EFFECT OF EDGE-SUBDIVISION ON VERTEX-DOMINATION IN A GRAPH

Amitava Bhattacharya

and

Gurusamy Rengasamy Vijayakumar

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba
Mumbai 400 005, India

e-mail: amitava@math.tifr.res.in

e-mail: vijay@math.tifr.res.in

Abstract

Let $G$ be a graph with $Δ(G) > 1$. It can be shown that the domination number of the graph obtained from $G$ by subdividing every edge exactly once is more than that of $G$. So, let $ξ(G)$ be the least number of edges such that subdividing each of these edges exactly once results in a graph whose domination number is more than that of $G$. The parameter $ξ(G)$ is called the subdivision number of $G$. This notion has been introduced by S. Arumugam and S. Velammal. They have conjectured that for any graph $G$ with $Δ(G) > 1$, $ξ(G) ≤ 3$. We show that the conjecture is false and construct for any positive integer $n ≥ 3$, a graph $G$ of order $n$ with $ξ(G) > 3 \log_2 n$. The main results of this paper are the following: (i) For any connected graph $G$ with at least three vertices, $ξ(G) ≤ γ(G) + 1$ where $γ(G)$ is the domination number of $G$. (ii) If $G$ is a connected graph of sufficiently large order $n$, then $ξ(G) ≤ 4 \sqrt{n \ln n} + 5$.

Keywords: domination number, subdivision number, matching.

2000 Mathematics Subject Classification: 05C69.
1. Introduction

All graphs considered in this paper are finite and have neither loops nor multiple edges. For definitions not given here and notations not explained, we refer to [2]. For a graph $G$, unless otherwise specified, $V(G)$ and $E(G)$ denote respectively the vertex-set and the edge-set of $G$.

Let $G = (V, E)$ be a graph. For any $a \in V$, its neighbourhood—the set of all vertices which are joined to $a$—is denoted by $N(a)$. (Sometimes it is denoted by $N_G(a)$ to avoid ambiguity when more graphs are under consideration.) The closed neighbourhood of $a$—the set $N(a) \cup \{a\}$—is denoted by $N[a]$. Its degree—the number of vertices in $N(a)$—is denoted by $\deg a$. Occasionally we use $I(a)$ or $I_G(a)$ to denote the set of all edges incident with $a$. By $\delta(G)$ and $\Delta(G)$, we mean $\min_{x \in V} \deg x$ and $\max_{x \in V} \deg x$ respectively. For any $A \subseteq V$, $N(A) = \bigcup_{x \in A} N(x)$. The induced subgraph defined on $A$ is denoted by $G[A]$.

A dominating set of a graph $G$ with vertex-set $V$, is a subset $D$ of $V$ such that each vertex of $V - D$ has a neighbour in $D$. The domination number of $G$ is the least number that can be the cardinality of a dominating set. The domination number of a graph $G$ is denoted by $\gamma(G)$ or simply $\gamma$ when there is no ambiguity regarding the graph whose domination number is referred to by $\gamma$. (This convention will be adopted for other parameters also.)

**Remark 1.1.** Let $G$ be a connected graph with at least two vertices. Since any spanning tree is bipartite, $V(G)$ has a bipartition $\{X, Y\}$ such that every vertex of $X$ has a neighbour in $Y$ and vice versa. Therefore both $X$ and $Y$ are dominating sets of $G$ and it follows that $\gamma(G) \leq \min\{|X|, |Y|\} \leq \frac{1}{2}|V(G)|$.

**Definition 1.2.** Let $G$ be a graph and $uv$ be an edge of $G$. By subdividing the edge $uv$ we mean forming a graph $H$ from $G$ by adding a new vertex $w$ and replacing the edge $uv$ by $uw$ and $wv$. (Formally, $V(H) = V(G) \cup \{w\}$ and $E(H) = (E(G) - \{uv\}) \cup \{uw, wv\}$.) The graph obtained from $G$ by subdividing each edge exactly once is denoted by $S(G)$.

**Remark 1.3.** If $G$ is a graph and $H$ is any graph obtained from $G$ by subdividing some edges of $G$, then $\gamma(H) \geq \gamma(G)$. (From a minimum dominating set $D$ of $H$, by replacing each vertex $x$ of $D - V(G)$ by a vertex of $V(G)$ which is adjacent to $x$, we get a dominating set $D'$ of $G$ such that $|D'| \leq |D|$.)
In [6] it has been observed that for a connected graph $G$ with at least 3 vertices, $\gamma(S(G)) > \gamma(G)$. (A lengthy argument has been given to prove this. A simpler proof is the following: Let $V$ and $E$ be respectively the vertex-set and the edge-set of $G$ and $n$ be the number of vertices. Let $D$ be a minimum dominating set of $S(G)$. Let $D_1 = V \cap D$ and $D_2 = D - D_1$. In $S(G)$, since each vertex of $D_1$ dominates exactly one vertex of $V$ and each vertex of $D_2$ dominates exactly two vertices of $V$, it follows that $|D_1| + 2|D_2| \geq n$. If $D_1 \neq \emptyset$, then $2\gamma(S(G)) = 2|D_1| + 2|D_2| \geq n + 1$; otherwise, $D = V' - V$ where $V'$ is the vertex-set of $S(G)$ and $\gamma(S(G)) = |V' - V| = |E| \geq n - 1$. In either case, $\gamma(S(G)) > \frac{n}{2}$ and by Remark 1.1, it follows that $\gamma(S(G)) > \gamma(G)$.)

By the above observation, obviously for any graph $G$ with $\Delta > 1$, $\gamma(G) < \gamma(S(G))$. This has prompted S. Arumugam to ask the following question: For a graph $G$ with $\Delta > 1$, what is the minimum number of edges to be subdivided exactly once so that the domination number of the resulting graph exceeds that of $G$?

**Definition 1.4.** Let $G$ be a graph with $\Delta > 1$. The least number that can be the cardinality of a set of edges such that subdividing each of them exactly once results in a graph with domination number more than that of $G$, is called the subdivision number of $G$ and is denoted by $\xi(G)$.

In [6], S. Velammal has computed the above parameter for a number of graphs. An interesting result of [6] in this regard is the following.

**Proposition 1.5.** For any tree $T$ of order $\geq 3$, $\xi \leq 3$.

Finding that $\xi \leq 3$ holds for each of the graphs considered in this regard in [6], S. Arumugam and S. Velammal have conjectured that for any connected graph $G$ with at least 3 vertices, $\xi(G) \leq 3$.

In [3], an upper bound for the subdivision number of a graph in terms of the minimum degrees of adjacent vertices has been found.

In this paper we show that the above conjecture is false by exhibiting a graph with $\xi > 3$. Using the method for constructing this graph we prove the following result.

**Proposition 1.6.** For any integer $n \geq 3$, there exists a graph of order $n$ such that $\xi > \frac{1}{3} \log_2 n$.

The main results of this paper are the following theorems.
Theorem 1.7. For a connected graph with at least 3 vertices, \( \xi \leq \gamma + 1 \).

In [4], a different proof of the above result is given.

Theorem 1.8. For a connected graph of large order \( n \), \( \xi \leq 4\sqrt{n} \ln n + 5 \).

We also give a proof of Proposition 1.5, since the argument given in [6] to prove this result is incorrect.

2. Results

First let us prove Proposition 1.5.

If \( T \) is a path, then it is easy to verify that the conclusion holds. So, assume that \( \Delta(T) \geq 3 \). If \( P = (v_0, v_1, \ldots, v_n) \) is a path in \( T \) such that \( \deg v_0 > 2, \deg v_i = 2 \) for \( 0 < i < n \) and \( \deg v_n = 1 \), then \( P \) is said to be a hanging path and \( v_0 \) is called the support of \( P \). If any hanging path is of length more than 2, then subdividing three of its edges shows that \( \xi(T) \leq 3 \).

So we assume the following.

\[ (** \) Length of any hanging path is at most 2.\]

Clearly removal of all the hanging paths but retaining their supports yields a tree \( T' \). Let \( u \) be a pendant vertex of \( T' \). Then \( u \) supports at least two hanging paths. Now by \( (** \) \) we have two cases.

Case a. \( u \) is incident with a pendant edge of \( T \). Subdivide this pendant edge. If \( u \) is incident with one more pendant edge of \( T \), then we find that \( \xi(T) = 1 \); otherwise subdividing the two edges of any other hanging path supported by \( u \) shows that \( \xi(T) \leq 3 \).

Case b. Every hanging path supported by \( u \) is of length 2. Now subdivide the two edges of one hanging path supported by \( u \). If \( V(T') = \{ u \} \), then we find that \( \xi(T) = 2 \). Otherwise, subdividing the edge of \( T' \) which is incident with \( u \) shows that \( \xi(T) \leq 3 \).

This completes the proof.

Remark 2.1. Let \( T \) be as above and \( H \) be any graph. If a graph \( G \) is formed by joining a pendant vertex \( a \) of \( T \) with a vertex \( b \) of \( H \) (formally \( V(G) = V(T) \cup V(H), V(T) \cap V(H) = \emptyset \) and \( E(G) = E(T) \cup E(H) \cup \)))
\{ab\}, then \(\xi(G) \leq 3\). The above proof works with slight modification just before choosing \(u\): We can assume that \(|V(T')| > 1\) for otherwise \(T\) is simply a graph obtained from a star by subdividing some of its edges and the conclusion can be easily verified; now let \(u\) be a pendant vertex of \(T'\) such that \(a\) does not lie on any hanging path supported by \(u\). (Note that hanging paths supported by different vertices are vertex-joint.) With this modification, in Case b the possibility that \(V(T') = \{u\}\) does not arise.

**Disproving the Conjecture.** Now let us construct a graph with \(\xi > 3\). Let 
\[X = \{1, 2, \ldots, 10\}\]
Let \(S = \{A \subset X : |A| = 4\}\). \(S\) has \(\binom{10}{4}\) elements.

Let \(G\) be the bipartite graph with bipartition \(\{X, S\}\) and adjacency defined as follows: For any \(x \in X\) and \(A \in S\), \(x\) is adjacent to \(A \iff x \in A\).

Let \(D\) be any dominating set of \(G\). If \(|D \cap X| \leq 4\), then \(|D| \geq |D \cap S| \geq \binom{6}{4}\). If \(|D \cap X| = 5\), then \(|D \cap S| \geq 5\) implying \(|D| \geq 10\). If \(|D \cap X| = 6\), then \(D \cap S \neq \emptyset\) implying \(|D| \geq 7\). Therefore, it can be easily seen that

\[\gamma(G) = 7.\]

Let \(\alpha_i A_i, 1 \leq i \leq 3\) be three edges of \(G\). Let \(H\) be the graph obtained from \(G\) by subdividing these three edges. For \(i = 1, 2, 3\), choose an element \(\beta_i \in A_i - \{\alpha_1, \alpha_2, \alpha_3\}\). Let \(D_1\) be a subset of \(X\) such that \(\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \subseteq D_1\) and \(|D_1| = 6\). Let \(D = D_1 \cup \{X - D_1\}\). It can be verified that \(D\) is a dominating set of \(H\). Now by (1) and Remark 1.3, it follows that

\[\gamma(H) = 7.\]

Now (1) and (2) \(\Rightarrow \gamma(H) = \gamma(G)\). Therefore \(\xi(G) > 3\).

**Remark 2.2.** In the above example taking \(X = \{1, 2, \ldots, 9\}\) also works but needs a little more computations.

**Proof of Proposition 1.6.** Let \(n\) be any positive integer. The proposition trivially holds when \(n \leq 7\) since \(\frac{1}{3} \log_2 n < 1\). When \(n = 8\) or 9, we can construct a graph of order \(n\) with \(\xi = 2\) and the conclusion holds. So, let us assume that \(n \geq 10\).

Let \(k\) be the positive integer such that

\[3k - 2 + \binom{3k - 2}{k} \leq n < 3k + 1 + \binom{3k + 1}{k + 1}.\]

Note that \(k \geq 2\). Now let
Let $G$ be as defined in the above example. Then by construction, order of $G$ is $n$. Let us show that $\gamma(G) = 2k - 1$. Let $D$ be a dominating set of $G$ and $\ell = |D \cap X|$. We can assume that $\ell \leq 2k - 2$. Then

$$|D| \geq \ell + \binom{3k - 2 - \ell}{k} \geq \ell + 2k - 2 - \ell + 1$$

(by using the fact that $\binom{k+m}{k} \geq m + 1$ when $m \geq 0$.)

$$= 2k - 1.$$ 

Therefore $\gamma(G) \geq 2k - 1$; since a dominating set $D$ of cardinality $2k - 1$ can be easily constructed such that $|D \cap X| = 2k - 2$, it follows that $\gamma(G) = 2k - 1$.

Let $\{\alpha_i A_i : 1 \leq i \leq k - 1\}$ be a set of $k - 1$ edges and let $H$ be the graph obtained from $G$ by subdividing these $k - 1$ edges. For any $i \leq k - 1$, choose a positive integer $\beta_i \in A_i - \{\alpha_j : 1 \leq j \leq k - 1\}$. Let $D$ be a subset of $X$ such that $\{\alpha_i : 1 \leq i \leq k - 1\} \cup \{\beta_i : 1 \leq i \leq k - 1\} \subseteq D$ and $|D| = 2k - 2$. It can be verified that $D \cup \{X - D\}$ is a dominating set of $H$. Therefore by Remark 1.3, $\gamma(H) = 2k - 1$. Thus we have $\gamma(H) = \gamma(G)$ and it follows that $\xi(G) \geq k$.

Since $n < 3k + 1 + \binom{3k + 1}{k+1} = 1 + 3k + \binom{3k}{k} + \binom{3k}{k+1} < 2^{3k}$, we have $3k > \log_2 n$ implying that $\xi > \frac{1}{3} \log_2 n$. This completes the proof. 

A set $M$ of edges in a graph $G$ is called a matching of $G$ (sometimes an independent set of edges in $G$) if no two edges of $M$ have a common end-vertex. The cardinality of a largest matching of $G$ is denoted by $\mu(G)$. The following result is quite well known. (cf. [5, p. 58]; for the sake of completeness, we give a proof of this result.)

**Lemma 2.3.** If $G$ is a graph without isolated vertices, then $\gamma \leq \mu$.

**Proof.** Let $M$ be a maximum matching. Let $S$ be the set of vertices which are not end-vertices of the edges in $M$. If $a$ is any vertex in $S$, then $a$ is not joined to any other vertex in $S$ since $M$ is a maximum matching; therefore $a$ is joined to an end-vertex, say $x$, of an edge in $M$, since $G$ does not have isolated vertices. Let $y$ be the other end of this edge. If $b$ is any other vertex in $S$, then $b$ is not joined to $y$ for otherwise $(M - \{xy\}) \cup \{ax, by\}$ would
be a matching of size $|M| + 1$. Hence it is possible to choose a dominating set $D$ of cardinality $\mu$ having exactly one end of each edge in $M$. Therefore $\gamma \leq \mu$.

**Remark 2.4.** Let $G$ be a graph with vertex set $V$; suppose $A$ is a subset of $V$ such that $G[V - A]$ has no isolated vertex and $\mu(G[A]) > \gamma(G[A])$. Then because of

$$
\mu(G) \geq \mu(G[A]) + \mu(G[V - A]),
\gamma(G) \leq \gamma(G[A]) + \gamma(G[V - A]) \text{ and }
\mu(G[V - A]) \geq \gamma(G[V - A]) \text{ (by Lemma 2.3)}
$$

we have $\mu(G) > \gamma(G)$.

**Lemma 2.5.** Suppose $G$ is a graph with vertex-set $V$ which can be partitioned as $\{A_1, B_1, A_2, B_2\}$ such that the following hold:

- For $i = 1, 2$, every vertex of $A_i$ is adjacent to every vertex of $B_i$.
- $|A_1|, |A_2| \geq 2, |B_1| \geq 3$ and $|B_2| \geq 1$.
- A vertex of $B_1$ is adjacent to a vertex of $A_2$.

Then $\mu(G) > \gamma(G)$.

**Proof.** If $|A_2| = 2$ or $|B_2| = 1$ then $\gamma(G) \leq 3$ and $\mu(G) \geq 4$. So suppose $|A_2| \geq 3$ and $|B_2| \geq 2$. Then $\mu(G) \geq 5$ and $\gamma(G) \leq 4$. Thus it follows that $\mu(G) \geq 5$ and $\gamma(G) \leq 4$. Thus it follows that $\mu(G) > \gamma(G)$.

**Definition 2.6.** Let $G$ be a graph with vertex-set $V$; a subset $X$ of $V$ is said to be modular in $G$, if all the vertices in $X$ have same neighbourhood and $G[V - (X \cup N(X))]$ has no isolated vertices. If in addition $X$ dominates $G$, then $G$ is called a module; $G$ is a proper module if $|X| \geq 2$ and $|N(X)| \geq 3$.

Note that any modular set $X$ of a graph is independent; i.e., no two vertices of $X$ are adjacent. If $G = (V, E)$ is a module with $E \neq \emptyset$, then $\gamma(G) \leq 2$. ($G$ can be imagined as a graph obtained from the complete bipartite graph with bipartition $\{X, V - X\}$ by adding edges having end-vertices in $V - X$ only.)

**Lemma 2.7.** For a graph $G$ without isolated vertices, one of the following holds.
(i) \( \mu(G) > \gamma(G) \).

(ii) Each connected component is a proper module.

(iii) There exists a modular subset \( A \) of \( V(G) \) such that either \(|A| = 1\) or \(|N(A)| \leq 2\).

**Proof.** By induction; assume that for any graph of order less than that of \( G \), the theorem holds. Let \( \alpha \) be any vertex of \( G \) such that \( \deg \alpha = \delta(G) \). Let \( A = \{ x \in V(G) : N(x) = N(\alpha) \} \). Let \( H = G[V(G) - (A \cup N(A))] \). If \( V(H) = \emptyset \) then \( G \) is a module and therefore either (ii) or (iii) holds.

When \( V(H) \neq \emptyset \), by the construction of \( A \), \( H \) has no isolated vertex. We can assume the following for otherwise (iii) holds.

\( (**) \quad |A| \geq 2 \) and \( |N(A)| \geq 3 \).

Applying the induction hypothesis for \( H \) we have the following cases.

**Case 1.** \( \mu(H) > \gamma(H) \).

Since \( G[A \cup N(A)] \) has no isolated vertex, by Remark 2.4, (i) holds.

**Case 2.** Each component of \( H \) is a proper module.

If there is one component \( J \) such that \( N(V(J)) \cap N(A) \neq \emptyset \), then by Lemma 2.5 and \( (**), \mu(G[A \cup N(A) \cup V(J)]) > \gamma(G[A \cup N(A) \cup V(J)]) \) and (i) holds by Remark 2.4; otherwise the components of \( G \) are those of \( H \) and \( G[A \cup N(A)] \) and therefore (ii) holds.

**Case 3.** A subset \( B \) of \( V(H) \) is modular in \( H \) such that either \(|B| = 1\) or \(|N_H(B)| \leq 2\).

Let \( X = A \cup N(A) \cup B \cup N_H(B) \). Note that \( G[V(G) - X] \) has no isolated vertex. First suppose \(|B| \geq 2\). If there is any edge from \( N(A) \) to \( B \) then by \( (**)) \) and Lemma 2.5, \( \mu(G[X]) > \gamma(G[X]) \) and by Remark 2.4, (i) holds; otherwise \( N(B) = N_H(B) \) and (iii) holds with \( B \) in place of \( A \).

Now suppose \( B \) has only one vertex, say \( \alpha \). If \( \alpha \) is not joined to every vertex of \( N(A) \), then (iii) holds with \( B \) in place of \( A \). So assume that \( \alpha \) is joined to every vertex of \( N(A) \). Then \( \gamma(G[X]) \leq 2 \) and \( \mu(G[X]) \geq 3 \); therefore again we have \( \mu(G[X]) > \gamma(G[X]) \) and (i) holds. This completes the proof.

**Proof of Theorem 1.7.** For a graph \( G \), by using induction on its order, let us show the following:
(**) If $G$ is connected and has at least three vertices then there exists a set $F$ of edges of order $\gamma$ or $\gamma + 1$ such that $F$ contains a matching of order $\gamma$ and subdividing the edges of $F$ results in a graph whose domination number is more than that of $G$.

When $|V(G)| = 3$, (**) is obvious. So, let $|V(G)| > 3$ and assume that for any graph $H$ with $|V(H)| < |V(G)|$, (**) holds with $H$ in place in $G$. By Lemma 2.7, we have three cases.

*Case 1.* $G$ has a matching $M$ of order $\gamma + 1$.
Then (**) holds with $M$ in place of $F$.

*Case 2.* $G$ is a module.
Then there exists a modular subset $A$ of $V(G)$ such that $V(G) = A \cup N(A)$. If $\gamma(G) = 1$, it is easy to see that (**) holds. So let $\gamma(G) = 2$. Let $a, b$ be two distinct vertices in $A$.
Suppose $N(A)$ has two adjacent vertices $x, y$. Since $\gamma(G) = 2$, there must be one more vertex $z \in N(A)$. If $|N(A)| \geq 4$, then $\mu(G) \geq 3$. So, let $N(A) = \{x, y, z\}$. Then neither $x$ nor $y$ is joined to $z$. If $|A| \geq 3$, then also we have $\mu(G) \geq 3$. So, let $A = \{a, b\}$. Now subdividing the edges of the matching $\{a, b\}$ shows that $\xi(G) = 2$ and (**) holds.
If $N(A)$ is a set of independent vertices, then subdividing the edges $ax, ay, bx$ where $x, y$ are any two arbitrary vertices in $N(A)$ shows that (**) holds.

*Case 3.* (iii) of Lemma 2.7 holds.
Let $H_1 = G[A \cup N(A)]$ and $H_2 = G[V(G) - (A \cup N(A))]$. We can assume that $\gamma(H_1) + \gamma(H_2) = \gamma(G)$ for otherwise $\mu(G) \geq \mu(H_1) + \mu(H_2) \geq \gamma(H_1) + \gamma(H_2) > \gamma(G)$ and (**) holds.

*Subcase a.* $|A| = 1$.
Let $A = \{a\}$ and $x$ be any vertex in $N(A)$.
If $H_2$ has a component $K$ of order $\geq 3$, then by induction hypothesis, there exists a set $F' \subseteq E(K)$ such that (**) holds with $K$ and $F'$ in places of $G$ and $F$ respectively. Since $H_2[V(H_2) - V(K)]$ has no isolated vertex, it has a matching of size $\gamma(H_2) - \gamma(K)$. Now taking $F = \{ax\} \cup F' \cup M$ it can be verified that (**) holds.
So assume that $H_2$ is a union of copies of $K_2$. If $\deg a > 1$, then (**) holds with $F = \{ax\} \cup E(H_2)$. So suppose $N(a) = \{x\}$. For any $e \in E(H_2)$, we can assume that both of its end-vertices are not joined to $x$,
A. Bhattacharya and G.R. Vijayakumar

for otherwise we would have $\gamma(G) < \mu(G)$. Therefore by connectivity of $G$, $x$ is joined to exactly one vertex of each edge in $E(H_2)$ and $(**)$ holds with $F = \{ax\} \cup \{xy\} \cup E(H_2)$ where $y$ is a vertex in $V(H_2)$ which is joined to $x$.

Subcase b. $|A| \geq 2$.

Then $|N(A)| \leq 2$. Let $M$ be any matching in $H_2$ of size $\gamma(H_2)$. Let $a, b$ be two distinct vertices in $A$. If $|N(A)| = 1$, then $a, b$ are pendant with the same support, say $x$; subdividing the edge $ax$ shows that $\xi(G) = 1$ and obviously $(**)$ holds with $F = \{ax\} \cup M$.

So suppose $N(A)$ contains one more vertex, say $y$. If $\gamma(H_1) = 1$, then $F = \{ax, by\} \cup M$ is a matching of size $\gamma(G) + 1$ and $(**)$ holds; so let $\gamma(H_1) = 2$. Subdividing the edges $ax, ay, bx$ shows that $\xi(G) \leq 3$ and $(**)$ holds with $F = \{ax, ay, bx\} \cup M$. $\blacksquare$

Now we prove the second main result of this paper. The main tool used in the proof is Alon’s result (cf. [1, Page 4]) on domination number of a graph: Any graph $G$ has a dominating set of size $\leq n\frac{1+\ln(\delta+1)}{\delta+1}$ where $n$ is the number of vertices.

**Proof of Theorem 1.8.** First we settle a few simple cases. (Throughout this proof, we consider a number of cases. Whenever a case is under consideration, it is assumed that the previous cases do not hold.)

**Case 1.** $G$ has two pendant vertices with same support.

By subdividing one of them, we find $\xi = 1$.

**Case 2.** There is an edge $e \in E(G)$ such that $G - e$ has two connected components $G_1$ and $G_2$ with the property that $G_1$ is a tree with at least 3 vertices.

Then by Remark 2.1, $\xi \leq 3$.

**Case 3.** There is a path $(u, v, w, x)$ such that $\deg(u) = \deg(x) = 1$.

Subdividing the three edges of this path shows that $\xi \leq 3$.

So let us assume that none of the above cases holds. Removing all the hanging paths but retaining their supports results in a connected graph $G'$ such that the following hold:

Every pendant vertex in $G$ is connected to a vertex in $G'$ by a path of length at most 2. Any such path of length 1 cannot have a vertex in
common with any other path. Any such path of length 2 cannot have an edge in common with any other path.

Let \( m = \lceil \sqrt{n} \ln n \rceil \). For any pendant vertex \( u \) of \( G \) let \( u^* \) denote its support.

Let \( S_1 = \{ v \in V(G') \mid \deg(v) \leq m \} \),
\[
S_2 = \{ v \in V(G') \mid \deg(v) > m \},
\]
\( V_1 = \{ v \in V(G) \mid \deg(v) = 1 \text{ and } v^* \in V(G') \} \),
\( V_2 = \{ v \in V(G) \mid \deg(v) = 1, \deg(v^*) = 2, \text{ and } N(v^*) \cap V(G') \neq \emptyset \} \)
and \( S = \{ v \in S_1 \mid N(v) \cap V_1 \neq \emptyset \} \).

Let \( \ell \) be the number of vertices in \( V_2 \) which are joined to vertices in \( S_2 \) by paths of length 2. If there is a vertex \( v \in V_2 \) which is joined by a path of length 2 to a vertex \( u \in S_1 \); i.e., then by subdividing the edges of this path and all the edges of \( E(G') \) which are incident with \( u \), we find that \( \xi \leq m + 2 \).

Let \( k \) be the number of vertices in \( V_1 \) with supports in \( S_2 \). Now let us settle three more cases.

**Case 4.** There exist \( u, v \in S_1 \) which are adjacent.
In this case by subdividing all the edges in \( I(u) \cup I(v) \) we find that \( \xi \leq 2m \).

**Case 5.** There exist \( u \in S_1 \) and \( v \in S_1 - S \) such that \( N(u) \cap N(v) \neq \emptyset \).
Fix a vertex \( a \in N(u) \cap N(v) \). Then subdivide all edges in \( I_G(u) \cup I_G(v) \).
Also for each vertex \( x \in (N(u) \cup N(v)) - \{a\} \) we subdivide an edge \( xy \in G' \) where \( y \notin \{u, v, a\} \). If there is any edge \( ab \) with \( b \in V_1 \), it is also subdivided.
If there are paths \( (a, b, c) \) with \( c \in V_2 \) then both the edges of one such path are also subdivided. Hence in this case the subdivision number is at most \( 4m + 1 \).

**Case 6.** There exist \( u, v \in S_1 - S \) and \( v_1 v_2 \in E(G') \) with \( v_1 \in N(u) \) and \( v_2 \in N(v) \).
We subdivide \( v_1 v_2 \) and the edges in \( I_{G'}(u) \cup I_{G'}(v) \). Also for each vertex \( x \in N(u) \cup N(v) \) we subdivide an edge \( xy \in E(G') \) with \( y \notin \{u, v, v_1, v_2\} \). If there is any edge \( a_1 a_2 \in E(G') \) with \( a_1 \in V_1 \) and \( a_2 \in \{v_1, v_2\} \) then subdivide it. If there are paths \( (a_1, a_2, v_1) \) with \( a_1 \in V_2 \), then subdivide both the edges of one such path; this is repeated with \( v_2 \) in place of \( v_1 \). Hence the subdivision number is at most \( 4m + 5 \).
If none of the cases considered so far holds, then by using Alon’s result mentioned above we have

\[ \ell + k + |S_1| \leq \gamma(G) \leq \ell + k + \rho + |S_1| \]

where \( \rho = |S_2| \frac{\ln(m+1)+1}{m+1}. \)

**Case 7.** \( k > \frac{\sqrt{n}}{\ln n}. \)

Then there exist two pendant vertices \( u, v \) with supports in \( S_2 \) such that \( N(N(u)) \cap N(N(v)) \neq \emptyset. \) Let \( x \) and \( y \) be supports of \( u \) and \( v \) respectively. Let \( p \) be a vertex in \( N(x) \cap N(y). \) Now we subdivide the edges \( ux, xp, py \) and \( yv. \) We also subdivide \( m \) edges in \( I(x). \) If any such edge is incident with a vertex \( a \in N(S_1 - S) \) then we also subdivide the edge \( ab \) where \( b \) is in \( S_1 - S. \) Now the domination number of the resulting graph is at least \( |S_1| + k + \ell + m \) which is more than that of \( G \) (if \( n \geq 8). \) Hence the subdivision number is at most \( 2m + 4. \)

**Case 8.** \( |S_1 - S| \leq \frac{m}{3}. \)

We take a matching \( M \) of size \( \min \left( \frac{m}{3}, |V(G') - (S_1 - S)| \right) \) in \( G'[V(G') - (S_1 - S)] \) and subdivide all the edges in the matching and all edges \( uu' \) where \( u' \) is an end-vertex of an edge in this matching. For the resulting graph \( H, \) the size of the dominating set is at least \( \gamma(H) \geq \ell + |S| + \min \left( \frac{m}{3}, |V(G') - (S_1 - S)| \right). \) If \( |M| < \frac{m}{3} \) then \( \gamma(H) > |S| + \ell = \gamma(G); \) otherwise for \( n \geq 235, \) by using (\( \ast \)), it can be verified that \( \ell + |S| + \frac{m}{3} > \ell + k + \rho + |S_1| \geq \gamma(G). \) Thus, when \( n \geq 235, \) \( \gamma(G) < \gamma(H). \)

**Case 9.** \( |S_1 - S| > \frac{m}{3}. \)

First we fix a set \( S'_1 \subset S_1 - S \) with \( |S'_1| = \frac{m}{4}. \) Next we get a matching \( M \) of size \( \frac{m}{4} \) in \( S_2 \) such that each edge in \( M \) has an end in \( N(S'_1) \) and for any \( a \in S'_1, \) there is at most one edge in \( M \) having an end-vertex in \( N(a). \) Now we subdivide all edges in \( M \) and for every vertex \( b \in S'_1 \) an edge \( bb_1 \in E(G'). \)

We also subdivide all edges of the form \( uv \in E(G) \) where \( u \) is an end-vertex of an edge in \( M \) and \( v \in S'_1. \) In the resulting graph \( H \) the domination number is at least \( |S_1| + \frac{m}{4} + \ell. \) So as in the last case, if \( n \geq 235 \) then \( \gamma(H) > \gamma(G). \) Therefore the subdivision number in this case is at most \( \frac{3m}{4}. \)

Thus we conclude that if \( n \geq 235 \) then \( \xi \leq 4\sqrt{n} \ln n + 5. \)
Acknowledgement
We thank the referee for giving suggestions and pointing out the errors in the earlier version of this paper.

References


Received 8 June 2001
Revised 20 October 2001