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GENERALIZED CHROMATIC NUMBERS AND ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P}) = n$ iff $\mathcal{P} \subseteq \mathcal{Q}^n$ but $\mathcal{P} \not\subseteq \mathcal{Q}^{n-1}$. We investigate the generalized chromatic numbers of the well-known properties of graphs $\mathcal{I}_k$, $\mathcal{O}_k$, $\mathcal{W}_k$, $\mathcal{S}_k$ and $\mathcal{D}_k$.

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1. Introduction

Following [1] we denote the class of all finite simple graphs by $\mathcal{I}$. A *property* of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$; $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$.

Throughout the text we will call a component of a graph that is a spanning supergraph of a path P_k of order k a *k-component*. Let G be a graph and $V_1 \subseteq V(G)$. We say that a vertex $v \in V(G) - V_1$ *is adjacent to a k-component of* $G[V_1]$ if v is adjacent to a vertex of some k-component of $G[V_1]$.

Example. For a positive integer k we define the following well-known properties:

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,

$\mathcal{I}_k = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$,

$\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$,

$\mathcal{W}_k = \{G \in \mathcal{I} :$ each path in G has at most $k+1$ vertices$\}$,

$\mathcal{S}_k = \{G \in \mathcal{I} :$ the maximum degree of G is at most $k\}$,

$\mathcal{T}_k = \{G \in \mathcal{I} : G$ contains no subgraph homeomorphic to K_{k+2} or $K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}$,

$\mathcal{D}_k = \{G \in \mathcal{I} : G$ is k-degenerate, i.e., every subgraph of G has a vertex of degree at most $k\}$.

For every additive hereditary property $\mathcal{P} \neq \mathcal{I}$ there is a smallest integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the *completeness* of $\mathcal{P}$. Note that all the properties in the above example, except $\mathcal{O}$, are of completeness k. The set $\mathbf{F}(\mathcal{P})$ of *minimal forbidden subgraphs* is defined by $\{G \in \mathcal{I} : G \in \overline{\mathcal{P}}$ and $H \in \mathcal{P}$ for all $H \subset G\}$.

Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ be arbitrary hereditary properties of graphs. A *vertex* $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$*-partition* of a graph G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{Q}_i$. The property $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$ is defined as the set of all graphs having a vertex $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$-partition. It is easy to see that if $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$ is additive and hereditary too. If $\mathcal{Q}_1 = \mathcal{Q}_2 = \cdots = \mathcal{Q}_n = \mathcal{Q}$, then we write $\mathcal{Q}^n = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$.

The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P}) = n$ iff $\mathcal{P} \subseteq \mathcal{Q}^n$ but $\mathcal{P} \nsubseteq \mathcal{Q}^{n-1}$.

As an example of the non-existence of $\chi_{\mathcal{Q}}(\mathcal{P})$ we have $\chi_{\mathcal{O}}(\mathcal{I}_1)$ since it is well known that there exist triangle-free graphs of arbitrary chromatic number. The following theorem, due to J. Nešetřil and V. Rödl (see [12]), implies that for some additive hereditary properties $\mathcal{P}$ we have that $\chi_{\mathcal{Q}}(\mathcal{P})$ exists if and only if $\chi_{\mathcal{Q}}(\mathcal{P}) = 1$. In particular, $\chi_{\mathcal{Q}}(\mathcal{I}_k)$ exists if and only $\chi_{\mathcal{Q}}(\mathcal{I}_k) = 1$.

Theorem 1.1 [12]**.** *Let* $\mathbf{F}(\mathcal{P})$ *be a finite set of* 2*-connected graphs. Then for every graph* $G \in \mathcal{P}$ *there exists a graph* $H \in \mathcal{P}$ *such that for any partition* $\{V_1, V_2\}$ *of* $V(H)$ *there is an* i, $i = 1$ *or* $i = 2$, *for which* $G \leq H[V_i]$. ■

Corollary 1.2. *If* $\mathbf{F}(\mathcal{P})$ *is a finite set of* 2*-connected graphs, then for any additive hereditary property* $\mathcal{Q}$ *it follows that* $\chi_{\mathcal{Q}}(\mathcal{P})$ *exists if and only if* $\mathcal{P} \subseteq \mathcal{Q}$. ■

The value of $\chi_{\mathcal{Q}}(\mathcal{P})$ is known for various choices of $\mathcal{P}$ and $\mathcal{Q}$. In the remainder of this section we mention some simple results, most of which are known or follow immediately from well-known results. See for example [2] and [5].

It is easy to see that $\mathcal{O}_{a+b+1} \subseteq \mathcal{O}_a \circ \mathcal{O}_b$ and $\mathcal{D}_{a+b+1} \subseteq \mathcal{D}_a \circ \mathcal{D}_b$ (see for example [9]), which implies that $\chi_{\mathcal{Q}}(\mathcal{O}_k) = \left\lceil \frac{k+1}{n+1} \right\rceil$ for any property $\mathcal{Q}$ of completeness n, and $\chi_{\mathcal{D}_n}(\mathcal{P}) = \left\lceil \frac{k+1}{n+1} \right\rceil$ for any property $\mathcal{P}$ such that $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{D}_k$. Note that Corollary 1.2 implies that the latter equality does not extend to $c(\mathcal{P}) = n$.

The well-known theorem of Lovász states:

Theorem 1.3 [10]**.** $\mathcal{S}_{a+b+1} \subseteq \mathcal{S}_a \circ \mathcal{S}_b$ *for all* $a, b \geq 0$. ■

This implies that $\chi_{\mathcal{S}_n}(\mathcal{S}_k) = \left\lceil \frac{k+1}{n+1} \right\rceil$. (See [5].)

It is also easy to see that if $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{O}^{k+1}$, then $\chi_{\mathcal{I}_n}(\mathcal{P}) = \left\lceil \frac{k+1}{n+1} \right\rceil$.

The next result is interesting since it shows that the value of $\chi_{\mathcal{S}_n}(\mathcal{D}_k)$ is independent of n.

Theorem 1.4. *For all* k *and* n *we have* $\chi_{\mathcal{S}_n}(\mathcal{D}_k) = k + 1$.

Proof. Since $\mathcal{D}_k \subseteq \mathcal{O}^{k+1} \subseteq \mathcal{S}_n^{k+1}$ we have the upper bound. We prove the lower bound by induction on k. The result is true for $k = 1$ since $\mathcal{D}_1 \not\subseteq \mathcal{S}_n$. Assume, therefore, that $\mathcal{D}_k \not\subseteq \mathcal{S}_n^k$ and let $H \in \mathcal{D}_k$ such that $H \notin \mathcal{S}_n^k$. Let $G = (n+1)H + K_1$. Since every subgraph of $(n+1)H$ has a vertex of degree at most k, every subgraph of G has a vertex of degree at most $k+1$. Thus $G \in \mathcal{D}_{k+1}$.

Also, $G \notin \mathcal{S}_n^{k+1}$: Suppose, to the contrary, that $\{V_1, V_2, \ldots, V_{k+1}\}$ is an $\mathcal{S}_n^{k+1}$-partition of $V(G)$. Let v be the universal vertex of G and suppose, without loss of generality, that $v \in V_1$. Since $G[V_1] \in \mathcal{S}_n$ it follows that $|V_1| \leq n + 1$. Since there are $n + 1$ copies of H in G we have that for some copy F of H, $F \cap V_1 = \emptyset$. This contradicts the fact that $H \notin \mathcal{S}_n^k$. ■

The lattice of (additive) hereditary properties is discussed in [1] — we use the supremum and infimum of properties in our next result without further discussion.

Theorem 1.5. *Let $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{Q}$ be additive hereditary properties such that $\chi_{\mathcal{Q}}(\mathcal{P}_1)$ and $\chi_{\mathcal{Q}}(\mathcal{P}_2)$ are finite. The following hold:*

(i) $\chi_{\mathcal{Q}}(\mathcal{P}_1 \cup \mathcal{P}_2) = \chi_{\mathcal{Q}}(\mathcal{P}_1 \vee \mathcal{P}_2) = \max\{\chi_{\mathcal{Q}}(\mathcal{P}_1), \chi_{\mathcal{Q}}(\mathcal{P}_2)\}$.

(ii) $\chi_{\mathcal{Q}}(\mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\chi_{\mathcal{Q}}(\mathcal{P}_1), \chi_{\mathcal{Q}}(\mathcal{P}_2)\}$.

(iii) $\max\{\chi_{\mathcal{Q}}(\mathcal{P}_1), \chi_{\mathcal{Q}}(\mathcal{P}_2)\} \leq \chi_{\mathcal{Q}}(\mathcal{P}_1 \circ \mathcal{P}_2) \leq \chi_{\mathcal{Q}}(\mathcal{P}_1) + \chi_{\mathcal{Q}}(\mathcal{P}_2)$. ■

We remark that the inequality in Theorem 1.5(ii) may be strict. For example $\chi_{\mathcal{O}}(\mathcal{T}_3) = 4$ and $\chi_{\mathcal{O}}(\mathcal{I}_1)$ is infinite but $\chi_{\mathcal{O}}(\mathcal{T}_3 \cap \mathcal{I}_1) = 3$. (See [2].)

2. Results on $\mathcal{W}_k$

In this section we investigate the value of $\chi_{\mathcal{W}_n}(\mathcal{W}_k)$. The problem of determining it has been discussed in (or is related to problems in) several papers (see for example [3], [4], [6], [7], [8] and [11]) and the following conjecture has been made in at least three of them:

Conjecture 2.1 [3], [6], [7]**.** *$\mathcal{W}_{a+b+1} \subseteq \mathcal{W}_a \circ \mathcal{W}_b$ for all positive integers a and b.*

This conjecture implies the following for $\chi_{\mathcal{W}_n}(\mathcal{W}_k)$:

Conjecture 2.2. *For every $n, k \geq 1$, the following holds:*

$$\chi_{\mathcal{W}_n}(\mathcal{W}_k) = \left\lceil \frac{k+1}{n+1} \right\rceil .$$

In [6] the bound $\chi_{\mathcal{W}_n}(\mathcal{W}_k) \leq \lfloor \frac{k-n+1}{2} \rfloor + 2$ is proved. The following theorem will enable us to improve on this bound.

Theorem 2.3. *$\mathcal{W}_{\lceil \frac{2a}{3} \rceil + b + 1} \subseteq \mathcal{W}_a \circ \mathcal{W}_b$ for all $a \geq 15$ and $b \geq 1$.*

Proof. Consider any graph G in $\mathcal{W}_{\lceil \frac{2a}{3} \rceil + b + 1}$. Take V_1 to be a maximal subset of $V(G)$ such that $G[V_1]$ is in $\mathcal{W}_a$. Let $V_2 = V(G) - V_1$. Suppose that there is a path P in $G[V_2]$ of length $b+1$ and let v_1 and v_2 denote the end-vertices of P. Since V_1 is maximal in $\mathcal{W}_a$ it follows that there is a path P_1 of length $a+1$ in $G[V_1 \cup \{v_1\}]$ and a path P_2 of length $a+1$ in $G[V_1 \cup \{v_2\}]$. Note that if either v_1 or v_2 is an end-vertex of P_1 or P_2 respectively, then in both cases we get a path of length at least $a+b+3$

in G, a contradiction. Therefore the vertices v_1 and v_2 are not end-vertices of their respective paths. Let P_{11} and P_{12} denote the paths on either side of v_1 such that $P_{11} \cup \{v_1\} \cup P_{12} = P_1$. Similarly, let $P_{21} \cup \{v_2\} \cup P_{22} = P_2$. Now suppose, without loss of generality, that $x = |E(P_{11})| + 1 \leq y = |E(P_{12})| + 1$, so that $x + y = a + 1$.

It is easily seen that if $y \geq \lfloor \frac{2a+2}{3} \rfloor + 1$, then by simply taking the path $P_{12} \cup P$, we get a path of length at least $\lfloor \frac{2a+2}{3} \rfloor + 1 + b + 1 \geq \frac{2a+2-2}{3} + b + 2 > \lceil \frac{2a}{3} \rceil + b + 1$ in G, a contradiction. Therefore $\lceil \frac{a+1}{2} \rceil \leq y \leq \lfloor \frac{2a+2}{3} \rfloor$. Moreover, each P_{ij}, $i, j \in \{1, 2\}$ has length at least $\lfloor \frac{a-5}{3} \rfloor$, since $x = a + 1 - y \geq a - \lfloor \frac{2a+2}{3} \rfloor + 1 \geq a - \frac{2a+5}{3} = \frac{a-5}{3} \geq \lfloor \frac{a-5}{3} \rfloor$.

Note that P_{11} and P_{12} are neccessarily disjoint as are P_{21} and P_{22}, and that v_1 and v_2 are not on any of these paths.

P_{12} must intersect both P_{21} and P_{22}: Firstly, P_{12} must intersect the longer of P_{21} and P_{22} since otherwise we get a too long path in G; containing the two longer paths and P. Furthermore, if P_{12} does not intersect the shorter of P_{21} and P_{22}, then we get a path of length at least $\lceil \frac{a+1}{2} \rceil + b + 1 + \lfloor \frac{a-5}{3} \rfloor \geq \frac{a+1}{2} + \frac{a-7}{3} + b + 1 = \frac{5}{6}(a-1) > \lceil \frac{2a}{3} \rceil + 1 + b$ (since $a \geq 15$) in G; containing P_{12}, P and the shorter of P_{21} and P_{22}, a contradiction. Similarly, the longer of P_{21} and P_{22} must intersect both P_{11} and P_{12}.

Note that since P_{11} and P_{12} are disjoint and P_{21} and P_{22} are disjoint, P_{2i}, $i \in \{1, 2\}$ can only intersect one of P_{11} and P_{12} first and vice-versa.

Suppose that both P_{21} and P_{22} intersect P_{12} first. Then we obtain a path of length at least $x + b + 1 + 1 + \lfloor \frac{y}{2} \rfloor \geq a + 1 - y + \frac{y-1}{2} + b + 2 \geq a - \frac{1}{2} \lfloor \frac{2a+2}{3} \rfloor + \frac{5}{2} + b \geq a - \frac{1}{2}(\frac{2a+2}{3}) + \frac{5}{2} + b = \frac{2a}{3} + \frac{13}{6} + b > \lceil \frac{2a}{3} \rceil + 1 + b$ in G; containing P_{11}, P, at least one edge of either P_{21} or P_{22} and at least a half of P_{12}, a contradiction.

Now, suppose that P_{21} or P_{22} intersects P_{11} first, say P_{21}. Then we obtain a path of length at least $y + \lfloor \frac{x}{2} \rfloor + b + 1 + 1 = y + \lfloor \frac{1}{2}(a + 1 - y) \rfloor + b + 2 \geq y + \frac{a+1-y-1}{2} + b + 2 = \frac{y}{2} + \frac{a}{2} + b + 2 \geq \frac{1}{2} \lceil \frac{a+1}{2} \rceil + \frac{a}{2} + b + 2 \geq \frac{3a+9}{4} + b > \lceil \frac{2a}{3} \rceil + 1 + b$ in G; containing P_{12}, P, at least one edge of P_{21} and at least a half of P_{11}, a contradiction. ∎

Theorem 2.4. *$\chi_{\mathcal{W}_n}(\mathcal{W}_k) \leq \left\lceil \frac{3k}{2n+3} \right\rceil$ for all $n \geq 15$ and $k \geq 1$.*

Proof. $\mathcal{W}_{c\lceil \frac{2n+3}{3} \rceil} \subseteq \mathcal{W}_n^c$ for all positive integers c and n: the proof is by induction on c. The result holds for $c = 1$. Suppose now that the result holds for c. Note that $\mathcal{W}_{(c+1)\lceil \frac{2n+3}{3} \rceil} = \mathcal{W}_{\lceil \frac{2n}{3} \rceil + 1 + c\lceil \frac{2n+3}{3} \rceil}$ which by Theorem 2.3 is

contained in $\mathcal{W}_n \circ \mathcal{W}_{c\lceil \frac{2n+3}{3} \rceil}$ which by the induction hypothesis is contained in $\mathcal{W}_n \circ \mathcal{W}_n^c = \mathcal{W}_n^{c+1}$.
Now, with $c = \lceil \frac{3k}{2n+3} \rceil$, since $k \leq \lceil \frac{3k}{2n+3} \rceil \lceil \frac{2n+3}{3} \rceil$ we have that $\mathcal{W}_k \subseteq \mathcal{W}_{c\lceil \frac{2n+3}{3} \rceil}$ $\subseteq \mathcal{W}_n^c$. ■

This result is close to the bound $\chi_{\mathcal{W}_n}(\mathcal{W}_k) \leq \left\lceil \frac{3(k-n)}{2n+2} \right\rceil + 1$ presented in [7] but our method of proof is completely different.

3. Results Relating $\mathcal{S}_k$ and $\mathcal{W}_n$

Theorem 3.1. *For positive integers n and k we have that*

$$\left\lceil \frac{k+1}{n+1} \right\rceil \leq \chi_{\mathcal{W}_n}(\mathcal{S}_k) \leq \left\lceil \frac{k+1}{2} \right\rceil .$$

Proof. The left inequality holds since $K_{k+1} \in \mathcal{S}_k$. The right inequality follows as a corollary to Theorem 1.3. ■

The first inequality in Theorem 3.1 may be strict, for example $\chi_{\mathcal{W}_2}(\mathcal{S}_2) = 2 > \lceil \frac{2+1}{2+1} \rceil$ (since $\mathcal{S}_2 \not\subseteq \mathcal{W}_2$). Equality in both the inequalities may be achieved, for example, by Theorem 1.3 we have that $\chi_{\mathcal{S}_n}(\mathcal{S}_k) = \left\lceil \frac{k+1}{n+1} \right\rceil$ and therefore $\chi_{\mathcal{W}_1}(\mathcal{S}_k) = \left\lceil \frac{k+1}{2} \right\rceil$.

Note that whether or not the second inequality proved in Theorem 3.1 may be strict still remains an open problem.

We now start working towards bounds on $\chi_{\mathcal{S}_n}(\mathcal{W}_k)$.

Theorem 3.2. $\mathcal{W}_4 \subseteq \mathcal{S}_2 \circ \mathcal{S}_1$.

Proof. Consider any graph G in $\mathcal{W}_4$. Take V_1 to be a subset of $V(G)$ such that, in order of priority:

(i) $G[V_1]$ is in $\mathcal{S}_2$,

(ii) $G[V_1]$ contains a maximum number of 4-components,

(iii) $G[V_1]$ contains a maximum number of components isomorphic to K_3,

(iv) $G[V_1]$ contains a maximum number of 2-components and

(v) $G[V_1]$ contains a maximum number of isolated vertices.

(In other words, we consider all subsets V of $V(G)$ such that $G[V] \in \mathcal{S}_2$. Amongst these we consider all subsets V for which $G[V]$ has a maximum

number of 4-components. Amongst these we consider all subsets inducing a maximum number of components isomorphic to K_3 etc.)
Let $V_2 = V(G) - V_1$. We will show that $G[V_2] \in \mathcal{S}_1$. Suppose, to the contrary, that $G[V_2] \notin \mathcal{S}_1$ and let v be a vertex in $G[V_2]$ of degree at least two with u and w two of its neighbours in $G[V_2]$. Note that by choice of V_1 every component in $G[V_1]$ is a 4-component, K_3, K_2 or K_1.

Moreover, by (v) it follows that u, v and w each have at least one neighbour in V_1. Furthermore, v is adjacent to a nontrivial component in $G[V_1]$: If this is not the case, then we can replace the vertices in V_1 that are adjacent to v with a 2-component; still satisfying (i) through (iii) but contradicting (iv). Similarly, u and w are adjacent to nontrivial components in $G[V_1]$.

Suppose that v is adjacent to a triangle in $G[V_1]$. Note that neighbours of both u and w in V_1 can only lie on this triangle, otherwise we obtain at least a P_6 in G. However then we obtain a P_6 in G; containing all three vertices of the triangle in $G[V_1]$ as well as the P_3 formed by u, v and w. Thus v cannot be adjacent to a triangle in $G[V_1]$.

Furthermore, v cannot be adjacent to a 4-component in $G[V_1]$. This case is analogous to the above case since a 4-component will also contribute three vertices to give a P_6 in G. Moreover, neither u nor w are adjacent to 4-components or triangles in $G[V_1]$, since otherwise we obtain at least a P_6 in G.

Therefore v must be adjacent to a K_2 in $G[V_1]$. Note that u and w must each have at least one neighbour on the K_2 adjacent to v in $G[V_1]$, otherwise we obtain a P_6 in G. If v is adjacent to both vertices on the K_2 in $G[V_1]$, then we can replace the components in $G[V_1]$ that are adjacent to u, v and w with a triangle; still satisfying (i) and (ii), but contradicting (iii).

Thus v has only one neighbour on any K_2 in $G[V_1]$. If u or w is adjacent to the same vertex as v on the K_2 adjacent to v in $G[V_1]$, then once again we can replace the components in $G[V_1]$ that are adjacent to u, v and w with a triangle; still satisfying (i) and (ii), but contradicting (iii). Therefore, both u and w are adjacent to the vertex on the K_2 in $G[V_1]$ that is not adjacent to v. However, then we can replace the components in $G[V_1]$ that are adjacent to u, v and w with a 4-component; containing the K_2 in $G[V_1]$ and the vertices v and either u or w; still satisfying (i), but contradicting (ii). Therefore $G[V_2] \in \mathcal{S}_1$. ■

Corollary 3.3. *For all $n \geq 2$ and k, we have that $\chi_{\mathcal{S}_n}(\mathcal{W}_k) \leq 2\left\lceil \frac{k+1}{5} \right\rceil$.*

Proof. It is known that $\mathcal{W}_{4+k+1} \subseteq \mathcal{W}_4 \circ \mathcal{W}_k$ (see [3]). Similar to the proof of Theorem 1.3 it follows that $\mathcal{W}_k \subseteq \mathcal{W}_4^{\lceil \frac{k+1}{5} \rceil}$. The result now follows from Theorem 3.2. ■

The inequality in Corollary 3.3 may be strict, for example $\chi_{\mathcal{S}_2}(\mathcal{W}_1) = 1 < 2 = 2\lceil \frac{2}{5} \rceil$. Equality may also be obtained, for example $\chi_{\mathcal{S}_2}(\mathcal{W}_2) = 2 = 2\lceil \frac{3}{5} \rceil$.

Having proved Corollary 3.3 we naturally ask: Can this bound be improved and if so under what conditions? Corollary 3.5 gives us an answer for $n \geq 5$ and Theorem 3.7 for $n \geq 9$.

Theorem 3.4. *For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 5$, the following holds: $\mathcal{W}_k \subseteq \mathcal{Q}^{\lceil \frac{k}{3} \rceil} \circ \mathcal{O}$.*

Proof. Let $c = \left\lceil \frac{k}{3} \right\rceil$. Consider any graph G in $\mathcal{W}_k$. Take V_1 to be a subset of $V(G)$ such that, in order of priority:

(i) $G[V_1]$ is in $\mathcal{Q}$,

(ii) $G[V_1]$ contains a maximum number of 6-components,

(iii) $G[V_1]$ contains a maximum number of 4-components,

(iv) $G[V_1]$ contains a maximum number of 2-components and

(v) $G[V_1]$ contains a maximum number of isolated vertices.

Now, for $2 \leq i \leq c$ take V_i to be a subset of $V(G) - \bigcup_{j=1}^{i-1} V_j$ such that for each i, $G[V_i]$ satisfies the above list. Let $S = V(G) - \bigcup_{j=1}^{c} V_j$. We will show that $G[S] \in \mathcal{O}$. Suppose, to the contrary, that $G[S] \notin \mathcal{O}$ and let v be a vertex in $G[S]$ of degree at least one and u be a neighbour of v in $G[S]$. Suppose that v is not an end-vertex of a P_4 in $G[V_c \cup \{v\}]$ and that u is not an end-vertex of a P_5 in $G[V_c \cup \{u\}]$. Note that for every i, the choice of V_i gives that every component in $G[V_i]$ is a 6-component, a 4-component, K_2 or K_1. Moreover, by (v) it follows that u and v have at least one neighbour in V_c each and by (iv) both u and v are adjacent to nontrivial components in $G[V_c]$. Since v is not an end-vertex of a P_4 in $G[V_c \cup \{v\}]$ it follows that v is not adjacent to a 6-component or a 4-component in $G[V_c]$. Similarly, u is not adjacent to a 6-component in $G[V_c]$.

Suppose that u is adjacent to a 4-component in $G[V_c]$. Then, since v is not adjacent to a 4-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to u and v with a 6-component; still satisfying (i) since $K_6 \in \mathcal{Q}$, but contradicting (ii), since neither u nor v is adjacent to a 6-component in $G[V_c]$.

Therefore u is adjacent to a 2-component in $G[V_c]$. However, then we can replace the components in $G[V_c]$ that are adjacent to u and v with a 4-component; satisfying (i) and (ii) but contradicting (iii).

Therefore, u is an end-vertex of a P_5 in $G[V_c \cup \{u\}]$ or v is an end-vertex of a P_4 in $G[V_c \cup \{v\}]$. In both cases it follows that there is a path P of length four in $G[S \cup V_c]$. Let x be the end-vertex of P in V_c and y the neighbour of x on P. By repeating this argument it follows that x is an end-vertex of a P_4 in $G[V_{c-1} \cup \{x\}]$ or y is an end-vertex of a P_5 in $G[V_{c-1} \cup \{y\}]$. Continuing in this way we obtain a path of length at least $3c+1 \geq k+1$ in G, a contradiction. Therefore, $G[S] \in \mathcal{O}$. ■

Corollary 3.5. *For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 5$, the following holds:* $\chi_{\mathcal{Q}}(\mathcal{W}_k) \leq \left\lceil \frac{k}{3} \right\rceil + 1$. ■

The inequality in Corollary 3.5 may be strict, for example we have that $\chi_{\mathcal{I}_5}(\mathcal{W}_k) = \lceil \frac{k+1}{6} \rceil < \frac{k+3}{3} \leq \lceil \frac{k}{3} \rceil + 1$ with $K_6 \in \mathcal{I}_5$ and $K_7 \notin \mathcal{I}_5$. Equality can also be obtained: In Theorem 3.8 (still to follow) we prove that $\chi_{\mathcal{S}_n}(\mathcal{W}_k) \geq \lfloor \log_2(k+2) \rfloor$ thus for all n we have that $\chi_{\mathcal{S}_n}(\mathcal{W}_6) \geq 3$ and by Corollary 3.5 we have $\chi_{\mathcal{S}_n}(\mathcal{W}_6) \leq \lceil \frac{6}{3} \rceil + 1 = 3$.

Theorem 3.6. *For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 9$, the following holds: $\mathcal{W}_k \subseteq \mathcal{Q}^{\lceil \frac{k-1}{4} \rceil} \circ \mathcal{S}_1$.*

Proof. Let $c = \left\lceil \frac{k-1}{4} \right\rceil$. Consider any graph G in $\mathcal{W}_k$. Take V_1 to be a subset of $V(G)$ such that, in order of priority:

(i) $G[V_1]$ is in $\mathcal{Q}$,

(ii) $G[V_1]$ contains a maximum number of 10-components,

(iii) $G[V_1]$ contains a maximum number of 8-components,

(iv) $G[V_1]$ contains a maximum number of 6-components,

(v) $G[V_1]$ contains a maximum number of 4-components,

(vi) $G[V_1]$ contains a maximum number of 2-components and

(vii) $G[V_1]$ contains a maximum number of isolated vertices.

Now, for $2 \leq i \leq c$ take V_i to be a subset of $V(G) - \bigcup_{j=1}^{i-1} V_j$ such that for each i, $G[V_i]$ satisfies the above list. Let $S = V(G) - \bigcup_{j=1}^{c} V_j$. We will show that $G[S] \in \mathcal{S}_1$. Suppose, to the contrary, that $G[S] \notin \mathcal{S}_1$ and let v be a vertex in $G[S]$ of degree at least two with u and w neighbours of v in $G[S]$.

Suppose that u is not an end-vertex of a P_7 in $G[V_c \cup \{u\}]$ and that v is not an end-vertex of a P_6 in $G[V_c \cup \{v\}]$ and that w is not an end-vertex of a P_5 in $G[V_c \cup \{w\}]$. Note that for every i, the choice of V_i gives that every component in $G[V_i]$ is a 10-component, an 8-component, a 6-component, a 4-component, K_2 or K_1. Moreover, by (vii) it follows that u, v and w have at least one neighbour in V_c each and by (vi) each of u, v and w is adjacent to a nontrivial component in $G[V_c]$. Since u is not an end-vertex of a P_7 in $G[V_c \cup \{u\}]$ it follows that u is not adjacent to a 10-component in $G[V_c]$. Similarly, v is not adjacent to a 10-component or an 8- component in $G[V_c]$ and w is not adjacent to a 10-component, an 8- component or a 6-component in $G[V_c]$.

Suppose that u is adjacent to an 8-component in $G[V_c]$. Then, since neither v nor w are adjacent to an 8-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to u, v and w with a 10-component; still satisfying (i) but contradicting (ii).

Suppose that v is adjacent to a 6-component in $G[V_c]$. Then, since w is not adjacent to a 6-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to u, v and w with an 8-component; still satisfying (i) and (ii) but contradicting (iii), since none of u, v and w is adjacent to a 10-component or an 8-component in $G[V_c]$, Similarly, u is not adjacent to a 6-component in $G[V_c]$.

Suppose that v is adjacent to a 4-component in $G[V_c]$. Note that since u, v and w are not adjacent to 6-components in $G[V_c]$ it follows that neither u nor w is adjacent to a 4-component in $G[V_c]$ — otherwise we can replace the components in $G[V_c]$ that are adjacent to u, v and w with a 6-component; satisfying (i) through (iii) but contradicting (iv). Therefore, since u and w are not adjacent to 4-components in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to u, v and w with a 6-component; containing three vertices of the 4-component, u and its neighbour in V_c.

Therefore v is adjacent to a 2-component in $G[V_c]$. However, then we can replace the components in $G[V_c]$ that are adjacent to u, v and w with a 4-component; satisfying (i) through (iv) but contradicting (v).

Therefore, u is an end-vertex of a P_7 in $G[V_c \cup \{u\}]$ or v is an end-vertex of a P_6 in $G[V_c \cup \{v\}]$ or w is an end-vertex of a P_5 in $G[V_c \cup \{w\}]$. In each case it follows that there is a path P of length 6 in $G[S \cup V_c]$. Let z be the end-vertex of P in V_c, y the neighbour of z on P and x the other neighbour of y on P. By repeating the above argument it follows that z is an end-vertex of a P_5 in $G[V_{c-1} \cup \{z\}]$ or y is an end-vertex of a P_6 in $G[V_{c-1} \cup \{y\}]$ or x

is an end-vertex of a P_7 in $G[V_{c-1} \cup \{x\}]$. Continuing in this way we obtain a path of length at least $4c+2 \geq k+1$ in G, a contradiction. Therefore, $G[S] \in \mathcal{S}_1$. ■

Theorem 3.7. *For $n \geq 9$, the following holds:*

$$\left\lceil \frac{k+1}{n+1} \right\rceil \leq \chi_{\mathcal{S}_n}(\mathcal{W}_k) \leq \left\lceil \frac{k-1}{4} \right\rceil + 1.$$

Proof. The left inequality holds since $K_{k+1} \in \mathcal{W}_k$. The right inequality follows as a corollary of Theorem 3.6 since $K_{10} \in \mathcal{S}_n$ for each $n \geq 9$. ■

Our next result improves on the lower bound in Theorem 3.7 for large values of n.

Theorem 3.8 *For all positive integers k and n, $\chi_{\mathcal{S}_n}(\mathcal{W}_k) \geq \lfloor \log_2(k+2) \rfloor$.*

Proof. We first prove, by induction on m, that for all positive integers m and n, $\mathcal{W}_{2^{m+1}-2} \not\subseteq \mathcal{S}_n^m$: For the case where $m=1$ the result holds since $\mathcal{W}_2 \not\subseteq \mathcal{S}_n$. Assume therefore that the result holds for $m-1$, thus there exists a graph H such that $H \in \mathcal{W}_{2^m-2}$ and $H \notin \mathcal{S}_n^{m-1}$. Now let $G = (n+1)H + K_1$. Clearly $G \in \mathcal{W}_{2(2^m-2)+2} = \mathcal{W}_{2^{m+1}-2}$. As in the proof of Theorem 1.4 $G \notin \mathcal{S}_n^m$.

Now, let k and n be any positive integers. We have that $\mathcal{W}_k \supseteq \mathcal{W}_{2^{\lfloor \log_2(k+2) \rfloor}-2} \not\subseteq \mathcal{S}_n^{\lfloor \log_2(k+2) \rfloor - 1}$ and the result follows. ■

Corollary 3.5 and Theorem 3.7 seem to suggest that for every k and m we can get $\mathcal{W}_k \subseteq \mathcal{S}_n^{\lceil \frac{k}{m} \rceil + 1}$ for all n sufficiently large. However, Theorem 3.8 implies that $\mathcal{W}_6 \not\subseteq \mathcal{S}_n^{\lceil \frac{6}{6} \rceil + 1}$ for all n since $\chi_{\mathcal{S}_n}(\mathcal{W}_6) \geq \lfloor \log_2(8) \rfloor = 3$. The method of proof in Theorem 3.6 does not extend. If we try to maximize with respect to 12-components, 10-components etc. the argument fails, and assuming that k is large makes no difference.

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References

[1] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.

[2] M. Borowiecki and P. Mihók, Hereditary properties of graphs, in: V.R. Kulli, ed., Advances in Graph Theory (Vishwa International Publication, Gulbarga, 1991) 41–68.

[3] I. Broere, M.J. Dorfling, J.E Dunbar and M. Frick, *A path(ological) partition problem*, Discuss. Math. Graph Theory **18** (1998) 113–125.

[4] I. Broere, P. Hajnal and P. Mihók, *Partition problems and kernels of graphs*, Discuss. Math. Graph Theory **17** (1997) 311–313.

[5] S.A. Burr and M.S. Jacobson, *On inequalities involving vertex-partition parameters of graphs*, Congr. Numer. **70** (1990) 159–170.

[6] G. Chartrand, D.P. Geller and S.T. Hedetniemi, *A generalization of the chromatic number*, Proc. Camb. Phil. Soc. **64** (1968) 265–271.

[7] M. Frick and F. Bullock, *Detour chromatic numbers*, manuscript.

[8] P. Hajnal, Graph partitions (in Hungarian), Thesis, supervised by L. Lovász (J.A. University, Szeged, 1984).

[9] T.R. Jensen and B. Toft, Graph colouring problems (Wiley-Interscience Publications, New York, 1995).

[10] L. Lovász, *On decomposition of graphs*, Studia Sci. Math. Hungar **1** (1966) 237–238; MR34#1715.

[11] P. Mihók, Problem 4, p. 86 in: M. Borowiecki and Z. Skupień (eds), Graphs, Hypergraphs and Matroids (Zielona Góra, 1985).

[12] J. Nešetřil and V. Rödl, *Partitions of vertices*, Comment. Math. Univ. Carolinae **17** (1976) 85–95; MR54#173.

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