

## FAMILIES OF STRONGLY PROJECTIVE GRAPHS

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### Abstract

We give several characterisations of strongly projective graphs which generalise in many respects odd cycles and complete graphs [7]. We prove that all known families of projective graphs contain only strongly projective graphs, including complete graphs, odd cycles, Kneser graphs and non-bipartite distance-transitive graphs of diameter  $d \geq 3$ .

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## 1. Introduction

In this paper all graphs are finite and undirected. For basic terminology and notation we shall follow [4] (see also [7]). If  $G$  and  $H$  are graphs, a *homomorphism* from  $G$  to  $H$  is an edge-preserving map from the vertex-set of  $G$  to the vertex-set of  $H$ , i.e., a map  $f : G \rightarrow H$  such that  $f(g)f(g')$  is an edge of  $H$  whenever  $gg'$  is an edge of  $G$ . The *product*  $G \times H$  of two graphs has vertex set  $G \times H$  and two vertices  $(g, h)$  and  $(g', h')$  are adjacent if  $gg'$

and  $hh'$  are edges of  $G$  and  $H$  respectively. For  $n \geq 1$  we let  $G^n$  denote the product of  $G$  with itself  $n$  times.

Consider the following problem: if  $G$ ,  $H$  and  $K$  are connected graphs, then under which conditions are the homomorphisms from  $G \times H$  to  $K$  determined by the homomorphisms from  $G$  to  $K$  and those from  $H$  to  $K$ ? Here are a few instances and aspects of this problem:

1. In [3], Greenwell and Lovász prove that if  $G$  is connected and  $\chi(G) \geq n+1$ , i.e.,  $G$  admits no homomorphism into  $K$ , where  $K$  is the complete graph on  $n \geq 3$  vertices, then the homomorphisms from  $G \times K$  to  $K$  are of the form  $(g, k) \mapsto \sigma(k)$  where  $\sigma$  is an automorphism of  $K$ . In other words, we have a bijection between  $\text{Hom}(G \times K, K)$  and  $\text{Hom}(K, K)$ .

2. C. Tardif [16] has recently proved the following: if  $H$  and  $H'$  are uniquely 3-colourable, i.e., admit, up to automorphisms, a unique homomorphism to the complete graph on 3 vertices, then  $H \times H'$  admits exactly two 3-colourings. More generally, Duffus, Sands and Woodrow [1] have conjectured the following: *if  $H$  and  $H'$  are uniquely  $n$ -colourable then  $H \times H'$  admits exactly two  $n$ -colourings.* They show that this conjecture implies Hedetniemi's conjecture for fixed  $n$ : if  $H$  and  $H'$  are  $(n+1)$ -chromatic then  $\chi(H \times H') = n+1$ .

3. Let  $K$  be a *core* graph, i.e.,  $K$  has no proper retracts, or equivalently, every homomorphism  $f : K \rightarrow K$  is an automorphism. Then the homomorphisms from  $K^s$  to  $K$  are determined by those from  $K$  to  $K$  if they are of the form  $\sigma \circ \pi_i$  where  $\sigma$  is an automorphism of  $K$  and  $\pi_i$  is the projection onto the  $i$ -th factor ( $i = 1, \dots, s$ ). This is equivalent to saying that  $K$  is projective: in general, a graph  $K$  is *projective* if, for every  $s \geq 2$ , the only homomorphisms from  $K^s$  to  $K$  that satisfy  $f(x, \dots, x) = x$  (i.e., are *idempotent*) for all  $x \in K$  are the projections [8].

4. In [7] we extend the result of Greenwell and Lovász as follows: let  $K$  be an odd cycle or a complete graph on  $n \geq 3$  vertices, and let  $s \geq 1$ . If  $G$  is connected and admits no homomorphism to  $K$  then there are, up to automorphisms of  $K$ , only  $s$  homomorphisms from  $G \times K^s$  to  $K$ . It follows from results in [8] that this sets up a bijection between  $\text{Hom}(G \times K^s, K)$  and  $\text{Hom}(K^s, K)$ . Notice that in the case where  $K$  is a complete graph and  $s = 2$ , this result verifies the conjecture of Duffus, Sands and Woodrow where  $H = G \times K$  and  $H' = K$ .

In view of the above, we are led to the following restricted version of the problem:

**Problem 1.** Let  $K$  be a core graph. Under what conditions on  $K$  may we conclude that, if  $G$  is any connected graph which admits no homomorphism to  $K$ , then there are, up to automorphisms of  $K$ , only  $s$  homomorphisms from  $G \times K^s$  to  $K$  ?

This problem splits naturally into 2 distinct problems, which we now describe. We shall require the notion of exponential graph, first introduced by Lovász [11] (see also [2]): Let  $G$  and  $K$  be two graphs. Define  $K^G$  as follows: the vertices are all the functions from  $G$  to  $K$ , and two such functions  $f$  and  $g$  are adjacent if they satisfy the following condition: if  $x$  and  $y$  are adjacent in  $G$  then  $f(x)$  and  $g(y)$  are adjacent in  $K$ .<sup>1</sup> This definition sets up a natural bijection between  $\text{Hom}(G \times H, K)$  and  $\text{Hom}(H, K^G)$  for any graph  $H$ .

**Problem 1a.** Let  $K$  be a core graph. Under what conditions on  $K$  may we conclude that, if  $G$  is any connected graph which admits no homomorphism to  $K$ , then there is a unique homomorphic image of  $K$  in  $K^G$  (namely, the one induced by the constant maps) ?

**Problem 1b.** Let  $K$  be a core graph. Under what conditions on  $K$  may we conclude the following: if  $G$  is any connected graph which admits no homomorphism to  $K$ , and if  $s \geq 2$ , then the only homomorphisms  $f : G \times K^s \rightarrow K$  that satisfy the identity  $f(g, x, \dots, x) = x$  for all  $x$  and  $g$ , are projections ?

We have the following:

**Proposition 1.1.** *A core graph  $K$  satisfies the condition of Problem 1 if and only if it satisfies the conditions of Problems 1a and 1b.*

**Proof.** Let  $K$  satisfy the conditions of Problems 1a and 1b. Let  $G$  be any connected graph which admits no homomorphism to  $K$ , and let  $f : G \times K^s \rightarrow K$  be a homomorphism. Since we may embed  $K$  in  $K^s$  as the diagonal, we obtain a map  $\Phi : K \rightarrow K^s \rightarrow K^G$  using the natural property of the exponential graph. By the property of Problem 1a, the image of this map must be the set of constant maps. Since  $K$  is a core, it follows that there exists an automorphism  $\sigma$  of  $K$  such that  $f(g, x, \dots, x) = \sigma(x)$  for all  $x \in K$  and  $g \in G$ . Hence  $h = \sigma^{-1} \circ f$  satisfies the identity  $h(g, x, \dots, x) = x$  for all  $x$  and  $g$ , and by the condition of Problem 1b it follows that  $h$  is a

<sup>1</sup>These graphs may contain loops, namely, the homomorphisms from  $G$  to  $K$ .

projection. Consequently,  $f$  is a projection up to an automorphism of  $K$  and there are exactly  $s$  of these.

Now suppose that  $K$  satisfies the condition of Problem 1. For  $s = 1$ , this is precisely the condition of Problem 1a; and it is immediate that  $K$  satisfies the condition of Problem 1b. ■

It is easy to see that a core graph  $K$  that satisfies the condition of Problem 1b must be projective: if  $f : K^s \rightarrow K$  is idempotent, let  $G$  be any connected graph that admits no homomorphism into  $K$ , and define a homomorphism  $F : G \times K^s \rightarrow K$  by  $F(g, x_1, \dots, x_s) = f(x_1, \dots, x_s)$ . It follows that  $f$  is a projection. On the other hand, consider the following stronger property.

**Problem 1b'.** Let  $K$  be a core graph. Under what conditions on  $K$  may we conclude the following: if  $G$  is any connected graph, and if  $s \geq 2$ , then the only homomorphisms  $f : G \times K^s \rightarrow K$  that satisfy the identity  $f(g, x, \dots, x) = x$  for all  $x$  and  $g$ , are projections?

In the first part of the paper we give several characterisations of those graphs  $K$  that satisfy the condition of Problem 1b' (Theorem 2.3), which we shall refer to as *strongly projective graphs*. In the second half of the paper we shall investigate the following question: *is every projective graph in fact strongly projective?* We shall prove that all known families of projective graphs contain only strongly projective graphs (Theorems 3.2, 3.3, 3.4, 3.5, 3.9 and 3.10). These include, among others, complete graphs (with at least 3 vertices), odd cycles, Kneser graphs, and non-bipartite distance-transitive graphs of diameter  $d \geq 3$ . Obviously, if every projective graph is strongly projective, then the conditions of Problems 1b and 1b' are equivalent.

Although we shall not investigate Problem 1a here, a few comments are in order. Let  $K$  be a core graph. For convenience, call a graph  $G$  *uniquely  $K$ -colourable* if, up to automorphisms of  $K$ , there exists a unique homomorphism from  $G$  to  $K$ . We adapt the result of Duffus, Sands and Woodrow [1] mentioned above to the more general setting of graph homomorphisms. Consider the following properties that a core graph  $K$  might possess:

- (A) If  $G$  and  $H$  are uniquely  $K$ -colourable, then  $G \times H$  admits only two homomorphisms to  $K$  (up to automorphisms of  $K$ ).
- (B) If  $G$  is uniquely  $K$ -colourable and  $H$  admits no homomorphism to  $K$  then  $G \times H$  is uniquely  $K$ -colourable.
- (C) If  $G$  and  $H$  admit no homomorphism to  $K$  then neither does  $G \times H$ , i.e.,  $K$  is *multiplicative*.

**Proposition 1.2.** *Let  $K$  be a core graph that satisfies the property of Problem 1a. If  $K$  satisfies property (A) then it must satisfy property (B). If it satisfies property (B), then  $K$  is multiplicative.*

**Proof.** This is a direct adaptation of [1], Theorem 3.3. Notice first that if a graph  $K$  satisfies the property of Problem 1a then by the natural property of the exponential graph, for every graph  $G$  that admits no homomorphisms to  $K$ ,  $G \times K$  is uniquely  $K$ -colourable.

Suppose that  $G$  is uniquely  $K$ -colourable, that  $H$  admits no homomorphism to  $K$  but  $G \times H$  admits more than one homomorphism to  $K$ . Consider the graph  $(G \times H) \times K$ : it admits at least three homomorphisms to  $K$  (two from the factor  $G \times H$  and one from the factor  $K$ ). However, when viewed as  $G \times (H \times K)$ , it admits only two homomorphisms to  $K$ , since (A) holds and  $G$  and  $H \times K$  are uniquely  $K$ -colourable. This is a contradiction so (A) implies (B).

Now suppose that  $G$  and  $H$  admit no homomorphism to  $K$  but that  $G \times H$  does. Consider the graph  $(G \times H) \times K$ : it admits at least two homomorphisms, one from the factor  $G \times H$  the other from the factor  $K$ . However, when viewed as  $G \times (H \times K)$ , it must be uniquely  $K$ -colourable, by (B). Hence (B) implies that  $K$  is multiplicative. ■

## 2. Strongly Projective Graphs

Let  $K$  be any graph and let  $s \geq 1$ . Let  $E_s(K)$  denote the graph  $K^{K^s}$  and let  $I_s(K)$  denote the subgraph of  $E_s(K)$  consisting of the idempotent functions, i.e., those  $f$  that satisfy  $f(x, \dots, x) = x$  for all  $x$ . Call a graph  $K$  *strongly projective* if, for every  $s \geq 2$ , the only  $f \in I_s(K)$  with at least one neighbour (in  $I_s(K)$ ) are the projections (see [7] for details).

We shall require the notion of *probe*, which was introduced in [10] and used by Nešetřil and Zhu in [13] to construct sparse graphs with prescribed colourings. Let  $K$  be any graph. A triple  $(P, Q, q)$  is a  *$K$ -coloured graph* if  $P$  is a graph,  $Q$  is a (possibly empty) set of vertices of  $P$  and  $q$  is a function with domain  $Q$  and codomain  $K$ . Let  $\theta$  be a  $k$ -ary relation on  $K$ ,  $k \geq 1$ . A quadruple  $(P, Q, q, R)$  is a  *$K$ -probe* for  $\theta$  (or  $\theta$  *admits* the probe  $(P, Q, q, R)$ ) if (i)  $(P, Q, q)$  is a  $K$ -coloured graph and (ii)  $R = (v_1, \dots, v_k)$  is a  $k$ -tuple of vertices of  $P$  such that

$$\theta = \{(f(v_1), \dots, f(v_k)) : f \text{ is a map from } P \text{ to } K \text{ that extends } q\}.$$

A  $K$ -probe  $(P, Q, q, R)$  is said to be *bipartite* if  $P$  is bipartite. When  $k = 1$ , which is the only case we'll use in this paper, we'll assume that  $R = v$  is a vertex of  $K$ .

We shall need the following:

**Lemma 2.1** [9]. *A graph  $K$  with at least 3 vertices is projective if and only if every pair of vertices admits a probe.* ■

Let  $K$  be a graph. If  $a$  is a vertex of  $K$  let  $N_a$  denote its neighbourhood in  $K$ , i.e., the set of all vertices of  $K$  adjacent to  $a$ . We shall say that a graph is *ramified* if for all vertices  $a$  and  $b$  of  $K$  we have that  $N_a \subseteq N_b$  implies  $a = b$ . If  $K$  is a ramified, connected, non-bipartite graph, the *neighbourhood poset* of  $K$  is the poset  $P = \mathcal{P}_K$  of all non-empty intersections of neighbourhoods of  $K$  ordered by inclusion. A poset is *projective* if the only idempotent order-preserving operations on  $P$  are projections.

**Lemma 2.2** [7]. *Let  $K$  be a ramified, connected, non-bipartite graph and let  $f \in E_s(K)$  have at least one neighbour. Then*

1. *If  $f$  is adjacent to a projection in  $E_s(K)$  then  $f$  is equal to that projection.*
2. *If  $f \in I_s(K)$  and  $g$  is adjacent to  $f$  then  $g \in I_s(K)$ .*
3. *If  $f$  satisfies  $f(x_1, \dots, x_s) \in \{x_1, \dots, x_s\}$  for all  $x_i$  then  $f$  is a projection.* ■

We can now state and prove our first result:

**Theorem 2.3.** *Let  $K$  be a graph with at least 3 vertices. Then the following statements are equivalent:*

1.  *$K$  is strongly projective.*
2.  *$\mathcal{P}_K$  is a projective poset.*
3.  *$K$  is projective and  $I_2(K)$  consists of isolated vertices.*
4. *Every pair of vertices of  $K$  admits a bipartite probe.*
5. *For every connected graph  $G$  with at least 2 vertices, and every  $s \geq 2$ , the only maps  $f : G \times K^s \rightarrow K$  satisfying  $f(y, x, \dots, x) = x$  for all  $y$  and  $x$ , are the projections  $f(y, x_1, \dots, x_s) = x_i$ .*

**Proof.** The equivalence of (1), (2) and (3) is proved in [7].

(1)  $\Rightarrow$  (5) Let  $f : G \times K^s \rightarrow K$  satisfy  $f(y, x, \dots, x) = x$  for all  $y$  and  $x$ . By the natural property of exponential graphs we obtain a homomorphism  $F : G \rightarrow I_s(K)$  such that  $F(g)(x_1, \dots, x_s) = f(g, x_1, \dots, x_s)$  for all  $g$  and  $x_i$ . Since  $G$  is connected and contains at least two vertices,  $F(g)$  has at least one neighbour, and so must be a projection, for any  $g \in G$ . Hence by Lemma 2.2 (1) no two distinct projections can be adjacent, so there is an  $i = 1, \dots, s$  such that  $F(g)$  is the  $i$ -th projection, and (5) follows.

(5)  $\Rightarrow$  (4) Let  $\theta = \{u, v\}$  be a pair of vertices of  $K$ . Our probe is the following: let  $P$  be  $K_2 \times K^2$  where  $K_2$  is the complete graph on two vertices (we shall denote these by 0 and 1 for convenience). Let  $Q$  be all elements of  $P$  of the form  $(i, x, x)$ , and define  $q$  by  $q(i, x, x) = x$  for all  $x \in K$  and  $i = 0, 1$ . Let  $R$  be  $(0, u, v)$ . Now any map  $f$  from  $P$  to  $K$  that extends  $q$  satisfies  $f(i, x, x) = x$  for all  $x$  and all  $i$  so  $f$  must be a projection. Hence  $f(0, u, v)$  is  $u$  or  $v$ . Since both projections do extend  $q$ , it follows that we have a bipartite probe for  $\theta$ .

(4)  $\Rightarrow$  (3) It follows from Lemma 2.1 that  $K$  is projective. Let  $\phi \in I_2(K)$  have a neighbour  $\psi$ . It will suffice by Lemma 2.2 (3) to show that  $\phi(x, y) \in \{x, y\}$  for all  $x, y \in K$ . Note that by Lemma 2.2 (2) we know that  $\psi \in I_2(K)$ . Pick  $x$  and  $y$  in  $K$  distinct, and let  $(P, Q, q, R)$  be a bipartite probe for the pair  $\{x, y\}$ , where  $R = u$ . We show that  $\phi(x, y) \in \{x, y\}$  as follows: let  $A$  and  $B$  be the colour classes of  $P$  and assume wlog that  $u \in A$ . There are maps  $\mu$  and  $\nu$  from  $P$  to  $K$  extending  $q$  such that  $\mu(u) = x$  and  $\nu(u) = y$ . Consider the map  $\eta$  from  $P$  to  $K$  defined by

$$\eta(p) = \begin{cases} \phi(\mu(p), \nu(p)) & \text{if } p \in A, \\ \psi(\mu(p), \nu(p)) & \text{if } p \in B. \end{cases}$$

Notice that  $\eta$  is a homomorphism because  $\phi$  and  $\psi$  are adjacent, and  $\eta$  extends  $q$  because both  $\phi$  and  $\psi$  are in  $I_2(K)$ . It follows that  $\eta(u) = \phi(x, y)$  must be in  $\{x, y\}$ , and this completes the proof. ■

### 3. Families of Strongly Projective Graphs

We now proceed to show that all known projective graphs are in fact strongly projective. For our first result we require the following: a poset  $P$  has *length* 1 if every element of  $P$  is either minimal or maximal (in other words, every chain in  $P$  has at most 2 elements). A finite poset is *ramified* if every

non-maximal element is covered by at least two elements, and dually, every non-minimal element covers at least two elements. The following result is a special case of Corollary 1 of [5].

**Theorem 3.1** [5]. *Let  $P$  be a finite, connected, ramified poset of length 1. Then  $P$  is projective.* ■

We shall say that a graph is *square-free* if it contains no (not necessarily induced) 4-cycle.

**Theorem 3.2.** *Let  $K$  be a square-free, connected, ramified, non-bipartite graph. Then  $K$  is strongly projective. In particular, odd cycles are strongly projective.*

**Proof.** It is easy to see that the neighbourhood poset of a square-free graph has length 1 (see [17], Corollary 12.2). Since  $K$  is connected, ramified and non-bipartite, it follows easily that its neighbourhood poset  $P$  is connected and ramified (see [7]). Hence  $P$  is projective by Theorem 3.1. We conclude by Theorem 2.3 that  $K$  is strongly projective. ■

Let  $r, l$  be positive integers such that  $r < l/2$ . The *circular graph*  $\text{Circ}(r, l)$  is defined as follows:

$$\begin{aligned} V(\text{Circ}(r, l)) &= \mathbb{Z}_l = \{0, 1, \dots, l-1\}, \\ E(\text{Circ}(r, l)) &= \{[i, j] : i - j \in \{r, r+1, \dots, l-r\}\}. \end{aligned}$$

**Theorem 3.3.** *If  $K$  is a circular graph, then  $K$  is strongly projective. In particular, complete graphs on at least 3 vertices are strongly projective.*

**Proof.** We prove that every pair of vertices admits a bipartite probe. In Theorem 6 of [10] it is proved that every pair of vertices admits a probe. It is easy to see that in fact, the probe constructed in that proof is bipartite, if one notices the following:

(i) if two sets of vertices admit bipartite probes then so does their intersection. Indeed, if  $\theta$  admits the bipartite probe  $(P, Q, q, R)$  and  $\theta'$  admits the bipartite probe  $(P', Q', q', R')$ , then we construct a probe for  $\theta \cap \theta'$  as follows. Let  $R = v$  and  $R' = v'$ . Let  $(P'', Q'', q'')$  be the  $K$ -coloured graph obtained from the disjoint union of  $(P, Q, q)$  and  $(P', Q', q')$  by identifying  $v$  and  $v'$ . Let  $v''$  denote this vertex, and let  $R'' = v''$ . Then it is easy to see that  $(P'', Q'', q'', R'')$  is a bipartite probe for  $\theta \cap \theta'$ .



(ii) if a set  $A$  of vertices admits a bipartite probe  $(P, Q, q, R)$ , then the set  $B$  of all vertices adjacent to some member of  $A$  also admits a bipartite probe  $(P', Q, q, R')$ , obtained by adding a new vertex  $u$  adjacent to the vertex  $R$ , keeping  $Q$  and  $q$  as is and letting  $R' = u$ . Since every pair of vertices admits a bipartite probe, it follows from Theorem 2.3 that the circular graphs are strongly projective. ■

The graphs we call *truncated simplices* were defined in [10]: Let  $n \geq 4$  be an integer. The truncated  $n$ -simplex  $T_n$  is defined as follows:

$$V(T_n) = \{(i, j) \in \{0, 1, \dots, n - 1\}^2 : i \neq j\},$$

$$E(T_n) = \{[(i, j), (k, l)] : i = k, j \neq l \text{ or } i = l, j = k\}.$$

**Theorem 3.4.** *If  $K$  is a truncated simplex, then  $K$  is strongly projective.*

**Proof.** Let  $K = T_n$  be a truncated simplex. If  $n = 4$  then  $K$  is ramified, non-bipartite and square-free so by Theorem 3.2 we may assume that  $n \geq 5$ . For convenience, in what follows we shall denote the vertex  $(a, b)$  of  $K$  by  $ab$ . Define binary relations  $\lambda, \sigma, \rho, E, I$  and  $M$  on  $V(K)^2$  as follows:

- $ab \lambda cd$  iff  $a = c$ ;
- $ab \rho cd$  iff  $b = d$ ;
- $ab \sigma cd$  iff  $a = d$  and  $c = b$ ;
- $ab E cd$  iff  $a = d$ ;
- $ab I cd$  iff  $b = c$ ;
- $ab M cd$  iff  $a, b, c, d$  are all distinct.

Notice that the union of these relations is equal to  $V(K)^2$ . We prove that  $K$  is strongly projective using the definition. Let  $f$  and  $g$  be adjacent in  $I_2(K)$ .

**Claim.** Let  $B$  be a block of  $\lambda$ . Then  $f|_{B^2} = g|_{B^2}$  is a projection. Indeed, we have that  $f(ab, ac)$  is adjacent to  $g(ad, ad) = ad$  for all  $d \neq a, b, c$ . Since there are at least two such  $d$ 's it follows that  $f(ab, ac)$  is either  $ab$  or  $ac$ . This means that the restriction of  $f$  to  $B^2$  is a member of  $I_2(B)$  which satisfies the condition of Lemma 2.2 (3). Since  $B$  is a complete graph with at least 4 vertices, and by Theorem 3.3 these graphs are strongly projective, it follows from Lemma 2.2 (1) and (3) that  $f|_{B^2} = g|_{B^2}$  is a projection on each block of  $\lambda$ .

Now we prove that  $f$  and  $g$  are the same projection on all blocks. Assume wlog that  $f(ab, ac) = ab$  for all  $b, c$ , for some  $a$ . Then  $g(ab, ba)$  is adjacent to  $f(ac, ab) = ac$  for all  $b, c \neq a$ , which forces  $g(ab, ba) = ab$ , and similarly for  $g$ . Now  $ab = f(ab, ba)$  is adjacent to  $g(ba, bc)$  for all  $c \neq a, b$ , and this forces  $g(ba, bc) = ba$ . Hence  $f$  and  $g$  are the first projection when restricted to any block of  $\lambda$ .

We may now assume without loss of generality that  $f|_{B^2} = g|_{B^2}$  is the first projection for every block  $B$  of  $\lambda$ . We show that  $f(x, y) = g(x, y) = x$  for all  $x, y$ . Notice first that the result holds for pairs in  $\sigma$  as was shown in the proof of the claim. Next we prove it for pairs in  $\rho$ :  $f(ac, bc)$  is adjacent to  $g(ab, ba) = ab$  and to  $g(ca, cb) = ca$ , and again we find  $f(ac, bc) = ac$ . Now for  $I$ :  $f(ab, bc)$  is adjacent to  $g(ad, bd) = ad$  for any  $d$  distinct from  $a, b, c$  so this forces  $f(ab, bc) = ab$  or  $f(ab, bc) = ac$ . However  $f(ab, bc)$  is adjacent to  $g(ba, ba) = ba$  so we're done. Next for  $E$ :  $f(ab, ca)$  is adjacent to  $g(ad, ac) = ad$  for any  $d$  different from  $a, b$  so this takes care of this case. Finally we take care of  $M$ :  $f(ab, cd)$  is adjacent to  $g(ad, dc) = ad$  and is also adjacent to  $g(ac, dc) = ac$ . ■

If  $u, v$  are vertices of a connected graph  $K$ , let  $\partial(u, v)$  denote the usual distance between  $u$  and  $v$  in  $K$ . Recall that a vertex in a graph  $K$  is *universal* if it is adjacent to all other vertices.

**Theorem 3.5.** *Let  $K$  be a graph obtained from a connected, ramified, bipartite graph by adding a universal vertex. Then  $K$  is strongly projective.*

**Proof.** Let  $K_0$  be a connected, ramified bipartite graph, and let  $K$  be the graph obtained from  $K_0$  by adding a vertex  $u$  such that  $u$  is adjacent to every vertex in  $K_0$ . We construct a bipartite probe for every pair in  $K$ .

It is usually more convenient to depict probes by diagrams. We shall use the following conventions (see Figures 1 and 2): let  $(P, Q, q, R)$  be a probe for  $\theta$ . Vertices in  $Q$  are depicted by darkened vertices, and the value of the function  $q$  is indicated. The element  $R = v$  is surrounded by a square. If there exists a bipartite probe  $(P', Q', q', R')$  for the set  $A$ , we may use the glueing method used in the proof of Theorem 3.3 above to force values of extensions of  $q$  at certain vertices of  $P$ . More precisely, if  $p \in P$ , we may construct a bipartite probe for the set  $\theta'$  defined by

$$\theta' = \{f(v) : f \text{ is a map from } P \text{ to } K \text{ that extends } q \text{ and } f(p) \in A\}$$

simply by attaching a copy of the probe for  $A$  at the vertex  $p$ . The process may of course be repeated. In diagrams, such glued probes will be omitted,

but the possible values that a homomorphism  $f : P \rightarrow K$  extending  $q$  may take at a vertex  $p$  (i.e., the elements in  $A$ ) will be indicated.

Let  $P$  be a path of length  $t$ , i.e., a graph with vertices  $\{0, 1, 2, \dots, t\}$  and edges  $[i, i + 1]$   $0 = i, \dots, t - 1$ . Let  $Q = \{0\}$  and  $R = t$ . If  $\theta$  admits the probe  $(P, Q, q, R)$  we say that  $\theta$  is a *ball*. Notice that neighbourhoods are balls, and that intersection of balls admit bipartite probes.

(i) Let  $x \in K_0$  and consider the pair  $\{x, u\}$ . Then this is an intersection of balls, namely, if  $N$  is the set of all neighbours of  $x$  in  $K_0$  then  $\{x, u\}$  is the set of all vertices of  $G$  that are adjacent to every vertex in  $N$ . Hence  $\{x, u\}$  admits a bipartite probe.

Let  $\{x, y\} \subseteq K_0$ . Let  $\{a_1, \dots, a_k\}$  and  $\{b_1, \dots, b_r\}$  denote the respective neighbourhoods of  $x$  and  $y$  in  $K_0$ .

(ii) Suppose first that  $\partial(x, y)$  is odd. We construct a probe  $(P, Q, q, R)$  for  $\{x, y\}$  as follows (see Figure 1): let  $P'$  be the subset of  $K^2 \times K_2$  consisting of the following triples:  $(x, y, 0), (y, x, 1), (u, u, 0), (u, u, 1); (a_i, u, 1), (u, b_j, 1), (u, a_i, 0)$  and  $(b_j, u, 0)$  for all  $i, j$ . The edges of  $P'$  are:  $(u, u, 1)$  adjacent to  $(x, y, 0)$ , which is adjacent to each of  $(a_i, u, 1)$  and  $(u, b_j, 1)$ . Each  $(a_i, u, 1)$  is adjacent only to  $(u, a_i, 0)$  and each  $(u, b_j, 1)$  is adjacent only to  $(b_j, u, 0)$ . Then each  $(u, a_i, 0)$  and  $(b_j, u, 0)$  is adjacent to  $(y, x, 1)$  which is adjacent to  $(u, u, 0)$ . To obtain  $P$ , it remains to glue bipartite probes at each  $(a_i, u, 0)$  and  $(u, b_j, 1)$  to insure that the possible values are only  $u$  and  $a_i$  ( $u$  and  $b_j$ ). Also,  $Q$  consists of  $(u, u, 0)$  and  $(u, u, 1)$  with  $q(u, u, i) = u$ . Finally let  $R = (x, y, 0)$ .

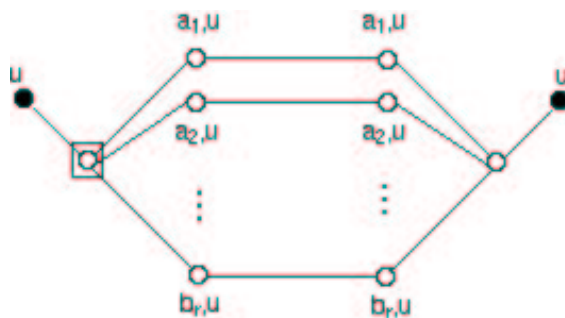


Figure 1: A probe for  $\{x, y\}$  when  $\partial(x, y)$  is odd

It is easy to see that the set  $\theta$  that admits this probe contains  $x$  and  $y$  (just choose the correct projection). Now suppose it contains some  $z \in K$ , and let

$f$  be a map from  $P$  to  $K$  such that  $f(R) = z$ . Clearly  $z \in K_0$ , so there exist  $a_i$  and  $b_j$  not adjacent to  $z$  since  $K_0$  is ramified. This means  $f(a_i, u, 1) = u$  and  $f(u, b_j, 1) = u$ . This in turn forces  $f(u, a_i, 0) = a_i$  and  $f(b_j, u, 0) = b_j$ , and both of these are adjacent to  $f(y, x, 1) \in K_0$ . But this is impossible since  $\partial(x, y)$  is odd.

(iii) Now suppose that  $\partial(x, y)$  is even. Consider a path  $\{c_0, c_1, \dots, c_{2n-1}, c_{2n}\}$  such that  $c_0 = x$ ,  $c_1 = a_1$ ,  $c_{2n-1} = b_1$  and  $c_{2n} = y$  (it must exist since  $K_0$  is bipartite and  $x$  and  $y$  are in the same colour block). We construct a probe  $(P, Q, q, R)$  for  $\{x, y\}$  as follows (see Figure 2): first let  $P'$  consist of vertices labeled by  $(x, y, 0), (x, y, 1); (a_i, u, 1), (u, b_j, 1), (b_j, u, 0), (u, a_i, 0)$  for all  $i, j$ ;  $(y, a_1, 1), (b_1, x, 1)$ ; and  $(u, x, 0), (y, a_i, 0)$  for all  $i$ ; and vertices labeled by  $C_1, \dots, C_{2n-2}$  and vertices labeled by  $B_0, \dots, B_{2n-1}$ . The edges are as follows:  $(x, y, 0)$  is adjacent to each  $(a_i, u, 1), (u, b_j, 1)$ ; each  $(a_i, u, 1)$  is adjacent only to  $(u, a_i, 0)$  and each  $(u, b_j, 1)$  is adjacent only to  $(b_j, u, 0)$ .  $(y, a_1, 1)$  is adjacent to every  $(b_j, u, 0)$  and to  $(u, x, 0)$ ;  $(b_1, x, 1)$  is adjacent to every  $(u, a_i, 0)$  and to every  $(y, a_i, 0)$ ;  $(x, y, 1)$  is adjacent to  $(u, x, 0)$  and every  $(y, a_i, 0)$ ; the  $C_i$  form a path, with  $C_1$  adjacent to  $(x, y, 0)$  and  $C_{2n-2}$  adjacent to  $(x, y, 1)$ ; finally, the  $B_i$  form a path with  $B_0$  coloured with the value  $x$  (i.e.,  $q(B_0) = x$ ) and  $B_{2n-1}$  adjacent to  $(x, y, 0)$ .

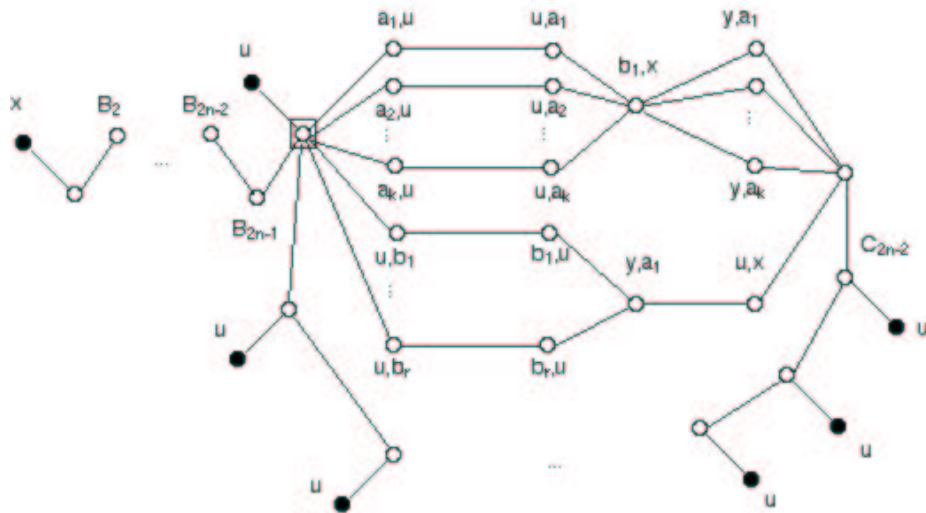


Figure 2: A probe for  $\{x, y\}$  when  $\partial(x, y)$  is even

Next, glue bipartite probes at every vertex other than  $(x, y, i)$ ,  $B_i$  and  $C_i$  to insure that the values are contained in the corresponding 2-subset (note that this is always possible: either one of the labels is  $u$  or the labels are at odd distance in  $K_0$ ). Finally glue a bipartite probe at each  $C_i$ ,  $B_i$  and  $(x, y, 0)$  to insure that their values will lie in  $K_0$  (i.e., just add a vertex coloured by  $u$  adjacent to it). Let  $R = (x, y, 0)$ .

First we show that  $x$  and  $y$  are in the set  $S$  constructed by  $P$ : first consider the map  $f$  which is the first projection on all tuples except  $f(x, y, 1) = b_1$ ,  $f(C_i) = c_i$  for all  $i$ , and  $f(B_i) = a_1$  if  $i$  is odd and  $f(B_i) = x$  otherwise. It is straightforward to verify that  $f$  is a homomorphism. Thus  $x \in S$ .

Now consider the map  $g$  which is the second projection on all tuples except  $g(x, y, 1) = u$ ,  $g(B_i) = c_i$  for all  $i$  and  $g(C_i) = y$  if  $i$  is even and  $g(C_i) = b$  otherwise. It is easy to see that  $g$  is a homomorphism and extends  $q$  so  $y \in S$ .

Next we show that no other element is in  $S$ . Suppose that there exists an  $f$  from  $P$  to  $K$  with  $f(x, y, 0) = z$  different from  $x$  and  $y$ . Then certainly  $z \in K_0$  so there exist  $i, j$  such that  $f(u, b_j, 1) = u$  and  $f(a_i, u, 1) = u$ , which forces in turn  $f(b_j, u, 0) = b_j$  and  $f(u, a_i, 0) = a_i$ . Now each  $b_j$  is at even distance from each  $a_i$  (in  $K_0$ ) so this forces  $f(y, a_1) = y$  and  $f(b_1, x, 1) = x$ . This forces  $f(u, x, 0) = u$  and  $f(y, a_i, 0) = a_i$  for all  $i$ , which forces  $f(x, y, 1) = x$ . But now we've got a path  $f(C_i)$  of odd length from  $x$  to  $z$  and a path  $f(B_i)$  of even length from  $x$  to  $z$ , both in  $K_0$ , a contradiction. ■

The last two families of graphs we shall consider are primitive and distance-transitive graphs. Recall that a graph  $K$  is *primitive* if there exists no non-trivial partition of the vertices of  $K$  which is invariant under all automorphisms of  $K$ . A connected graph is *distance-transitive* if, for all vertices  $a, b, c, d \in K$  such that  $\partial(a, b) = \partial(c, d)$  there exists an automorphism  $\sigma$  of  $K$  such that  $\sigma(a) = c$  and  $\sigma(b) = d$ . The *diameter* of a connected graph  $K$  is the maximum value of  $\partial(u, v)$  taken over all vertices  $u, v$  of  $K$ . For our purposes, call a graph  $K$  *directly indecomposable* if  $K$  is not isomorphic to a product  $A \times B$  where  $A$  and  $B$  are graphs (possibly with loops) with more than one vertex. Notice that a projective graph must be directly indecomposable: the so-called *decomposition* operation

$$((a_1, b_1), (a_2, b_2)) \longmapsto (a_1, b_2)$$

is a non-trivial idempotent operation on the graph  $A \times B$ .

The proofs of the last two theorems will rely on some algebraic machinery that has been developed in [8]. An (universal) algebra is a pair  $\mathbb{A} = \langle A, F \rangle$  where  $A$  is a non-empty set called the *universe* of  $\mathbb{A}$  and  $F$  is a set of operations  $f : A^s \rightarrow A$  on  $A$  called the *basic operations* of  $\mathbb{A}$ . An algebra is *surjective* if all its basic operations are surjective. A *subalgebra* of an algebra  $\mathbb{A}$  is a non-empty subset  $B$  of  $A$  invariant under all operations in  $F$ . The algebra  $\mathbb{A}$  is *simple* if there is no non-trivial partition of  $A$  invariant under all operations in  $F$  (such a partition is called a *congruence* of  $\mathbb{A}$ ). A *term operation* of  $\mathbb{A}$  is an operation on  $A$  which can be obtained from the operations in  $F$  and projections by composition. A *Maltsev term* is a 3-ary operation  $f$  that satisfies

$$f(x, y, y) = f(y, y, x) = x$$

for all  $x, y$ . We shall require a result of A. Szendrei [15, Theorem 3.4] that we reformulate slightly for our purposes.

**Lemma 3.6.** *Let  $\mathbb{A}$  be a finite, simple, surjective algebra with no proper subalgebras. If  $\mathbb{A}$  has no Maltsev term, then there exist an integer  $m \geq 1$ , a set  $X$  and a group of permutations  $\mathcal{S}$  acting primitively on  $X$  such that*

- (i)  $A = X^m$ ,
- (ii) *the  $s$ -ary idempotent term operations of  $\mathbb{A}$  are precisely those of the form*

$$(1) \quad f((x_1^1, \dots, x_m^1), (x_1^2, \dots, x_m^2), \dots, (x_1^s, \dots, x_m^s)) = (x_1^{i_1}, \dots, x_m^{i_m})$$

*for some  $1 \leq i_1, \dots, i_m \leq s$ , and*

- (iii) *the unary term operations of  $\mathbb{A}$  are precisely those of the form*

$$(2) \quad f((x_1, \dots, x_m)) = (\sigma_1(x_{j_1}), \dots, \sigma_m(x_{j_m}))$$

*for some  $\sigma_i \in \mathcal{S}$  and  $1 \leq j_1, \dots, j_m \leq m$ .*

**Proof.** By Szendrei's theorem a finite, simple, surjective algebra with no proper subalgebras either has a Maltsev term or is isomorphic to an algebra term-equivalent to a matrix power of a primitive permutation group (see [8]). The identities (1) and (2) follow directly from the definition of matrix product. ■

In what follows  $K$  shall denote a connected, non-bipartite, ramified vertex-transitive graph without loops. We consider the algebra  $\mathbb{A}(K) = \langle A, F \rangle$  where  $A$  is the vertex set of  $K$ , and  $F$  consists of all functions (of all arities  $s$ )  $f : K^s \rightarrow K$  that are surjective and admit at least one neighbour in  $E_s(K)$ . Notice that for  $s = 1$ , these are precisely the automorphisms of the graph. Notice also that the terms of  $\mathbb{A}(K)$  are precisely its basic operations. Clearly  $\mathbb{A}(K)$  is surjective, and since  $K$  is vertex-transitive  $\mathbb{A}(K)$  has no proper subalgebras.

**Lemma 3.7.** *Let  $K$  be a connected, ramified, non-bipartite graph. Let  $s \geq 3$  and let  $f \in E_s(K)$  that has at least one neighbour and satisfies*

$$f(a_1, \dots, a_{s-2}, x, x) = a_1$$

for all  $a_i \in K$  and all  $x \in K$ . Then  $f$  is a projection onto the first variable. In particular, no function in  $E_3(K)$  with a neighbour can be a Maltsev term.

**Proof.** Let  $P$  be the neighbourhood poset of  $K$ . Consider the operation  $\hat{f}$  defined on  $P$  by

$$\hat{f}(X_1, \dots, X_s) = \bigcap_{a_i \in N_{X_i}} N_{f(a_1, \dots, a_s)}$$

for all  $X_i \in P$  (see [7] Lemma 2.2) It is straightforward to see that  $\hat{f}$  satisfies the identity

$$\hat{f}(Y_1, \dots, Y_{s-2}, X, X) = Y_1$$

for all  $Y_i \in P$  and all  $X \in P$ . By Lemma 2.2 (Claim 3) in [7] it follows that  $\hat{f}$  is the projection onto the first variable, and hence that  $f$  is the projection onto the first variable. ■

**Lemma 3.8.** *Let  $K$  be a connected, ramified, non-bipartite vertex-transitive graph. If the algebra  $\mathbb{A}(K)$  is simple, then there exists a primitive graph  $H$  and an integer  $m \geq 1$  such that  $K$  is isomorphic to  $H^m$ .*

**Proof.** Let  $\mathbb{A} = \mathbb{A}(K)$ . By Lemma 3.6 there exist a set  $H$  and an integer  $m \geq 1$  such that  $V(K) = H^m$ . We shall define the correct graph structure on the set  $H$ . First we must prove that the terms of  $\mathbb{A}$  are actually graph homomorphisms. Let  $f$  be an idempotent term of the algebra  $\mathbb{A}$ . Then

$f \in I_s(K)$  and has a neighbour  $g$ . By Lemma 2.2 (2)  $g \in I_s(K)$ . We show that  $f = g$ . By Lemma 3.6  $f$  and  $g$  satisfy (1) so

$$f((x_1^1, \dots, x_m^1), (x_1^2, \dots, x_m^2), \dots, (x_1^s, \dots, x_m^s)) = (x_1^{i_1}, \dots, x_m^{i_m})$$

$$g((x_1^1, \dots, x_m^1), (x_1^2, \dots, x_m^2), \dots, (x_1^s, \dots, x_m^s)) = (x_1^{j_1}, \dots, x_m^{j_m})$$

for some  $1 \leq i_1, \dots, i_m, j_1, \dots, j_m \leq s$ . Assume wlog that  $i_m \neq j_m$ . Choose 2 vertices  $u = (x_1^1, \dots, x_m^1)$  and  $v = (x_1^2, \dots, x_m^2)$  of  $K$  that have the greatest number of identical coordinates and are adjacent. Because the unary term operations of  $\mathbb{A}$  are automorphisms of  $K$ , and in view of Lemma 3.6 (iii), permutation of the entries of  $m$ -tuples are automorphisms of  $K$ , and hence we may assume that  $x_m^1 \neq x_m^2$ .

Let  $u' = f(u, \dots, u, v, u, \dots, u)$  where  $v$  is in position  $i_m$  and let  $v' = g(v, \dots, v, u, v, \dots, v)$  where  $u$  is in position  $i_m$ . Notice that these vertices are adjacent. It is easy to verify that, if  $u$  and  $v$  have the same  $l$ -th coordinate, then  $u'$  and  $v'$  have the same  $l$ -th coordinate. Furthermore, it is clear that, since  $i_m \neq j_m$ ,  $u'$  and  $v'$  also have the same last coordinate, contradicting our choice of  $u$  and  $v$ .

Let  $h_1, h_2 \in H$ . Define a graph structure on  $H$  by stipulating that  $h_1$  and  $h_2$  are adjacent if there exist  $(x_1, \dots, x_m)$  and  $(y_1, \dots, y_m)$  that are adjacent in  $K$  such that  $x_i = h_1$  and  $y_i = h_2$  for some  $1 \leq i \leq m$ . To show that  $K = H^m$ , let  $u = (x_1^1, \dots, x_m^1)$  and  $v = (x_1^2, \dots, x_m^2)$  be tuples such that  $x_i^1$  is adjacent to  $x_i^2$  in  $H$  for all  $i$ . We must show that  $u$  and  $v$  are adjacent in  $K$ . Fix  $1 \leq i \leq m$ . By definition of adjacency in  $H$ , there exist  $u^i = (y_1^1, \dots, y_m^1)$  and  $v^i = (y_1^2, \dots, y_m^2)$  adjacent in  $K$  such that  $y_j^1 = x_i^1$  and  $y_j^2 = x_i^2$ . By permuting the entries we may suppose that  $j = 1$ . Now the following operation  $f$  is a term of  $\mathbb{A}$ , and hence a homomorphism  $f : K^m \rightarrow K$ :

$$f((z_1^1, \dots, z_m^1), (z_1^2, \dots, z_m^2), \dots, (z_1^m, \dots, z_m^m)) = (z_1^1, z_1^2, \dots, z_1^m).$$

We then have that  $u = f(u^1, \dots, u^m)$  is adjacent to  $v = f(v^1, \dots, v^m)$  and we are done.

Finally we prove that the group of permutations  $\mathcal{S}$  that acts primitively on  $H$  by virtue of Lemma 3.6 is precisely the group of automorphisms of  $H$ . Let  $\sigma$  be an automorphism of  $H$ . Then

$$f((x_1, \dots, x_m)) = (\sigma(x_1), x_2, \dots, x_m)$$



is an automorphism of  $K$  and hence a term of  $\mathbb{A}$ . By Lemma 3.6 (iii) it follows that  $\sigma \in \mathcal{S}$ . Conversely if  $\sigma \in \mathcal{S}$  it is easy to see that  $\sigma$  is an automorphism of  $H$ . It follows that  $H$  is a primitive graph. ■

**Theorem 3.9.** *Let  $K$  be a directly indecomposable primitive graph on at least 3 vertices. Then  $K$  is strongly projective. In particular, Kneser graphs are strongly projective.*

**Proof.** Let  $K$  be a directly indecomposable primitive graph with 3 or more vertices. Clearly connected components are blocks of an invariant partition, so  $K$  must be connected.  $K$  cannot be bipartite, for otherwise the colour classes would form the blocks of an invariant partition. Now suppose that there exist vertices  $a$  and  $b$  such that  $N_a \subseteq N_b$ . Since  $K$  is vertex-transitive, it is regular so we must have that  $N_a = N_b$ . Define a partition  $\rho$  on  $K$  as follows:  $(x, y) \in \rho$  if  $N_x = N_y$ . It is obvious that  $\rho$  is an equivalence relation and is preserved by all automorphisms. Since  $\rho$  contains at least one non-trivial pair and  $K$  is primitive, it follows that  $\rho$  must equal  $K^2$ , i.e.,  $N_x = N_y$  for all  $x$  and  $y$ . This is obviously impossible if  $K$  contains an edge. Hence,  $K$  is ramified. So we may construct the algebra  $\mathbb{A} = \mathbb{A}(K)$  as above. Since  $K$  is primitive it follows that  $\mathbb{A}$  is simple. By Lemma 3.8 we must have  $m = 1$  and so by 3.6 (ii) the only idempotent terms of  $\mathbb{A}$  are projections. It follows that  $K$  is strongly projective. ■

The remainder of the section shall be devoted to proving the following result.

**Theorem 3.10.** *Let  $K$  be a non-bipartite distance-transitive graph of diameter at least 3. Then  $K$  is strongly projective.*

**Lemma 3.11.** *Let  $K$  be a non-bipartite distance-transitive graph of diameter  $d \geq 3$ . Then  $K$  is ramified and directly indecomposable.*

**Proof.** Notice that if  $N_x \subseteq N_y$  for some  $x \neq y$  in  $K$  then by transitivity we get that  $N_u = N_v$  for all vertices  $u$  and  $v$  with  $\partial(u, v) = 2$ . Let  $a, b \in K$  be elements at distance at least 3 and let  $a = x_0, x_1, \dots, x_n = b$  be a shortest path in  $K$  from  $a$  to  $b$ . Then  $x_3$  is adjacent to  $x_2$  which is at distance 2 from  $x_0$ , hence  $x_3$  must be adjacent to  $x_0$ , a contradiction.

It is known that we have unique factorisation into indecomposable factors for connected, non-bipartite ramified graphs, in the class of graphs with loops (see [6], Theorem 5.42). So let

$$K \simeq P_1^{m_1} \times P_2^{m_2} \times \dots \times P_k^{m_k}$$

where the  $P_i$  are pairwise non-isomorphic directly indecomposable graphs (possibly with loops). The  $P_i$  are ramified, connected and non-bipartite. Furthermore, the automorphism group of  $K$  is

$$\text{Aut}(K) = \text{Aut}(P_1) \wr S_{m_1} \times \cdots \times \text{Aut}(P_k) \wr S_{m_k}.$$

*Case 1.* Suppose some factor has loops at every vertex. Let  $P_1, \dots, P_r$  be the factors that have loops at every vertex (since  $K$  has no loops  $r \leq k-1$ ). Since  $K$  is ramified, so is  $P_i$ , and it follows that  $P_i$  must contain vertices  $u_i$  and  $v_i$  at distance 2. For  $r+1 \leq i \leq k$  let  $a_i$  and  $b_i$  be distinct vertices of  $P_i$  such that  $a_i$  has no loop and such that there is a path of length 2 from  $a_i$  to  $b_i$  (such elements must exist if  $P_i$  is connected and has more than 2 elements). Now consider the following elements of  $K$ : let  $\alpha$  be the element of  $K$  with  $u_i$  in each coordinate corresponding to  $P_i$  for  $1 \leq i \leq r$  and  $a_i$  in the coordinates corresponding to  $P_i$  with  $r+1 \leq i \leq k$ ; let  $\beta$  be obtained from  $\alpha$  by replacing each  $u_i$  by  $v_i$ . Let  $\gamma$  be obtained from  $\beta$  by replacing each  $a_i$  by  $b_i$ . Then it is easy to see that  $\partial(\alpha, \beta) = \partial(\alpha, \gamma) = 2$  but there is no automorphism of  $K$  fixing  $\alpha$  and mapping  $\beta$  to  $\gamma$ .

*Case 2.* Each factor has at least one loopless element, and there are at least two non-isomorphic factors. Notice that some factor  $P_i$  has a pair of elements at distance 2, otherwise each of these factors is a complete graph (with no loops, since they must be ramified). But then  $K$  would have diameter 2, a contradiction. So suppose that  $P_1$  has elements  $u$  and  $v$  at distance 2, and for  $i \geq 2$  choose distinct elements  $a_i$  and  $b_i$  in  $P_i$  just as in Case 1:  $a_i$  has no loop and there is a path of length 2 from  $a_i$  to  $b_i$ . The elements  $\alpha$ ,  $\beta$  and  $\gamma$  of  $K$  are defined in a manner similar to Case 1.

*Case 3.*  $K = P^n$  for some directly indecomposable graph  $P$ . We suppose that  $n \geq 2$ . Let  $a$  and  $b$  be elements of  $P$  such that  $\partial(a, b) = 2$ . Consider the following elements of  $K$ :  $\alpha = (a, b, b, \dots, b)$ ,  $\beta = (a, a, b, \dots, b)$  and  $\gamma = (b, a, b, \dots, b)$ . It is clear that  $\partial(\alpha, \beta) = \partial(\alpha, \gamma) = 2$ . It remains to show that no automorphism of  $K$  can fix  $\alpha$  and map  $\beta$  to  $\gamma$ : but this follows from the fact that  $\alpha$  and  $\gamma$  have two distinct coordinates, but  $\alpha$  and  $\beta$  have only one. By the structure of the automorphism group of  $K$  we conclude that  $n = 1$ . ■

Let  $K$  be a non-bipartite distance-transitive graph of diameter  $d \geq 3$ . By the last lemma, we may construct the algebra  $\mathbb{A} = \mathbb{A}(K)$ . If it is simple, then by Lemmas 3.8 and 3.11  $K$  is primitive and so it is strongly projective

by Theorem 3.9. So assume that the algebra  $\mathbb{A}$  is not simple. In particular, this means that there exists a non-trivial partition  $\theta$  of the vertices which is invariant under all automorphisms of  $K$ . The following result of Smith [14] will describe  $\theta$  precisely. A graph  $K$  of diameter  $d$  is *antipodal* if the following holds: for any vertices  $u, v, w$  such that  $\partial(u, v) = \partial(u, w) = d$  then either  $v = w$  or  $\partial(v, w) = d$ . In other words, the relation ‘being at distance  $d$  or equal’ is an equivalence relation.

**Theorem 3.12** [14]. *Let  $G$  be a non-bipartite distance-transitive graph of diameter  $d \geq 3$ . Then  $G$  is (i) an odd cycle, or (ii) primitive or (iii) antipodal. Furthermore, in the last case, the only partition preserved by the automorphisms is the antipodality relation. ■*

We may assume that  $K$  is neither an odd cycle nor primitive. It follows that the algebra  $\mathbb{A}$  has a unique non-trivial congruence that we shall denote by  $\theta$ . Let  $\mathbb{B}$  denote the quotient algebra  $\mathbb{A}/\theta$ , i.e., the algebra whose universe is the set  $B = A/\theta$  of blocks of the partition, and whose basic operations are those induced by the basic operations of  $\mathbb{A}$ , i.e., if  $f \in F$ , let  $f'$  be defined by

$$f'(a_1/\theta, \dots, a_s/\theta) = f(a_1, \dots, a_s)/\theta$$

for all  $a_i \in A$ .

**Lemma 3.13.** *The algebra  $\mathbb{B}$  is simple, surjective and has no proper subalgebras. Furthermore, it has no Maltsev term.*

**Proof.** By the correspondence theorem (see [12], Theorem 4.12), and since  $\mathbb{A}$  has a unique non-trivial congruence,  $\mathbb{B}$  is simple. Let  $f$  be a basic operation of  $\mathbb{A}$ . Then  $f$  is surjective, and hence the corresponding operation  $f'$  of  $\mathbb{B}$  is also surjective: if  $b/\theta \in \mathbb{B}$  then there exist  $a_i \in A$  such that  $f(a_1, \dots, a_s) = b$  and so  $f'(a_1/\theta, \dots, a_s/\theta) = f(a_1, \dots, a_s)/\theta = b/\theta$ . Let  $a/\theta$  and  $b/\theta$  be distinct elements of  $B$ . Then there exists an operation  $f \in F$  such that  $f(a) = b$  so  $f'(a/\theta) = b/\theta$ . It follows that  $\mathbb{B}$  cannot have proper subalgebras.

Now we prove that  $\mathbb{B}$  has no Maltsev term. We define a graph structure  $K/\theta$  on  $B$  as follows: two elements  $X$  and  $Y$  of  $B$  are adjacent if there exist  $a \in X$  and  $b \in Y$  such that  $a$  and  $b$  are adjacent in  $K$ . Clearly this relation is symmetric, and since  $\theta$  is the antipodality relation of  $K$  the graph  $K/\theta$  has no loops. It is easy to see that  $K/\theta$  is ramified, connected

and non-bipartite, since  $K$  has these properties. Let  $\phi$  be a term of  $\mathbb{B}$ . Since the set  $F$  of basic operations of  $\mathbb{A}$  is closed under composition, it follows that  $\phi = f'$  is induced by some  $f \in F$ , which has a neighbour  $g$  in  $E_s(K)$ . It is straightforward to verify that the operation  $g'$  induced by  $g$  is a neighbour of  $f'$ : if  $X_1, \dots, X_s$  are adjacent to  $Y_1, \dots, Y_s$  respectively, then there exist  $a_i \in X_i$  and  $b_i \in Y_i$  such that  $a_i$  is adjacent to  $b_i$  for all  $i$ . Hence  $f'(X_1, \dots, X_s) = f(a_1, \dots, a_s)/\theta$  is adjacent to  $g(b_1, \dots, b_s)/\theta = g'(Y_1, \dots, Y_s)$ . By Lemma 3.7, it follows that  $\mathbb{B}$  has no Maltsev term. ■

**Proof of Theorem 3.10.** Let  $K$  be a non-bipartite distance-transitive graph of diameter at least 3. By the above discussion we may assume that the algebra  $\mathbb{A}(K)$  has the unique congruence  $\theta$  induced by the antipodality relation. Define the algebra  $\mathbb{B}$  as above, and the graph  $K/\theta$  as in the proof of Lemma 3.13. This graph is connected, ramified, non-bipartite and vertex-transitive. In particular we may consider the algebra  $\mathbb{A}(K/\theta)$ .

To show that  $K$  is strongly projective, it will suffice to prove the following: if  $f \in I_2(K)$  has a neighbour, then it is a projection. Indeed, if this holds then  $K$  is projective by Theorem 1.1 of [9]; by Lemma 2.2 (1),  $I_2(K)$  has only isolated vertices, and thus by 2.3 (3)  $K$  is strongly projective.

Let  $f \in I_2(K)$  have a neighbour  $g$  (then  $g \in I_2(K)$  by Lemma 2.2 (2)). Let  $f'$  and  $g'$  be the induced terms of  $\mathbb{B}$ . Referring again to the proof of Lemma 3.13, we see that  $f'$  and  $g'$  are adjacent; and that every term of  $\mathbb{B}$  is a term of  $\mathbb{A}(K/\theta)$  which implies that this algebra is simple. We apply (the proof of) Lemma 3.8 to obtain that  $B = H^m$  and

$$f'((x_1^1, \dots, x_m^1), (x_1^2, \dots, x_m^2)) = (x_1^{i_1}, \dots, x_m^{i_m})$$

for some  $1 \leq i_1, \dots, i_m \leq 2$ , and that  $f' = g'$ . Hence  $f' = g'$  is a decomposition operation, i.e., satisfies the identities

$$f'(X, f'(Y, Z)) = f'(X, Z) = f'(f'(X, Y), Z)$$

for all  $X, Y, Z \in B$ . It follows that  $f$  has the following property: for all  $a, b, c \in K$ , the elements  $f(a, f(b, c))$ ,  $f(a, c)$  and  $f(f(a, b), c)$  are all in the same block of  $\theta$ . We shall show that in fact  $f$  is a decomposition operation. Let  $a, b, c \in K$ . Since this graph is connected and non-bipartite, there exists an integer  $n \geq 1$  and an element  $d \in K$  such that there are paths of length  $n$  from  $d$  to each one of  $a, b, c$ . We use induction on  $n$  to prove that  $f(a, f(b, c)) = f(a, c)$  (the other identity is similar). If  $n = 1$ , then

$f(a, f(b, c))$  is adjacent to  $g(d, g(d, d)) = d$  and  $f(a, c)$  is adjacent to  $g(d, d) = d$ . But since  $K$  has diameter at least 3, no two distinct elements of a block can have a common neighbour. It follows that  $f(a, f(b, c)) = f(a, c)$ , and similarly for  $g$ . Now suppose the result holds for  $n$ , and that  $a = x_0, x_1, \dots, x_{n+1} = d$ ,  $b = y_0, y_1, \dots, y_{n+1} = d$ ,  $c = z_0, z_1, \dots, z_{n+1} = d$  are paths of length  $n + 1$  from  $a, b, c$  to  $d$ . Then  $f(a, f(b, c))$  is adjacent to  $f(x_1, f(y_1, z_1))$ , and  $f(a, c)$  is adjacent to  $f(x_1, z_1)$ . By induction hypothesis  $f(x_1, f(y_1, z_1)) = f(x_1, z_1)$  and as before we conclude that  $f(a, f(b, c)) = f(a, c)$ .

Next we show that  $f = g$ . Indeed, if  $a$  and  $b$  are adjacent to  $c$  then both  $g(a, b)$  and  $f(a, b)$  are adjacent to  $c$ . But since  $f' = g'$ ,  $g(a, b)$  and  $f(a, b)$  are in the same block of  $\theta$  and hence must be equal. The rest follows as above by an easy induction.

It is known (and straightforward to verify) that if  $f$  is a decomposition operation on the set  $K$ , then there exists a (set-theoretic) decomposition  $K = A \times B$  such that  $f((a, b), (a', b')) = (a, b')$  for all  $a, a' \in A$  and all  $b, b' \in B$  [12, p. 162]. Define graph structures on  $A$  and  $B$  as follows: let  $a$  and  $a'$  be adjacent in  $A$  if there exist  $y, y' \in B$  such that  $(a, y)$  and  $(a', y')$  are adjacent in  $K$ , and similarly for  $B$ . We claim that  $K$  is isomorphic to the (graph) product  $A \times B$ . Indeed, let  $a$  and  $a'$  be adjacent in  $A$  and let  $b$  and  $b'$  be adjacent in  $B$ . Then there exist  $x, x' \in A$  and  $y, y' \in B$  such that  $(a, y)$  and  $(a', y')$  are adjacent in  $K$  and  $(x, b)$  and  $(x', b')$  are adjacent in  $K$ . This means that  $(a, b) = f((a, y), (x, b))$  is adjacent to  $f((a', y'), (x', b')) = (a', b')$ . By Lemma 3.11 the graph  $K$  is directly indecomposable, so one of  $A$  or  $B$  has only one vertex. It follows immediately that  $f$  is a projection, and this completes the proof of Theorem 3.10. ■

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