, j \in \{1, 2\}\) has length at least \(\left\lceil \frac{a}{3} \right\rceil\), since \(x = a - y \geq a - \left\lceil \frac{2a}{3} \right\rceil \geq a - \frac{2a}{3} = \frac{a}{3} \geq \left\lceil \frac{a}{3} \right\rceil\).

Note that \(T_{11}\) and \(T_{12}\) are necessarily edge disjoint as are \(T_{21}\) and \(T_{22}\). \(T_{12}\) must intersect \(T_{21}\) and \(T_{22}\), otherwise we get an open trail of length at least \(\left\lceil \frac{a}{3} \right\rceil + b - 2 + \left\lceil \frac{a}{3} \right\rceil \geq \frac{a}{3} + \frac{a-2}{3} + b - 2 = \frac{5a-16}{6} + b > \left\lceil \frac{2a-6}{3} \right\rceil + b\) in \(G\); containing \(T_{12}\), \(T - e_1 - e_2\) and \(T_{21}\) or \(T_{22}\).

In the following, when we say that \(T_{21}\) intersects \(T_{12}\) first we mean that there is a trail starting from an end-vertex of \(e_2\), following \(T_{21}\) and ending with an edge of \(T_{12}\), containing no edge of \(T_{11}\). Similarly for \(T_{22}\) intersecting \(T_{12}\) first or \(T_{21}\) intersecting \(T_{11}\) first. Note that since \(T_{11}\) and \(T_{12}\) are disjoint and \(T_{12}\) intersects \(T_{21}\) and \(T_{22}\), we must have that \(T_{21}, i \in \{1, 2\}\) intersects one of \(T_{11}\) and \(T_{12}\) first.

Suppose that both \(T_{21}\) and \(T_{22}\) intersect \(T_{12}\) first. Then we obtain an open trail of length at least \(x + b - 1 + \left\lceil \frac{y}{3} \right\rceil \geq a - y + \frac{y}{3} + b - 1 \geq a - \frac{1}{3} y - 1 + b \geq a - \frac{1}{3} \left\lceil \frac{2a}{3} \right\rceil - 1 + b \geq a - \frac{1}{3} \left(\frac{2a}{3}\right) - 1 + b = \frac{2a-3}{3} + b > \left\lceil \frac{2a-6}{3} \right\rceil + b\) in \(G\); containing \(T_{11}, T - e_1\) and at least a half of \(T_{12}\).

Now, suppose that \(T_{21}\) or \(T_{22}\) intersects \(T_{11}\) first, say \(T_{21}\). Then we obtain an open trail of length at least \(y + \left\lceil \frac{a}{3} \right\rceil + b - 2 = y + \left[\frac{1}{3}(a - y)\right] + b - 2 \geq y + \frac{a - y}{3} + b - 2 \geq \frac{a}{3} + \frac{1}{3} \left\lceil \frac{a}{3} \right\rceil + b - 2 \geq \frac{2a-6}{3} + b\) in \(G\); containing \(T_{12}, T - e_1 - e_2\) and at least a half of \(T_{11}\).

We remark that a similar result has been proved for vertex partitions and \(W_k\) in [5].

**Theorem 4.2**. For all positive integers \(k\) and \(n \geq 9\), \(\rho(W_n^*) \leq \left\lceil \frac{3k}{2n-6} \right\rceil\).

**Proof**. From Theorem 4.1 it follows by induction on \(c\) that \(W_n^* \subseteq cW_n^*\) for all positive integers \(c\) and \(n\). Now, with \(c = \left\lceil \frac{3k}{2n-6} \right\rceil\) we have that \(W_k^* \subseteq cW_n^* \subseteq cW_n^*\).

**Theorem 4.3**. For all positive integers \(k\) and \(n \geq 2\), \(\left\lceil \frac{k-2}{n-1} \right\rceil + 1 \leq \rho(W_k) \leq 2k\).

**Proof**. We first show that \(W_{2ac+2} \nsubseteq cW_{2a+1}\) for every positive integer \(c\): Clearly, \(G = K_{ac+1,t} \in W_{2ac+2}\) for every \(t\). Let \(t\) be large and suppose that \(G \in cW_{2a+1}\). Let \(\{E_1, E_2, \ldots, E_c\}\) be a corresponding decomposition
of $E(G)$. As in the proof of Theorem 3.2 we get, if $t$ is large enough, for some $i \in \{1, 2, \ldots, c\}$ that $K_{a+1,a+2} \subseteq G[E_i]$, a contradiction.

Now let $a = \frac{n-1}{2}$ and $c = \left\lfloor \frac{k-2}{n-1} \right\rfloor$. Since $k \geq 2ac + 2$ we have $W_k \supseteq W_{2ac+2} \not\subseteq cW_n$. Therefore $\rho_{W_k}(W_k) \geq c + 1$.

For the upper bound we have $W_k \subseteq D_k \subseteq kD_1 \subseteq 2kW_2 \subseteq 2kW_n$ from Theorem 3.3 and the well-known fact that every tree has a $2(W_2 \cap D_1)$ edge decomposition. □

Acknowledgement

The authors wish to thank their supervisor, Prof. I. Broere, for his criticism and assistance in the final preparation of this paper.

References


Received 13 June 2001
Revised 5 April 2002