EFFECT OF EDGE-SUBDIVISION ON
VERTEX-DOMINATION IN A GRAPH

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Abstract

Let $G$ be a graph with $\Delta(G) > 1$. It can be shown that the domination number of the graph obtained from $G$ by subdividing every edge exactly once is more than that of $G$. So, let $\xi(G)$ be the least number of edges such that subdividing each of these edges exactly once results in a graph whose domination number is more than that of $G$. The parameter $\xi(G)$ is called the subdivision number of $G$. This notion has been introduced by S. Arumugam and S. Velammal. They have conjectured that for any graph $G$ with $\Delta(G) > 1$, $\xi(G) \leq 3$. We show that the conjecture is false and construct for any positive integer $n \geq 3$, a graph $G$ of order $n$ with $\xi(G) > \frac{1}{2} \log_2 n$. The main results of this paper are the following: (i) For any connected graph $G$ with at least three vertices, $\xi(G) \leq \gamma(G) + 1$ where $\gamma(G)$ is the domination number of $G$. (ii) If $G$ is a connected graph of sufficiently large order $n$, then $\xi(G) \leq 4\sqrt{n} \ln n + 5$.

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1. Introduction

All graphs considered in this paper are finite and have neither loops nor multiple edges. For definitions not given here and notations not explained, we refer to [2]. For a graph \( G \), unless otherwise specified, \( V(G) \) and \( E(G) \) denote respectively the vertex-set and the edge-set of \( G \).

Let \( G = (V, E) \) be a graph. For any \( a \in V \), its neighbourhood—the set of all vertices which are joined to \( a \)—is denoted by \( N(a) \). (Sometimes it is denoted by \( N_G(a) \) to avoid ambiguity when more graphs are under consideration.) The closed neighbourhood of \( a \)—the set \( N(a) \cup \{a\} \)—is denoted by \( N[a] \). Its degree—the number of vertices in \( N(a) \)—is denoted by \( \text{deg} \, a \).

Occasionally we use \( I(a) \) or \( I_G(a) \) to denote the set of all edges incident with \( a \). By \( \delta(G) \) and \( \Delta(G) \), we mean \( \min_{x \in V} \text{deg} \, x \) and \( \max_{x \in V} \text{deg} \, x \) respectively. For any \( A \subseteq V \), \( N(A) = \bigcup_{x \in A} N(x) \). The induced subgraph defined on \( A \) is denoted by \( G[A] \).

A dominating set of a graph \( G \) with vertex-set \( V \), is a subset \( D \) of \( V \) such that each vertex of \( V - D \) has a neighbour in \( D \). The domination number of \( G \) is the least number that can be the cardinality of a dominating set. The domination number of a graph \( G \) is denoted by \( \gamma(G) \) or simply \( \gamma \) when there is no ambiguity regarding the graph whose domination number is referred to by \( \gamma \). (This convention will be adopted for other parameters also.)

**Remark 1.1.** Let \( G \) be a connected graph with at least two vertices. Since any spanning tree is bipartite, \( V(G) \) has a bipartition \( \{X, Y\} \) such that every vertex of \( X \) has a neighbour in \( Y \) and vice versa. Therefore both \( X \) and \( Y \) are dominating sets of \( G \) and it follows that \( \gamma(G) \leq \min\{|X|, |Y|\} \leq \frac{1}{2} |V(G)| \).

**Definition 1.2.** Let \( G \) be a graph and \( uv \) be an edge of \( G \). By subdividing the edge \( uv \) we mean forming a graph \( H \) from \( G \) by adding a new vertex \( w \) and replacing the edge \( uv \) by \( uw \) and \( wv \). (Formally, \( V(H) = V(G) \cup \{w\} \) and \( E(H) = (E(G) - \{uv\}) \cup \{uw, wv\} \).) The graph obtained from \( G \) by subdividing each edge exactly once is denoted by \( S(G) \).

**Remark 1.3.** If \( G \) is a graph and \( H \) is any graph obtained from \( G \) by subdividing some edges of \( G \), then \( \gamma(H) \geq \gamma(G) \). (From a minimum dominating set \( D \) of \( H \), by replacing each vertex \( x \) of \( D - V(G) \) by a vertex of \( V(G) \) which is adjacent to \( x \), we get a dominating set \( D' \) of \( G \) such that \( |D'| \leq |D| \).)
In [6] it has been observed that for a connected graph $G$ with at least 3 vertices, $\gamma(S(G)) > \gamma(G)$. (A lengthy argument has been given to prove this. A simpler proof is the following: Let $V$ and $E$ be respectively the vertex-set and the edge-set of $G$ and $n$ be the number of vertices. Let $D$ be a minimum dominating set of $S(G)$. Let $D_1 = V \cap D$ and $D_2 = D - D_1$. In $S(G)$, since each vertex of $D_1$ dominates exactly one vertex of $V$ and each vertex of $D_2$ dominates exactly two vertices of $V$, it follows that $|D_1| + 2|D_2| \geq n$. If $D_1 \neq \emptyset$, then $2\gamma(S(G)) = 2|D_1| + 2|D_2| \geq n + 1$; otherwise, $D = V' - V$ where $V'$ is the vertex-set of $S(G)$ and $\gamma(S(G)) = |V' - V| = |E| \geq n - 1$. In either case, $\gamma(S(G)) > \frac{n}{2}$ and by Remark 1.1, it follows that $\gamma(S(G)) > \gamma(G)$.)

By the above observation, obviously for any graph $G$ with $\Delta > 1$, $\gamma(G) < \gamma(S(G))$. This has prompted S. Arumugam to ask the following question: For a graph $G$ with $\Delta > 1$, what is the minimum number of edges to be subdivided exactly once so that the domination number of the resulting graph exceeds that of $G$?

**Definition 1.4.** Let $G$ be a graph with $\Delta > 1$. The least number that can be the cardinality of a set of edges such that subdividing each of them exactly once results in a graph with domination number more than that of $G$, is called the **subdivision number** of $G$ and is denoted by $\xi(G)$.

In [6], S. Velammal has computed the above parameter for a number of graphs. An interesting result of [6] in this regard is the following.

**Proposition 1.5.** For any tree $T$ of order $\geq 3$, $\xi \leq 3$.

Finding that $\xi \leq 3$ holds for each of the graphs considered in this regard in [6], S. Arumugam and S. Velammal have conjectured that for any connected graph $G$ with at least 3 vertices, $\xi(G) \leq 3$.

In [3], an upper bound for the subdivision number of a graph in terms of the minimum degrees of adjacent vertices has been found.

In this paper we show that the above conjecture is false by exhibiting a graph with $\xi > 3$. Using the method for constructing this graph we prove the following result.

**Proposition 1.6.** For any integer $n \geq 3$, there exists a graph of order $n$ such that $\xi > \frac{1}{3} \log_2 n$.

The main results of this paper are the following theorems.
Theorem 1.7. For a connected graph with at least 3 vertices, \( \xi \leq \gamma + 1 \).

In [4], a different proof of the above result is given.

Theorem 1.8. For a connected graph of large order \( n \), \( \xi \leq 4\sqrt{n} \ln n + 5 \).

We also give a proof of Proposition 1.5, since the argument given in [6] to prove this result is incorrect.

2. Results

First let us prove Proposition 1.5.

If \( T \) is a path, then it is easy to verify that the conclusion holds. So, assume that \( \Delta(T) \geq 3 \). If \( P = (v_0, v_1, \ldots, v_n) \) is a path in \( T \) such that \( \deg v_0 > 2 \), \( \deg v_i = 2 \) for \( 0 < i < n \) and \( \deg v_n = 1 \), then \( P \) is said to be a hanging path and \( v_0 \) is called the support of \( P \). If any hanging path is of length more than 2, then subdividing three of its edges shows that \( \xi(T) \leq 3 \). So we assume the following.

\((**)\) Length of any hanging path is at most 2.

Clearly removal of all the hanging paths but retaining their supports yields a tree \( T' \). Let \( u \) be a pendant vertex of \( T' \). Then \( u \) supports at least two hanging paths. Now by \((***)\) we have two cases.

Case a. \( u \) is incident with a pendant edge of \( T \).
Subdivide this pendant edge. If \( u \) is incident with one more pendant edge of \( T \), then we find that \( \xi(T) = 1 \); otherwise subdividing the two edges of any other hanging path supported by \( u \) shows that \( \xi(T) \leq 3 \).

Case b. Every hanging path supported by \( u \) is of length 2.
Now subdivide the two edges of one hanging path supported by \( u \). If \( V(T') = \{u\} \), then we find that \( \xi(T) = 2 \). Otherwise, subdividing the edge of \( T' \) which is incident with \( u \) shows that \( \xi(T) \leq 3 \).

This completes the proof.

Remark 2.1. Let \( T \) be as above and \( H \) be any graph. If a graph \( G \) is formed by joining a pendant vertex \( a \) of \( T \) with a vertex \( b \) of \( H \) (formally \( V(G) = V(T) \cup V(H), V(T) \cap V(H) = \emptyset \) and \( E(G) = E(T) \cup E(H) \cup \ldots \)))
\{ab\}, then \(\xi(G) \leq 3\). The above proof works with slight modification just before choosing \(u\): We can assume that \(|V(T')| > 1\) for otherwise \(T\) is simply a graph obtained from a star by subdividing some of its edges and the conclusion can be easily verified; now let \(u\) be a pendant vertex of \(T'\) such that \(a\) does not lie on any hanging path supported by \(u\). (Note that hanging paths supported by different vertices are vertex-joint.) With this modification, in Case b the possibility that \(V(T') = \{u\}\) does not arise.

### Disproving the Conjecture

Now let us construct a graph with \(\xi > 3\).

Let \(X = \{1, 2, \ldots, 10\}\). Let \(S = \{A \subset X : |A| = 4\}\). \(S\) has \(\binom{10}{4}\) elements. Let \(G\) be the bipartite graph with bipartition \(\{X, S\}\) and adjacency defined as follows: For any \(x \in X\) and \(A \in S\), \(x\) is adjacent to \(A\) \(\iff x \in A\).

Let \(D\) be any dominating set of \(G\). If \(|D \cap X| \leq 4\), then \(|D| \geq |D \cap S| \geq \binom{6}{4}\). If \(|D \cap X| = 5\), then \(|D \cap S| \geq 5\) implying \(|D| \geq 10\). If \(|D \cap X| = 6\), then \(D \cap S \neq \emptyset\) implying \(|D| \geq 7\). Therefore it can be easily seen that

\[
\gamma(G) = 7.
\]

Let \(\alpha_i A_i, 1 \leq i \leq 3\) be three edges of \(G\). Let \(H\) be the graph obtained from \(G\) by subdividing these three edges. For \(i = 1, 2, 3\), choose an element \(\beta_i \in A_i - \{\alpha_1, \alpha_2, \alpha_3\}\). Let \(D_1\) be a subset of \(X\) such that \(\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \subseteq D_1\) and \(|D_1| = 6\). Let \(D = D_1 \cup \{X - D_1\}\). It can be verified that \(D\) is a dominating set of \(H\). Now by (1) and Remark 1.3, it follows that

\[
\gamma(H) = 7.
\]

Now (1) and (2) \(\Rightarrow \gamma(H) = \gamma(G)\). Therefore \(\xi(G) > 3\).

### Remark 2.2

In the above example taking \(X = \{1, 2, \ldots, 9\}\) also works but needs a little more computations.

### Proof of Proposition 1.6

Let \(n\) be any positive integer. The proposition trivially holds when \(n \leq 7\) since \(\frac{1}{3} \log_2 n < 1\). When \(n = 8\) or \(9\), we can construct a graph of order \(n\) with \(\xi = 2\) and the conclusion holds. So, let us assume that \(n \geq 10\).

Let \(k\) be the positive integer such that

\[
3k - 2 + \binom{3k - 2}{k} \leq n < 3k + 1 + \binom{3k + 1}{k + 1}.
\]

Note that \(k \geq 2\). Now let
\[ X = \{1, 2, \ldots, 3k - 2\} \quad \text{and} \quad S = \{A \subset X : |A| = k\} \cup \{X \cup \{-i\} : 1 \leq i \leq n - (3k - 2) - \binom{3k - 2}{k}\}. \]

Let \( G \) be as defined in the above example. Then by construction, order of \( G \) is \( n \). Let us show that \( \gamma(G) = 2k - 1 \). Let \( D \) be a dominating set of \( G \) and \( \ell = |D \cap X| \). We can assume that \( \ell \leq 2k - 2 \). Then
\[
|D| \geq \ell + \left(\frac{3k - 2 - \ell}{k}\right) \geq \ell + 2k - 2 - \ell + 1
\]
(by using the fact that \( \binom{k + m}{k} \geq m + 1 \) when \( m \geq 0 \))
\[
= 2k - 1.
\]

Therefore \( \gamma(G) \geq 2k - 1 \); since a dominating set \( D \) of cardinality \( 2k - 1 \) can be easily constructed such that \( |D \cap X| = 2k - 2 \), it follows that \( \gamma(G) = 2k - 1 \).

Let \( \{\alpha_i A_i : 1 \leq i \leq k - 1\} \) be a set of \( k - 1 \) edges and let \( H \) be the graph obtained from \( G \) by subdividing these \( k - 1 \) edges. For any \( i \leq k - 1 \), choose a positive integer \( \beta_i \in A_i - \{\alpha_j : 1 \leq j \leq k - 1\} \). Let \( D \) be a subset of \( X \) such that \( \{\alpha_i : 1 \leq i \leq k - 1\} \cup \{\beta_i : 1 \leq i \leq k - 1\} \subseteq D \) and \( |D| = 2k - 2 \). It can be verified that \( D \cup \{X - D\} \) is a dominating set of \( H \). Therefore by Remark 1.3, \( \gamma(H) = 2k - 1 \). Thus we have \( \gamma(H) = \gamma(G) \) and it follows that \( \xi(G) \geq k \).

Since \( n < 3k + 1 + \binom{3k + 1}{k+1} = 1 + 3k + \binom{3k}{k} + \binom{3k}{k+1} < 2^{3k} \), we have \( 3k > \log_2 n \) implying that \( \xi > \frac{1}{3} \log_2 n \). This completes the proof. \( \square \)

A set \( M \) of edges in a graph \( G \) is called a matching of \( G \) (sometimes an independent set of edges in \( G \)) if no two edges of \( M \) have a common end-vertex. The cardinality of a largest matching of \( G \) is denoted by \( \mu(G) \). The following result is quite well known. (cf. [5, p. 58]; for the sake of completeness, we give a proof of this result.)

**Lemma 2.3.** If \( G \) is a graph without isolated vertices, then \( \gamma \leq \mu \).

**Proof.** Let \( M \) be a maximum matching. Let \( S \) be the set of vertices which are not end-vertices of the edges in \( M \). If \( a \) is any vertex in \( S \), then \( a \) is not joined to any other vertex in \( S \) since \( M \) is a maximum matching; therefore \( a \) is joined to an end-vertex, say \( x \), of an edge in \( M \), since \( G \) does not have isolated vertices. Let \( y \) be the other end of this edge. If \( b \) is any other vertex in \( S \), then \( b \) is not joined to \( y \) for otherwise \( (M - \{xy\}) \cup \{ax, by\} \) would
be a matching of size $|M| + 1$. Hence it is possible to choose a dominating set $D$ of cardinality $\mu$ having exactly one end of each edge in $M$. Therefore $\gamma \leq \mu$.

**Remark 2.4.** Let $G$ be a graph with vertex set $V$; suppose $A$ is a subset of $V$ such that $G[V - A]$ has no isolated vertex and $\mu(G[A]) > \gamma(G[A])$. Then because of

$$\mu(G) \geq \mu(G[A]) + \mu(G[V - A]),$$

$$\gamma(G) \leq \gamma(G[A]) + \gamma(G[V - A]) \text{ and}$$

$$\mu(G[V - A]) \geq \gamma(G[V - A]) \text{ (by Lemma 2.3)}$$

we have $\mu(G) > \gamma(G)$.

**Lemma 2.5.** Suppose $G$ is a graph with vertex-set $V$ which can be partitioned as $\{A_1, B_1, A_2, B_2\}$ such that the following hold:

- For $i = 1, 2$, every vertex of $A_i$ is adjacent to every vertex of $B_i$.
- $|A_1|, |A_2| \geq 2$, $|B_1| \geq 3$ and $|B_2| \geq 1$.
- A vertex of $B_1$ is adjacent to a vertex of $A_2$.

Then $\mu(G) > \gamma(G)$.

**Proof.** If $|A_2| = 2$ or $|B_2| = 1$ then $\gamma(G) \leq 3$ and $\mu(G) \geq 4$. So suppose $|A_2| \geq 3$ and $|B_2| \geq 2$. Then $\mu(G) \geq 5$ and $\gamma(G) \leq 4$. Thus it follows that $\mu(G) > \gamma(G)$.

**Definition 2.6.** Let $G$ be a graph with vertex-set $V$; a subset $X$ of $V$ is said to be modular in $G$, if all the vertices in $X$ have same neighbourhood and $G[V - (X \cup N(X))]$ has no isolated vertices. If in addition $X$ dominates $G$, then $G$ is called a module; $G$ is a proper module if $|X| \geq 2$ and $|N(X)| \geq 3$.

Note that any modular set $X$ of a graph is independent; i.e., no two vertices of $X$ are adjacent. If $G = (V, E)$ is a module with $E \neq \emptyset$, then $\gamma(G) \leq 2$. ($G$ can be imagined as a graph obtained from the complete bipartite graph with bipartition $\{X, V - X\}$ by adding edges having end-vertices in $V - X$ only.)

**Lemma 2.7.** For a graph $G$ without isolated vertices, one of the following holds.
(i) $\mu(G) > \gamma(G)$.

(ii) Each connected component is a proper module.

(iii) There exists a modular subset $A$ of $V(G)$ such that either $|A| = 1$ or $|N(A)| \leq 2$.

**Proof.** By induction; assume that for any graph of order less than that of $G$, the theorem holds. Let $\alpha$ be any vertex of $G$ such that $\deg \alpha = \delta(G)$. Let $A = \{x \in V(G) : N(x) = N(\alpha)\}$. Let $H = G[V(G) - (A \cup N(A))]$. If $V(H) = \emptyset$ then $G$ is a module and therefore either (ii) or (iii) holds. When $V(H) \neq \emptyset$, by the construction of $A$, $H$ has no isolated vertex. We can assume the following for otherwise (iii) holds.

(**) $|A| \geq 2$ and $|N(A)| \geq 3$.

Applying the induction hypothesis for $H$ we have the following cases.

**Case 1.** $\mu(H) > \gamma(H)$.

Since $G[A \cup N(A)]$ has no isolated vertex, by Remark 2.4, (i) holds.

**Case 2.** Each component of $H$ is a proper module.

If there is one component $J$ such that $N(V(J)) \cap N(A) \neq \emptyset$, then by Lemma 2.5 and (**), $\mu(G[A \cup N(A) \cup V(J)]) > \gamma(G[A \cup N(A) \cup V(J)])$ and (i) holds by Remark 2.4; otherwise the components of $G$ are those of $H$ and $G[A \cup N(A)]$ and therefore (ii) holds.

**Case 3.** A subset $B$ of $V(H)$ is modular in $H$ such that either $|B| = 1$ or $|N_H(B)| \leq 2$.

Let $X = A \cup N(A) \cup B \cup N_H(B)$. Note that $G[V(G) - X]$ has no isolated vertex. First suppose $|B| \geq 2$. If there is any edge from $N(A)$ to $B$ then by (**) and Lemma 2.5, $\mu(G[X]) > \gamma(G[X])$ and by Remark 2.4, (i) holds; otherwise $N(B) = N_H(B)$ and (iii) holds with $B$ in place of $A$.

Now suppose $B$ has only one vertex, say $\alpha$. If $\alpha$ is not joined to every vertex of $N(A)$, then (iii) holds with $B$ in place of $A$. So assume that $\alpha$ is joined to every vertex of $N(A)$. Then $\gamma(G[X]) \leq 2$ and $\mu(G[X]) \geq 3$; therefore again we have $\mu(G[X]) > \gamma(G[X])$ and (i) holds. This completes the proof.

**Proof of Theorem 1.7.** For a graph $G$, by using induction on its order, let us show the following:
(**) If $G$ is connected and has at least three vertices then there exists a set $F$ of edges of order $\gamma$ or $\gamma + 1$ such that $F$ contains a matching of order $\gamma$ and subdividing the edges of $F$ results in a graph whose domination number is more than that of $G$.

When $|V(G)| = 3$, (**) is obvious. So, let $|V(G)| > 3$ and assume that for any graph $H$ with $|V(H)| < |V(G)|$, (**) holds with $H$ in place in $G$. By Lemma 2.7, we have three cases.

Case 1. $G$ has a matching $M$ of order $\gamma + 1$. Then (**) holds with $M$ in place of $F$.

Case 2. $G$ is a module. Then there exists a modular subset $A$ of $V(G)$ such that $V(G) = A \cup N(A)$. If $\gamma(G) = 1$, it is easy to see that (**) holds. So let $\gamma(G) = 2$. Let $a, b$ be two distinct vertices in $A$.

Suppose $N(A)$ has two adjacent vertices $x, y$. Since $\gamma(G) = 2$, there must be one more vertex $z \in N(A)$. If $|N(A)| \geq 4$, then $\mu(G) \geq 3$. So, let $N(A) = \{x, y, z\}$. Then neither $x$ nor $y$ is joined to $z$. If $|A| \geq 3$, then also we have $\mu(G) \geq 3$. So, let $A = \{a, b\}$. Now subdividing the edges of the matching $\{ax, ay\}$ shows that $\xi(G) = 2$ and (**) holds.

If $N(A)$ is a set of independent vertices, then subdividing the edges $ax, ay, bx$ where $x, y$ are any two arbitrary vertices in $N(A)$ shows that (**) holds.

Case 3. (iii) of Lemma 2.7 holds. Let $H_1 = G[A \cup N(A)]$ and $H_2 = G[V(G) - (A \cup N(A))]$. We can assume that $\gamma(H_1) + \gamma(H_2) = \gamma(G)$ for otherwise $\mu(G) \geq \mu(H_1) + \mu(H_2) \geq \gamma(H_1) + \gamma(H_2) > \gamma(G)$ and (**) holds.


Let $A = \{a\}$ and $x$ be any vertex in $N(A)$.

If $H_2$ has a component $K$ of order $\geq 3$, then by induction hypothesis, there exists a set $F' \subseteq E(K)$ such that (**) holds with $K$ and $F'$ in places of $G$ and $F$ respectively. Since $H_2[V(H_2) - V(K)]$ has no isolated vertex, it has a matching of size $\gamma(H_2) - \gamma(K)$. Now taking $F = \{ax\} \cup F' \cup M$ it can be verified that (**) holds.

So assume that $H_2$ is a union of copies of $K_2$. If $\deg a > 1$, then (**) holds with $F = \{ax\} \cup E(H_2)$. So suppose $N(a) = \{x\}$. For any $e \in E(H_2)$, we can assume that both of its end-vertices are not joined to $x$,
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for otherwise we would have \( \gamma(G) < \mu(G) \). Therefore by connectivity of \( G \),
\( x \) is joined to exactly one vertex of each edge in \( E(H_2) \) and (**) holds with
\( F = \{ax\} \cup \{xy\} \cup E(H_2) \) where \( y \) is a vertex in \( V(H_2) \) which is joined to \( x \).

**Subcase b.** \( |A| \geq 2 \).
Then \( |N(A)| \leq 2 \). Let \( M \) be any matching in \( H_2 \) of size \( \gamma(H_2) \). Let \( a, b \)
be two distinct vertices in \( A \). If \( |N(A)| = 1 \), then \( a, b \) are pendant with
the same support, say \( x \); subdividing the edge \( ax \) shows that \( \xi(G) = 1 \) and
obviously (**) holds with \( F = \{ax\} \cup M \).

So suppose \( N(A) \) contains one more vertex, say \( y \). If \( \gamma(H_1) = 1 \), then
\( F = \{ax, by\} \cup M \) is a matching of size \( \gamma(G) + 1 \) and (**) holds; so let
\( \gamma(H_1) = 2 \). Subdividing the edges \( ax, ay, bx \) shows that \( \xi(G) \leq 3 \) and (**)
holds with \( F = \{ax, ay, bx\} \cup M \).

Now we prove the second main result of this paper. The main tool used
in the proof is Alon’s result (cf. [1, Page 4]) on domination number of a
graph: Any graph \( G \) has a dominating set of size \( \leq n \frac{1+\ln(\delta+1)}{\delta+1} \) where \( n \) is
the number of vertices.

**Proof of Theorem 1.8.** First we settle a few simple cases. (Throughout
this proof, we consider a number of cases. Whenever a case is under
consideration, it is assumed that the previous cases do not hold.)

1. **Case 1.** \( G \) has two pendant vertices with same support.

By subdividing one of them, we find \( \xi = 1 \).

2. **Case 2.** There is an edge \( e \in E(G) \) such that \( G - e \) has two connected
components \( G_1 \) and \( G_2 \) with the property that \( G_1 \) is a tree with at least 3
vertices.

Then by Remark 2.1, \( \xi \leq 3 \).

3. **Case 3.** There is a path \((u, v, w, x)\) such that \( \deg(u) = \deg(x) = 1 \).

Subdividing the three edges of this path shows that \( \xi \leq 3 \).

So let us assume that none of the above cases holds. Removing all the
hanging paths but retaining their supports results in a connected graph \( G' \)
such that the following hold:
Every pendant vertex in \( G \) is connected to a vertex in \( G' \) by a path
of length at most 2. Any such path of length 1 cannot have a vertex in
common with any other path. Any such path of length 2 cannot have an edge in common with any other path.

Let $m = \lceil \sqrt{n} \ln n \rceil$. For any pendant vertex $u$ of $G$ let $u^*$ denote its support.

Let $S_1 = \{ v \in V(G') \mid \deg(v) \leq m \}$,
$S_2 = \{ v \in V(G') \mid \deg(v) > m \}$,
$V_1 = \{ v \in V(G) \mid \deg(v) = 1 \text{ and } v^* \in V(G') \}$,
$V_2 = \{ v \in V(G) \mid \deg(v) = 1, \deg(v^*) = 2, \text{ and } N(v^*) \cap V(G') \neq \emptyset \}$
and $S = \{ v \in S_1 \mid N(v) \cap V_1 \neq \emptyset \}$.

Let $\ell$ be the number of vertices in $V_2$ which are joined to vertices in $S_2$ by paths of length 2. If there is a vertex $v \in V_2$ which is joined by a path of length 2 to a vertex $u \in S_1$; i.e., then by subdividing the edges of this path and all the edges of $E(G')$ which are incident with $u$, we find that $\xi \leq m + 2$.

Let $k$ be the number of vertices in $V_1$ with supports in $S_2$. Now let us settle three more cases.

**Case 4.** There exist $u, v \in S_1$ which are adjacent.

In this case by subdividing all the edges in $I(u) \cup I(v)$ we find that $\xi \leq 2m$.

**Case 5.** There exist $u \in S_1$ and $v \in S_1 - S$ such that $N(u) \cap N(v) \neq \emptyset$.

Fix a vertex $a \in N(u) \cap N(v)$. Then subdivide all edges in $I_G(u) \cup I_G(v)$.

Also for each vertex $x \in (N(u) \cup N(v)) - \{a\}$ we subdivide an edge $xy \in G'$ where $y \notin \{u, v, a\}$. If there is any edge $ab$ with $b \in V_1$, it is also subdivided.

If there are paths $(a, b, c)$ with $c \in V_2$ then both the edges of one such path are also subdivided. Hence in this case the subdivision number is at most $4m + 1$.

**Case 6.** There exist $u, v \in S_1 - S$ and $v_1v_2 \in E(G')$ with $v_1 \in N(u)$ and $v_2 \in N(v)$.

We subdivide $v_1v_2$ and the edges in $I_{G'}(u) \cup I_{G'}(v)$. Also for each vertex $x \in N(u) \cup N(v)$ we subdivide an edge $xy \in E(G')$ with $y \notin \{u, v, v_1, v_2\}$.

If there is any edge $a_1a_2 \in E(G')$ with $a_1 \in V_1$ and $a_2 \in \{v_1, v_2\}$ then subdivide it. If there are paths $(a_1, a_2, v_1)$ with $a_1 \in V_2$, then subdivide both the edges of one such path; this is repeated with $v_2$ in place of $v_1$.

Hence the subdivision number is at most $4m + 5$. 

If none of the cases considered so far holds, then by using Alon’s result mentioned above we have

\[ \ell + k + |S_1| \leq \gamma(G) \leq \ell + k + \rho + |S_1| \]

where \( \rho = |S_2| \frac{\ln(m+1)+1}{m+1} \).

**Case 7.** \( k > \sqrt{\frac{n}{\ln n}} \).

Then there exist two pendant vertices \( u, v \) with supports in \( S_2 \) such that \( N(N(u)) \cap N(N(v)) \neq \emptyset \). Let \( x \) and \( y \) be supports of \( u \) and \( v \) respectively. Let \( p \) be a vertex in \( N(x) \cap N(y) \). Now we subdivide the edges \( ux, xp, py \) and \( yv \). We also subdivide \( m \) edges in \( I(x) \). If any such edge is incident with a vertex \( a \in N(S_1 - S) \) then we also subdivide the edge \( ab \) where \( b \) is in \( S_1 - S \). Now the domination number of the resulting graph is at least \( |S_1| + k + \ell + m \) which is more than that of \( G \) (if \( n \geq 8 \)). Hence the subdivision number is at most \( 2m + 4 \).

**Case 8.** \( |S_1 - S| \leq \frac{m}{4} \).

We take a matching \( M \) of size \( \min \left( \frac{m}{4}, |V(G') - (S_1 - S)| \right) \) in \( G'[V(G') - (S_1 - S)] \) and subdivide all the edges in the matching and all edges \( uu^* \) where \( u^* \) is an end-vertex of an edge in this matching. For the resulting graph \( H \), the size of the dominating set is at least \( \gamma(H) \geq \ell + |S| + \min \left( \frac{m}{4}, |V(G') - (S_1 - S)| \right) \). If \( |M| < \frac{m}{2} \) then \( \gamma(H) > |S| + \ell = \gamma(G) \); otherwise for \( n \geq 235 \), by using (*) it can be verified that \( \ell + |S| + \frac{m}{4} > \ell + k + \rho + |S_1| \geq \gamma(G) \). Thus, when \( n \geq 235 \), \( \gamma(G) < \gamma(H) \).

**Case 9.** \( |S_1 - S| > \frac{m}{4} \).

First we fix a set \( S_1' \subset S_1 - S \) with \( |S_1'| = \frac{m}{4} \). Next we get a matching \( M \) of size \( \frac{m}{4} \) in \( S_2 \) such that each edge in \( M \) has an end in \( N(S_1') \) and for any \( a \in S_1' \), there is at most one edge in \( M \) having an end-vertex in \( N(a) \). Now we subdivide all edges in \( M \) and for every vertex \( b \in S_1' \) an edge \( bb_1 \in E(G') \). We also subdivide all edges of the form \( uv \in E(G) \) where \( u \) is an end-vertex of an edge in \( M \) and \( v \in S_1' \). In the resulting graph \( H \) the domination number is at least \( |S_1| + \frac{m}{4} + \ell \). So as in the last case, if \( n \geq 235 \) then \( \gamma(H) > \gamma(G) \). Therefore the subdivision number in this case is at most \( \frac{3m}{4} \).

Thus we conclude that if \( n \geq 235 \) then \( \xi \leq 4\sqrt{n} \ln n + 5 \). \( \blacksquare \)
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References


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