

## TREES WITH UNIQUE MINIMUM TOTAL DOMINATING SETS

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### Abstract

A set  $S$  of vertices of a graph  $G$  is a *total dominating set* if every vertex of  $V(G)$  is adjacent to some vertex in  $S$ . We provide three equivalent conditions for a tree to have a unique minimum total dominating set and give a constructive characterization of such trees.

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## 1. Introduction

For notation and graph theory terminology, we in general follow [1, 7]. Specifically, let  $G = (V, E)$  be a graph. For a vertex  $v \in V$ , the *open neighborhood of  $v$*  is the set  $N(v) = \{u \in V \mid uv \in E\}$ , and its *closed neighborhood* is the set  $N[v] = N(v) \cup \{v\}$ . For a set  $S \subseteq V$ , its *open neighborhood* is the set  $N(S) = \cup_{v \in S} N(v)$  and its *closed neighborhood* is

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the set  $N[S] = N(S) \cup S$ . The *private neighborhood*  $\text{pn}(v, S)$  of  $v \in S$  is defined by  $\text{pn}(v, S) = N(v) - N(S - \{v\})$ . Equivalently,  $\text{pn}(v, S) = \{u \in V \mid N(u) \cap S = \{v\}\}$ . Each vertex in  $\text{pn}(v, S)$  is called a *private neighbor* of  $v$ . The *external private neighborhood*  $\text{epn}(v, S)$  of  $v$  with respect to  $S$  consists of those private neighbors of  $v$  in  $V - S$ , while the *internal private neighborhood*  $\text{ipn}(v, S)$  of  $v$  with respect to  $S$  consists of those private neighbors of  $v$  in  $S$ . Thus,  $\text{epn}(v, S) = \text{pn}(v, S) \cap (V - S)$  and  $\text{ipn}(v, S) = \text{pn}(v, S) \cap S$ , while  $\text{pn}(v, S) = \text{epn}(v, S) \cup \text{ipn}(v, S)$ . If  $G$  has no isolated vertices, then the set  $S$  is a *total dominating set* if every vertex in  $V$  is adjacent to a vertex in  $S$ , that is,  $N(S) = V$ . Every graph without isolated vertices has a total dominating set, since  $S = V$  is such a set. The *total domination number* of  $G$ , denoted by  $\gamma_t(G)$ , is the minimum cardinality of any total dominating set of  $G$ . A total dominating set of cardinality  $\gamma_t(G)$  is called a  $\gamma_t(G)$ -*set*. Note that every  $\gamma_t(G)$ -set is also a dominating set of  $G$ , and so  $\gamma(G) \leq \gamma_t(G)$ . Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory (see, for example, [4] and [9]).

The literature on domination and its variations in graphs has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7, 8]. Gunther, Hartnell, Markus, and Rall [5] studied graphs with unique minimum dominating sets, and Hopkins and Staton [10] and Gunther, Hartnell, and Rall [6] studied graphs with unique maximum independent sets. We investigate graphs  $G$  with unique minimum total dominating sets, that is, unique  $\gamma_t(G)$ -sets. A graph  $G$  will be called a *unique total domination graph*, or just a *UTD-graph*, if  $G$  has a unique  $\gamma_t(G)$ -set.

Observe that the graph  $mK_2$  has its vertex set as its unique minimum total dominating set. For other examples of UTD-graphs, consider the paths  $P_n$  with  $n \equiv 0 \pmod{4}$ . Apart from a few minor results on UTD-graphs in general, we study UTD-trees. For ease of presentation, we mostly consider *rooted trees*. For a vertex  $v$  in a (rooted) tree  $T$ , we let  $C(v)$  and  $D(v)$  denote the set of children and descendants, respectively, of  $v$ , and we define  $D[v] = D(v) \cup \{v\}$ . The maximal subtree at  $v$  is the subtree of  $T$  induced by  $D[v]$ , and is denoted by  $T_v$ . A vertex of degree one is called an *endvertex* or a *leaf* and its neighbor is called a *support* vertex. The set of leaves in  $T$  is denoted by  $L(T)$  and the set of support vertices by  $S(T)$ . We define a *branch vertex* as a vertex of degree at least 3. The set of branch vertices of  $T$  is denoted by  $B(T)$ . A tree  $T$  is a *double star* if it contains exactly two vertices that are not leaves.

## 2. Known Results

We shall need the following properties of minimal total dominating sets established in [2] and [9].

**Theorem 1** (Cockayne et al. [2]). *If  $S$  is a minimal total dominating set of a connected graph  $G$ , then  $|\text{epn}(v, S)| \geq 1$  or  $|\text{ipn}(v, S)| \geq 1$  for each  $v \in S$ .*

**Theorem 2** (Henning [9]). *If  $G \neq K_n$  is a connected graph of order  $n \geq 3$ , then there exists a  $\gamma_t(G)$ -set  $S$  where for every vertex  $v \in S$ ,  $|\text{epn}(v, S)| \geq 1$  or there exists a vertex  $u \in \text{ipn}(v, S)$  with  $|\text{epn}(u, S)| \geq 1$ .*

Cockayne, Henning, and Mynhardt [3] characterized the set of vertices of a tree that are contained in all, or in no, respectively, minimum total dominating sets of the tree. To state this characterization, we introduce the following notation. We define the sets  $\mathcal{A}_t(G)$  and  $\mathcal{N}_t(G)$  of a graph  $G$  by

$$\mathcal{A}_t(G) = \{v \in V(G) \mid v \text{ is in every } \gamma_t(G)\text{-set}\}, \text{ and}$$

$$\mathcal{N}_t(G) = \{v \in V(G) \mid v \text{ is in no } \gamma_t(G)\text{-set}\}.$$

Let  $T$  be a tree rooted at a vertex  $v$ . The set of leaves in  $T = T_v$  distinct from  $v$  we denote by  $L(v)$ , that is,  $L(v) = D(v) \cap L(T)$ . For  $j = 0, 1, 2, 3$ , we define

$$L^j(v) = \{u \in L(v) \mid d(u, v) \equiv j \pmod{4}\}.$$

We next describe a technique called *tree pruning*, which will allow us to characterize the sets  $\mathcal{A}_t(T)$  and  $\mathcal{N}_t(T)$  for an arbitrary tree  $T$ .

Let  $T$  be a tree and let  $v$  be a vertex of  $T$  that is not a support vertex. The pruning of  $T$  is performed with respect to the root. Hence suppose  $T$  is rooted at  $v$ , that is,  $T = T_v$ . If  $\deg u \leq 2$  for each  $u \in V(T_v) - \{v\}$ , then let  $\bar{T}_v = T$ . Otherwise, let  $u$  be a branch vertex at maximum distance from  $v$ ; note that  $|C(u)| \geq 2$  and  $\deg x \leq 2$  for each  $x \in D(u)$ . We now apply the following pruning process:

- If  $|L^2(u)| \geq 1$ , then delete  $D(u)$  and attach a path of length 2 to  $u$ .
- If  $|L^1(u)| \geq 1$ ,  $L^2(u) = \emptyset$  and  $|L^3(u)| \geq 1$ , then delete  $D(u)$  and attach a path of length 2 to  $u$ .
- If  $|L^1(u)| \geq 1$  and  $L^2(u) = L^3(u) = \emptyset$ , then delete  $D(u)$  and attach a path of length 1 to  $u$ .

- If  $L^1(u) = L^2(u) = \emptyset$  and  $|L^3(u)| \geq 1$ , then delete  $D(u)$  and attach a path of length 3 to  $u$ .
- If  $L^1(u) = L^2(u) = L^3(u) = \emptyset$ , then delete  $D(u)$  and attach a path of length 4 to  $u$ .

This step of the pruning process, where all the descendants of  $u$  are deleted and a path of length 1, 2, 3, or 4 is attached to  $u$  to give a tree in which  $u$  has degree 2, is called a *pruning of  $T_v$  at  $u$* . Repeat the above process until a tree  $\overline{T}_v$  is obtained with  $\deg u \leq 2$  for each  $u \in V(\overline{T}_v) - \{v\}$ . Then,  $\overline{T}_v$  is called a *pruning of  $T_v$* . The tree  $\overline{T}_v$  is unique. Thus, to simplify notation, we write  $\overline{L}^j(v)$  instead of  $L_{\overline{T}_v}^j(v)$ . The following characterization of the sets  $\mathcal{A}_t(T)$  and  $\mathcal{N}_t(T)$  for an arbitrary tree  $T$  is presented in [3].

**Theorem 3** (Cockayne et al. [3]). *Let  $v$  be a vertex of a tree  $T$ . Then,*

- $v \in \mathcal{A}_t(T)$  if and only if  $v$  is a support vertex or  $|\overline{L}^1(v) \cup \overline{L}^2(v)| \geq 2$ ,
- $v \in \mathcal{N}_t(T)$  if and only if  $\overline{L}^1(v) \cup \overline{L}^2(v) = \emptyset$ .

### 3. Preliminary Results

We first consider induced subgraphs of UTD-graphs. In particular, we show that any graph  $G$  without isolated vertices is an induced subgraph of a UTD-graph. The *corona*  $\text{cor}(G)$  of a graph  $G$  is that graph obtained from  $G$  by adding a pendant edge to each vertex of  $G$ . Obviously, the graph  $G$  is an induced subgraph of  $\text{cor}(G)$  and if  $G$  has no isolated vertices, then  $V(G)$  is the unique  $\gamma_t(\text{cor}(G))$ -set. Therefore every graph without isolated vertices is an induced subgraph of a UTD-graph, and hence there does not exist a forbidden subgraph characterization of the class of UTD-graphs.

Every endvertex is uniquely dominated by the support vertex adjacent to it, and so any total dominating set contains every support vertex.

**Observation 4.** *Every support vertex of  $G$  is in every  $\gamma_t(G)$ -set.*

**Observation 5.** *A path  $P_n$  is a UTD-graph if and only if  $n \in \{2, 5\}$  or  $n \equiv 0 \pmod{4}$ .*

**Lemma 6.** *If a graph  $G$  has a unique  $\gamma_t(G)$ -set  $S$ , then every vertex  $v \in S$  is a support vertex or satisfies  $|\text{pn}(v, S)| \geq 2$ .*

**Proof.** By Theorem 1,  $|\text{pn}(v, S)| \geq 1$  for each  $v \in S$ . Suppose that  $v \in S$  is not a support vertex and  $|\text{pn}(v, S)| = 1$ . If  $|\text{epn}(v, S)| = 1$  (and so  $|\text{ipn}(v, S)| = 0$ ), then let  $u \in \text{epn}(v, S)$  and let  $w \in N(u) - \{v\}$ . Then,  $w \in V(G) - S$  and  $(S - \{v\}) \cup \{w\}$  is a  $\gamma_t(G)$ -set, contradicting the uniqueness of  $S$ . On the other hand, if  $|\text{ipn}(v, S)| = 1$  (and so  $|\text{epn}(v, S)| = 0$ ), then, by Theorem 2, there exists a vertex  $u \in \text{ipn}(v, S)$  with  $|\text{epn}(u, S)| \geq 1$ . Let  $w \in \text{epn}(u, S)$ . Then,  $(S - \{v\}) \cup \{w\}$  is a  $\gamma_t(G)$ -set, contradicting the uniqueness of  $S$ . Hence,  $|\text{pn}(v, S)| \geq 2$ . ■

As an immediate consequence of Lemma 6 we have the following observation.

**Observation 7.** *Let  $G$  be a connected graph of order  $n \geq 3$ . If any endvertex  $u$  of  $G$  is in a  $\gamma_t(G)$ -set, then  $G$  is not a UTD-graph.*

The converse of Lemma 6 is not true in general. For example, if  $G$  is the 8-cycle  $v_1, v_2, \dots, v_8, v_1$ , then  $S = \{v_2, v_3, v_6, v_7\}$  is a  $\gamma_t(G)$ -set; however,  $S$  is not a unique  $\gamma_t(G)$ -set.

Recall that  $S(G)$  is the set of support vertices of  $G$ .

**Lemma 8.** *If a graph  $G$  is a UTD-graph with  $\gamma_t(G)$ -set  $S$ , then  $\gamma_t(G - v) \geq \gamma_t(G)$  for every  $v \in S - S(G)$ .*

**Proof.** Let  $G$  be a UTD-graph with  $\gamma_t(G)$ -set  $S$ , and assume to the contrary that  $\gamma_t(G - v) < \gamma_t(G)$  for some  $v \in S - S(G)$ . Let  $R$  be a  $\gamma_t(G - v)$ -set. Since  $|R| < \gamma_t(G)$ ,  $R$  does not dominate  $v$ . Furthermore, since  $G$  has no isolates,  $v$  has a neighbor, say  $u$ , in  $V - R$ . Then  $R \cup \{u\}$  is a total dominating set of  $G$  that does not contain  $v$ . Hence,  $R \cup \{u\} \neq S$ , contradicting the uniqueness of  $S$ . ■

**Lemma 9.** *If a graph  $G$  has a  $\gamma_t(G)$ -set  $S$  for which  $\gamma_t(G - v) > \gamma_t(G)$  for every  $v \in S - S(G)$ , then  $S$  is the unique  $\gamma_t(G)$ -set of  $G$ .*

**Proof.** Suppose there exists a  $\gamma_t(G)$ -set  $D$  that is different from  $S$ . Let  $v \in S - D$ . By Observation 4,  $v$  is not a support vertex. In particular,  $G - v$  contains no isolated vertex. Since  $D$  is a total dominating set of  $G - v$ ,  $\gamma_t(G) = |D| \geq \gamma_t(G - v)$ , a contradiction. Hence,  $S$  is the unique  $\gamma_t(G)$ -set. ■

The converse of Lemma 9 is not true in general. For example, the set  $\{v, x, y\}$  is the unique  $\gamma_t(G)$ -set for the graph  $G$  in Figure 1, and  $\gamma_t(G - v) = \gamma_t(G)$ .

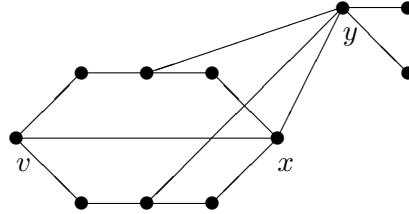


Figure 1: A graph  $G$  with the unique  $\gamma_t(G)$ -set  $\{v, x, y\}$

## 4. Trees

### 4.1 Equivalent Conditions for UTD-Trees

Our aim in this section is to provide three equivalent conditions for a tree to have a unique minimum total domination set. We begin with the following lemmas.

**Lemma 10.** *Let  $T_1$  and  $T_2$  be vertex disjoint trees, and let  $v \in \mathcal{A}_t(T_1)$ . Let  $T$  be a tree obtained from  $T_1 \cup T_2$  by joining  $v$  to a vertex of  $T_2$ . Then,  $v \in \mathcal{A}_t(T)$ .*

**Proof.** Since  $v \in \mathcal{A}_t(T_1)$ , Theorem 3 implies that  $v_1$  is a support vertex of  $T_1$  or  $|\bar{L}^1(v_1) \cup \bar{L}^2(v_1)| \geq 2$  in  $T_1$ . So, certainly,  $v_1$  is a support vertex of  $T$  or  $|\bar{L}^1(v_1) \cup \bar{L}^2(v_1)| \geq 2$  in  $T$ . Thus, by Theorem 3,  $v \in \mathcal{A}_t(T)$ . ■

Notice that if a vertex  $v$  in a tree  $T$  belongs to some but not all  $\gamma_t(T)$ -sets, then clearly  $T$  does not have a unique minimum total domination set. Hence we have the following observation.

**Observation 11.** *A tree  $T$  is a UTD-tree if and only if  $v \in \mathcal{A}_t(T) \cup \mathcal{N}_t(T)$  for every vertex  $v \in V(T)$ .*

As an immediate consequence of Theorem 3 and Observation 11, we have the following characterization of UTD-trees.

**Theorem 12.** *A tree  $T$  is a UTD-tree if and only if for every vertex  $v \in V(T)$ ,  $v$  is a support vertex or  $|\bar{L}^1(v) \cup \bar{L}^2(v)| \neq 1$ .*

We now establish three equivalent conditions for a tree to be a UTD-tree.

**Theorem 13.** *Let  $T$  be a tree of order  $n \geq 2$ . Then the following conditions are equivalent:*

- (i)  $T$  is a UTD-tree.
- (ii)  $T$  has a  $\gamma_t(T)$ -set  $S$  for which every vertex  $v \in S$  is a support vertex or satisfies  $|\text{pn}(v, S)| \geq 2$ .
- (iii)  $T$  has a  $\gamma_t(T)$ -set  $S$  for which  $\gamma_t(T-v) > \gamma_t(T)$  for every  $v \in S - S(T)$ .
- (iv) For every vertex  $v \in V(T)$ ,  $v$  is a support vertex or  $|\overline{L}^1(v) \cup \overline{L}^2(v)| \neq 1$ .

**Proof.** By Theorem 12, (i)  $\Leftrightarrow$  (iv). By Lemma 6, (i)  $\Rightarrow$  (ii), and by Lemma 9, (iii)  $\Rightarrow$  (i). Hence it suffices to prove that (ii)  $\Rightarrow$  (iii). Suppose, then, that  $T$  has a  $\gamma_t(T)$ -set  $S$  for which every vertex  $v \in S$  is a support vertex or satisfies  $|\text{pn}(v, S)| \geq 2$ . We show that condition (iii) holds. We proceed by induction on the order  $n$  of the tree  $T$ .

If every vertex of  $S$  is a support vertex, then condition (iii) is vacuously true. In particular, the base case when  $n = 2$  is true. Assume that for all trees of order less than  $n$ , where  $n \geq 3$ , that (ii)  $\Rightarrow$  (iii). Let  $T$  be a tree of order  $n$  that satisfies condition (ii). We may assume that  $S \neq S(T)$ , for otherwise condition (iii) is vacuously true. Let  $v \in S - S(T)$ . We show that  $\gamma_t(T - v) > \gamma_t(T)$ .

Since  $v$  is not a support vertex,  $|\text{pn}(v, S)| \geq 2$ . In particular,  $\deg v \geq 2$ . Let  $A = \text{epn}(v, S)$ ,  $B = N(v) - (A \cup S)$ ,  $C = \text{ipn}(v, S)$  and  $D = N(v) \cap (S - C)$ . Since  $S$  is a total dominating set of  $T$ ,  $|C \cup D| \geq 1$ . Let  $N(v) = \{v_1, v_2, \dots, v_m\}$ , where the subscripts are indexed so that if  $v_i \in A$ ,  $v_j \in B$ ,  $v_k \in C$  and  $v_\ell \in D$ , then  $i < j < k < \ell$ .

Let  $T$  be rooted at  $v$ . For  $i = 1, 2, \dots, m$ , let  $T_i = T_{v_i}$  (so  $T_i$  is the subtree of  $T$  induced by  $D[v_i]$ ), and let  $S_i = S \cap V(T_i)$ . Since  $v$  is not a support vertex, each component  $T_i$  of the forest  $T - v$  is a nontrivial tree. Let  $R$  be a  $\gamma_t(T - v)$ -set. For  $i = 1, 2, \dots, m$ , let  $R_i = R \cap V(T_i)$ . Then,  $R_i$  is a  $\gamma_t(T_i)$ -set for each  $i$ . We proceed further by proving four claims.

**Claim 1.** *If  $v_i \in A$ , then  $|R_i| = |S_i| + 1$ .*

**Proof.** Suppose  $v_i \in A$ . Then,  $S_i$  does not contain  $v_i$  or any child of  $v_i$ . Note that since  $v_i \notin S_i$ ,  $v_i$  is not a support vertex implying that each component of  $T_i - v_i$  is a nontrivial tree. We show first that  $S_i$  is the unique  $\gamma_t(T_i - v_i)$ -set. The set  $S_i$  is a total dominating set of  $T_i - v_i$ . If  $S_i$  is

not a  $\gamma_t(T_i - v_i)$ -set, then replacing  $S_i$  in  $S$  by a  $\gamma_t(T_i - v_i)$ -set produces a total dominating set of  $T$  of cardinality less than  $|S| = \gamma_t(T)$ , which is impossible. Hence,  $S_i$  is a  $\gamma_t(T_i - v_i)$ -set. Furthermore, every vertex  $v \in S_i$  is a support vertex in  $T_i - v_i$  or satisfies  $|\text{pn}(v, S)| \geq 2$  in  $T_i - v_i$ . Applying the inductive hypothesis to each component of  $T_i - v_i$ , each component of  $T_i - v_i$  satisfies condition (iii). Since (iii)  $\Rightarrow$  (i), it follows that  $S_i$  is the unique  $\gamma_t(T_i - v_i)$ -set.

Since  $R_i$  is a  $\gamma_t(T_i)$ -set,  $R_i$  contains a child of  $v_i$ , and so  $S_i \neq R_i$ . If  $|R_i| < |S_i|$ , then  $(S - S_i) \cup R_i$  is a total dominating set of  $T$  of cardinality less than  $|S| = \gamma_t(T)$ , which is impossible. Hence,  $|R_i| \geq |S_i|$ .

If  $v_i \in \mathcal{A}_t(T_i)$ , then, by Lemma 10,  $v_i \in \mathcal{A}_t(T)$  contradicting the fact that  $S$  is a  $\gamma_t(T)$ -set not containing  $v_i$ . Hence,  $v_i \notin \mathcal{A}_t(T_i)$ . We may assume therefore that  $R_i$  is chosen so that  $v_i \notin R_i$ . But then  $R_i$  is also a total dominating set of  $T_i - v_i$ . Since  $R_i$  is a  $\gamma_t(T_i)$ -set,  $R_i$  contains a child of  $v_i$ , and so  $S_i \neq R_i$ . Thus, since  $S_i$  is the unique  $\gamma_t(T_i - v_i)$ -set,  $R_i$  is not a  $\gamma_t(T_i - v_i)$ -set, and so  $|R_i| \geq |S_i| + 1$ . Since  $S_i \cup \{v'_i\}$  is a total dominating set of  $T_i$  where  $v'_i \in C(v_i)$ ,  $|S_i| + 1 \leq |R_i| = \gamma_t(T_i) \leq |S_i| + 1$ . Consequently,  $|R_i| = |S_i| + 1$ .  $\blacksquare$

**Claim 2.** *If  $v_i \in B$ , then  $|R_i| = |S_i|$ .*

**Proof.** Suppose  $v_i \in B$ . Then,  $S_i$  is a total dominating set of  $T_i$ , and so  $\gamma_t(T_i) \leq |S_i|$ . If  $|R_i| < |S_i|$ , then  $(S - S_i) \cup R_i$  is a total dominating set of  $T$  of cardinality less than  $|S| = \gamma_t(T)$ , which is impossible. Hence,  $|S_i| \leq |R_i| = \gamma_t(T_i) \leq |S_i|$ . Consequently,  $|R_i| = |S_i|$ .  $\blacksquare$

**Claim 3.** *If  $v_i \in C$ , then  $|R_i| = |S_i| + 1$ .*

**Proof.** Suppose  $v_i \in C$ . Then,  $v_i \in S_i$  but  $S_i$  does not contain any child of  $v_i$ . Let  $H$  be the tree obtained from  $T_i$  by joining  $v$  to  $v_i$  and to a new vertex  $u$ . Then,  $H$  is a tree of order less than  $n$  in which the vertex  $v$  is a support vertex and therefore belongs to every  $\gamma_t(H)$ -set.

The set  $S_i \cup \{v\}$  is a total dominating set of  $H$ . If  $S_i \cup \{v\}$  is not a  $\gamma_t(H)$ -set, then replacing  $S_i \cup \{v\}$  in  $S$  by a  $\gamma_t(H)$ -set (which necessarily contains  $v$ ) produces a total dominating set of  $T$  of cardinality less than  $|S| = \gamma_t(T)$ , which is impossible. Hence,  $S_i \cup \{v\}$  is a  $\gamma_t(H)$ -set. Furthermore, every vertex  $w \in S_i \cup \{v\}$  is a support vertex in  $H$  or satisfies  $|\text{pn}(w, S)| \geq 2$  in  $H$ . Applying the inductive hypothesis to  $H$ , the tree  $H$  satisfies condition (iii). Since (iii)  $\Rightarrow$  (i), it follows that  $S_i \cup \{v\}$  is the unique  $\gamma_t(H)$ -set.



We show now that  $v_i \notin \mathcal{N}_t(T_i)$ . Since  $S_i \cup \{v\}$  is the unique  $\gamma_t(H)$ -set, we know by Observation 11 that  $v_i \in \mathcal{A}_t(H)$ . Hence, by Theorem 3,  $v_i$  is a support vertex of  $H$  or  $|\bar{L}^1(v_i) \cup \bar{L}^2(v_i)| \geq 2$  in the tree  $H$ . If  $v_i$  is a support vertex in  $H$ , then, since  $v$  is not a leaf,  $v_i$  is also a support vertex in  $T_i$ , and so in the tree  $T_i$ ,  $|\bar{L}^1(v_i)| \geq 1$  and therefore  $|\bar{L}^1(v_i) \cup \bar{L}^2(v_i)| \geq 1$  in  $T_i$ . On the other hand, if  $|\bar{L}^1(v_i) \cup \bar{L}^2(v_i)| \geq 2$  in the tree  $H$ , then  $|\bar{L}^1(v_i) \cup \bar{L}^2(v_i)| \geq 1$  in the tree  $T_i$ . In any event,  $|\bar{L}^1(v_i) \cup \bar{L}^2(v_i)| \geq 1$  in the tree  $T_i$ . Hence, by Theorem 3,  $v_i \notin \mathcal{N}_t(T_i)$ .

Since  $v_i \notin \mathcal{N}_t(T_i)$ , there exists a  $\gamma_t(T_i)$ -set that contains  $v_i$ . We may assume that  $R_i$  is chosen so that  $v_i \in R_i$ . The desired result now follows as in the proof of Claim 1. ■

**Claim 4.** *If  $v_i \in D$ , then  $|R_i| = |S_i|$ .*

**Proof.** Suppose  $v_i \in D$ . Then,  $v_i \in S_i$  and  $S_i$  contains a child of  $v_i$ . Let  $H$  be defined as in Claim 3. Then, as shown in Claim 3,  $v_i \notin \mathcal{N}_t(T)$  and we may assume that  $v_i \in R_i$ . The desired result now follows as in the proof of Claim 2. ■

We now return to our proof of Theorem 13. Since  $|\text{pn}(v, S)| \geq 2$ ,  $|A| + |C| = |\text{epn}(v, S)| + |\text{ipn}(v, S)| = |\text{pn}(v, S)| \geq 2$ . By Claims 1, 2, 3 and 4,  $|R| = \sum_{i=1}^m |R_i| \geq 2 + \sum_{i=1}^m |S_i| = 2 + (|S| - 1) = |S| + 1$ . Thus,  $\gamma_t(T - v) = |R| > |S| = \gamma_t(T)$ . Since  $v$  is an arbitrary vertex of  $S - S(T)$ , the set  $S$  satisfies condition (iii). Hence, (ii)  $\Rightarrow$  (iii) as desired. ■

## 4..2 Combining UTD-Trees

Our aim in this section is to provide a constructive characterization of UTD-trees. For this purpose, we introduce the following notation. Let  $T$  be a UTD-tree of order at least 4 and let  $S$  be the unique  $\gamma_t(T)$ -set. Let the vertices of  $T$  be partitioned into sets  $S_A$ ,  $S_B$ ,  $S_C$ ,  $S_D$ , and  $S_E$  as follows:

$$S_A = \{v \in S \mid v \in \text{ipn}(w, S) \text{ for some } w \in S - S(T) \text{ with } |\text{pn}(w, S)| = 2\},$$

$$S_B = S - S_A,$$

$$S_C = \{v \in V - S \mid \text{pn}(w, S) = \{v\} \text{ for some } w \in S\},$$

$$S_D = \{v \in V - S \mid v \in \text{pn}(w, S) \text{ for some } w \in S - S(T - v) \text{ with } |\text{pn}(w, S)| = 2\},$$

$$S_E = (V - S) - (S_C \cup S_D).$$

Note that if  $v \in S_C$ , then  $v \in L(T)$ . We say that the vertices of  $S_X$  have status  $X$  where  $X \in \{A, B, C, D, E\}$ .

The following lemma will prove to be useful.

**Lemma 14.** *Let  $T_1$  and  $T_2$  be vertex disjoint trees, and let  $v \in \mathcal{A}_t(T_1)$ . Let  $T$  be a tree obtained from  $T_1 \cup T_2$  by joining  $v$  to a vertex of  $T_2$ . Let  $D$  be a  $\gamma_t(T)$ -set. If  $T_1$  is a UTD-tree of order at least 3, then  $|D \cap V(T_1)| \geq \gamma_t(T_1)$ .*

**Proof.** Let  $S_1$  be the unique  $\gamma_t(T_1)$ -set and let  $D_1 = D \cap V(T_1)$ . If  $D_1$  is a total dominating set of  $T_1$ , then  $|D_1| \geq \gamma_t(T_1)$ , as desired. Suppose, then, that  $D_1$  is not a total dominating set of  $T_1$ . Then,  $D_1$  contains no neighbor of  $v$ . Now,  $D_1 \cup \{v'\}$  is a total dominating set of  $T_1$  for any neighbor  $v'$  of  $v$  in  $T_1$ . Suppose in the tree  $T_1$ ,  $N(v) \subset S_1$ . Then,  $v$  cannot be a support vertex (since no leaf belongs to  $S_1$ ), and so  $\deg v \geq 2$ . Thus, since  $D_1 \cup \{v'\}$  contains only one neighbor of  $v$ , the uniqueness of  $S_1$  implies that  $D_1 \cup \{v'\}$  is not a  $\gamma_t(T_1)$ -set. Hence,  $|D_1| + 1 \geq \gamma_t(T_1) + 1 = |S_1| + 1$ , and so  $|D_1| \geq |S_1|$ . On the other hand, if in the tree  $T_1$ ,  $N(v) \not\subset S_1$ , then we choose  $v' \in N(v) - S_1$ . Since  $D_1 \cup \{v'\} \neq S_1$ , the uniqueness of  $S_1$  once again implies that  $|D_1| \geq |S_1|$ . The result follows. ■

In what follows, we shall adopt the following notation. Let  $T_1$  and  $T_2$  be two vertex disjoint UTD-trees each of order at least 4. For  $i \in \{1, 2\}$ , let  $S_i$  denote the unique  $\gamma_t(T_i)$ -set. Then,  $S_i$  consists of the vertices of status  $A$  and  $B$ . We now present three operations which allow us to link up  $T_1$  and  $T_2$  to produce a new UTD-tree  $T$ .

**Operation  $\mathcal{T}_1$ .** Join a vertex  $u_1$  of status  $D$  or  $E$  in  $T_1$  to a vertex  $u_2$  of status  $D$  or  $E$  in  $T_2$ .

**Operation  $\mathcal{T}_2$ .** Join a vertex  $u_1$  of  $S_1$  to a vertex  $u_2$  of status  $E$  in  $T_2$ .

**Operation  $\mathcal{T}_3$ .** Join a vertex  $u_1$  of status  $B$  in  $T_1$  to a vertex  $u_2$  of status  $B$  in  $T_2$ .

In the next lemma, for the tree  $T$  obtained from  $T_1 \cup T_2$  using one of these three operations, let  $D$  be a  $\gamma_t(T)$ -set, and let  $D_i = D \cap V(T_i)$  for  $i \in \{1, 2\}$ .

**Lemma 15.**  *$S_1 \cup S_2$  is the unique  $\gamma_t(T)$ -set of  $T$  produced by Operation  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  or  $\mathcal{T}_3$ .*

**Proof.** (i) Suppose  $T$  is produced by Operation  $\mathcal{T}_1$ . We show first that  $S_1 \cup S_2$  is a  $\gamma_t(T)$ -set. The set  $S_1 \cup S_2$  is a total dominating set of  $T$ , and so

$\gamma_t(T) \leq |S_1| + |S_2|$ . If  $|D_1| + |D_2| \geq |S_1| + |S_2|$ , then  $\gamma_t(T) \geq |S_1| + |S_2|$  and consequently  $\gamma_t(T) = |S_1| + |S_2|$ . Hence it suffices to show that  $|D_1| + |D_2| \geq |S_1| + |S_2|$ .

Suppose  $u_1, u_2 \notin D$ . Then  $D_i$  is a total dominating set of  $T_i$ , and so  $|D_i| \geq \gamma_t(T_i) = |S_i|$ . Thus,  $|D_1| + |D_2| \geq |S_1| + |S_2|$ .

Suppose  $u_1 \in D$  and  $u_2 \notin D$ . Then,  $D_1$  is a total dominating set of  $T_1$ . Since  $S_1$  is the unique  $\gamma_t(T_1)$ -set and  $u_1 \notin S_1$ ,  $|D_1| \geq |S_1| + 1$ . Also,  $D_2 \cup \{u'_2\}$  is a total dominating set of  $T_2$  where  $u'_2$  is any neighbor of  $u_2$  in  $T_2$ . Thus,  $|D_2| + 1 \geq \gamma_t(T_2) = |S_2|$ . Hence,  $|D_1| + |D_2| \geq |S_1| + |S_2|$ . Similarly, if  $u_1 \notin D$  and  $u_2 \in D$ , then  $|D_1| + |D_2| \geq |S_1| + |S_2|$ .

Suppose  $u_1, u_2 \in D$ . Then, for  $i \in \{1, 2\}$ ,  $D_i \cup \{u'_i\}$  is a total dominating set of  $T_i$  where  $u'_i$  is any neighbor of  $u_i$  in  $T_i$ . Since  $S_i$  is the unique  $\gamma_t(T_i)$ -set,  $D_i \cup \{u'_i\}$  is not a  $\gamma_t(T_i)$ -set, and so  $|D_i| + 1 \geq \gamma_t(T_i) + 1 = |S_i| + 1$ . Hence,  $|D_i| \geq |S_i|$  for each  $i$ . Thus,  $|D_1| + |D_2| \geq |S_1| + |S_2|$ . Hence,  $S_1 \cup S_2$  is a  $\gamma_t(T)$ -set.

Since  $u_1$  and  $u_2$  are vertices of status  $D$  or  $E$  in  $T_1$  and  $T_2$ , respectively, in the tree  $T$  every vertex  $v \in S_1 \cup S_2$  is a support vertex or satisfies  $|\text{pn}(v, S_1 \cup S_2)| \geq 2$ . Thus, since  $S_1 \cup S_2$  is a  $\gamma_t(T)$ -set, it follows from Theorem 13 that  $T$  is a UTD-tree and  $S_1 \cup S_2$  is the unique  $\gamma_t(T)$ -set.

(ii) Suppose  $T$  is produced by Operation  $\mathcal{T}_2$ . We show that  $|D_i| \geq |S_i|$  for each  $i$ . It follows from Lemma 14 that  $|D_1| \geq |S_1|$ . If  $D_2$  is a total dominating set of  $T_2$ , then  $|D_2| \geq |S_2|$  as desired. Suppose  $D_2$  is not a total dominating set of  $T_2$ . Then  $D_2$  contains no neighbor of  $u_2$  and  $D_2 \cup \{u'_2\}$  is a total dominating set of  $T_2$  for any neighbor  $u'_2$  of  $u_2$  in  $T_2$ . If  $D_2 \cup \{u'_2\} = S_2$ , then since  $u_2$  has status  $E$  in  $T_2$ ,  $|\text{pn}(u_2, S_2)| \geq 2$ , contradicting the fact that  $D_2 = S_2 - \{u'_2\}$  is a total dominating set of  $T_2 - u_2$ . Hence,  $D_2 \cup \{u'_2\} \neq S_2$ . Since  $S_2$  is the unique  $\gamma_t(T_2)$ -set,  $|D_2| + 1 \geq \gamma_t(T_2) + 1 = |S_2| + 1$ , and so  $|D_2| \geq |S_2|$ . Thus,  $|D_i| \geq |S_i|$  for each  $i$ .

The set  $S_1 \cup S_2$  is a total dominating set of  $T$ , and so  $\gamma_t(T) \leq |S_1| + |S_2|$ . However,  $\gamma_t(T) = |D| = |D_1| + |D_2| \geq |S_1| + |S_2|$ . Consequently,  $\gamma_t(T) = |S_1| + |S_2|$ .

Since  $u_1 \in S_1$  and  $u_2$  is a vertex of status  $E$  in  $T_2$ , in the tree  $T$  every vertex  $v \in S_1 \cup S_2$  is a support vertex or satisfies  $|\text{pn}(v, S_1 \cup S_2)| \geq 2$ . Thus, since  $S_1 \cup S_2$  is a  $\gamma_t(T)$ -set, it follows from Theorem 13 that  $T$  is a UTD-tree and  $S_1 \cup S_2$  is the unique  $\gamma_t(T)$ -set.

(iii) Suppose  $T$  is produced by Operation  $\mathcal{T}_3$ . It follows from Lemma 14 that  $|D_i| \geq |S_i|$  for each  $i$  and therefore that  $S_1 \cup S_2$  is a  $\gamma_t(T)$ -set. Since  $u_i \in S_i$  has status  $B$  in  $T_i$  for  $i \in \{1, 2\}$ , in the tree  $T$  every vertex  $v \in$

$S_1 \cup S_2$  is a support vertex or satisfies  $|\text{pn}(v, S_1 \cup S_2)| \geq 2$ . It follows from Theorem 13 that  $T$  is a UTD-tree and  $S_1 \cup S_2$  is the unique  $\gamma_t(T)$ -set. ■

Let  $\mathcal{T}$  be the family of trees  $T$  with  $V(T) = L(T) \cup S(T)$ ,  $|S(T)| \geq 2$ . Let  $\mathcal{F}$  be the family of trees that can be obtained from a star  $T$  with at least two leaves by adding at least one leaf adjacent to each leaf of  $T$  (so each leaf of  $T$  is a support vertex in the resulting tree). We are now in a position to present a constructive characterization of UTD-trees.

**Theorem 16.** *Let  $T$  be a tree of order at least 4. Then  $T$  is a UTD-tree if and only if  $T$  can be constructed from disjoint trees in  $\mathcal{T} \cup \mathcal{F}$  by a sequence of Operations  $\mathcal{T}_1$ ,  $\mathcal{T}_2$ , and  $\mathcal{T}_3$ .*

**Proof.** Each tree in  $\mathcal{T} \cup \mathcal{F}$  is a UTD-tree, and so the sufficiency follows from Lemmas 15. To prove the necessity, we proceed by induction on  $\gamma_t(T)$ . If  $\gamma_t(T) = 2$ , then  $T$  is a double star, and so  $T \in \mathcal{T}$ . Hence the base case holds. Assume the result is true for all UTD-trees  $T'$  with  $\gamma_t(T') < m$ , where  $m \geq 3$ . Let  $T = (V, E)$  be a UTD-tree with  $\gamma_t(T) = m$ . Let  $S$  be the unique  $\gamma_t(T)$ -set.

Let  $u_1u_2$  be an edge of  $T$ , and let  $T_1$  and  $T_2$  be the components of  $T - u_1u_2$  containing  $u_1$  and  $u_2$ , respectively. For  $i \in \{1, 2\}$ , let  $S_i = S \cap V(T_i)$  and let  $D_i$  be a  $\gamma_t(T_i)$ -set. We proceed further by proving three claims.

**Claim 5.** *For  $i \in \{1, 2\}$ , if  $S_i$  is a total dominating set of  $T_i$ , then  $S_i$  is the unique  $\gamma_t(T_i)$ -set.*

**Proof.** For  $i \in \{1, 2\}$ ,  $|D_i| \leq |S_i|$ . Now,  $D_1 \cup D_2$  is a total dominating set of  $T$ , and so  $|S_1| + |S_2| \geq |D_1| + |D_2| \geq \gamma_t(T) = |S_1| + |S_2|$ . Hence  $|D_i| = |S_i|$  for each  $i$ . Thus,  $D_1 \cup D_2$  is a  $\gamma_t(T)$ -set. The uniqueness of  $S$  implies that  $D_1 \cup D_2 = S$ , and therefore  $D_i = S_i$  for each  $i$ . Hence,  $T_i$  is a UTD-tree and  $S_i$  is the unique  $\gamma_t(T_i)$ -set. ■

**Claim 6.** *If  $u_1, u_2 \in V - S$ , then  $T$  can be constructed as claimed.*

**Proof.** Since  $S_i$  is a total dominating set of  $T_i$  for  $i \in \{1, 2\}$ ,  $T_i$  is a UTD-tree and  $S_i$  is the unique  $\gamma_t(T_i)$ -set by Claim 5. Applying the inductive hypothesis to  $T_i$ , each  $T_i$  can be constructed from disjoint trees in  $\mathcal{T} \cup \mathcal{F}$  by a sequence of Operations  $\mathcal{T}_1$ ,  $\mathcal{T}_2$ , and  $\mathcal{T}_3$ . Since  $u_i \notin S_i$ ,  $u_i$  has status  $C$ ,  $D$  or  $E$  in  $T_i$ . If  $u_i$  has status  $C$  in  $T_i$ , then in the tree  $T_i$ ,  $u_i$  is a leaf and  $\text{pn}(u_i, S_i) = \{u_i\}$  for some  $w \in S_i$ . Thus in  $T$ ,  $|\text{pn}(u_i, S)| = 1$  and

$w \in S - S(T)$ , contradicting Theorem 13. Hence,  $u_i$  has status  $D$  or  $E$  in  $T_i$ , and  $T$  can be obtained from  $T_1 \cup T_2$  by Operation  $\mathcal{T}_1$ . The result follows. ■

**Claim 7.** *If  $u_1 \in S$  and if  $u_2 \in V - S$  is not an external private neighbor of any vertex in  $S$ , then  $T$  can be constructed as claimed.*

**Proof.** Since  $S_i$  is a total dominating set of  $T_i$  for  $i \in \{1, 2\}$ ,  $T_i$  is a UTD-tree and  $S_i$  is the unique  $\gamma_t(T_i)$ -set by Claim 5. Applying the inductive hypothesis to  $T_i$ , each  $T_i$  can be constructed from disjoint trees in  $\mathcal{T} \cup \mathcal{F}$  by a sequence of Operations  $\mathcal{T}_1$ ,  $\mathcal{T}_2$ , and  $\mathcal{T}_3$ .

We show next that  $u_2$  has status  $E$  in  $T_2$ . Since  $u_2 \notin S_2$ ,  $u_2$  has status  $C$ ,  $D$  or  $E$  in  $T_2$ . If  $u_2$  has status  $C$  in  $T_2$ , then  $\text{pn}(w, S_2) = \{u_2\}$  for some  $w \in S_2$ . Thus in  $T$ ,  $|\text{pn}(w, S)| = 0$ , contradicting Theorem 1. If  $u_2$  has status  $D$  in  $T_2$ , then in the tree  $T_2$ ,  $u_2 \in \text{pn}(w, S_2)$  where  $w$  is adjacent to no leaf except possibly  $u_2$  and  $|\text{pn}(w, S_2)| = 2$ . Thus in  $T$ ,  $|\text{pn}(w, S)| = 1$  and  $w \in S - S(T)$ , contradicting Theorem 13. Hence,  $u_2$  has status  $E$  in  $T_2$ . Thus,  $T$  can be obtained from  $T_1 \cup T_2$  by Operation  $\mathcal{T}_2$ . The result follows. ■

We now return to the proof of Theorem 16. By Claim 6, we may assume that no edge joins two vertices of  $V - S$  and by Claim 7, we may assume that each vertex in  $V - S$  is the external private neighbor of some vertex in  $S$ . Hence, each vertex in  $V - S$  is a leaf in  $T$ . If  $S = S(T)$ , then  $T \in \mathcal{T}$ . Hence, we may assume that  $S \neq S(T)$ .

Let  $u_1 \in S - S(T)$ . By assumption,  $N(u_1) \subseteq S$ . Hence, by Theorem 13,  $|\text{ipn}(u_1, S)| \geq 2$ . For each  $w \in \text{ipn}(u_1, S)$ ,  $N(w) \cap S = \{u_1\}$  and  $w \in S(T)$ . If  $S = N[u_1]$ , then  $T \in \mathcal{F}$ . Hence we may assume that some neighbor  $u_2$  of  $u_1$  is not an internal private neighbor of  $u_1$ .

For  $i \in \{1, 2\}$ , let  $T_i$ ,  $S_i$ , and  $D_i$  be as defined earlier. Since  $S_i$  is a total dominating set of  $T_i$  for  $i \in \{1, 2\}$ ,  $T_i$  is a UTD-tree and  $S_i$  is the unique  $\gamma_t(T_i)$ -set by Claim 5. Applying the inductive hypothesis to  $T_i$ , each  $T_i$  can be constructed from disjoint trees in  $\mathcal{T} \cup \mathcal{F}$  by a sequence of Operations  $\mathcal{T}_1$ ,  $\mathcal{T}_2$ , and  $\mathcal{T}_3$ .

In the tree  $T_1$ ,  $|\text{ipn}(u_1, S_1)| \geq 2$ , and so  $u_1 \notin \text{ipn}(w, S_1)$  for any  $w \in S_1$ . Thus,  $u_1$  has status  $B$  in  $T_1$ . In the tree  $T_2$ , if  $u_2$  has status  $A$ , then  $u_2 \in \text{ipn}(w, S_2)$  for some  $w \in S_2 - S(T)$  where  $|\text{pn}(w, S_2)| = 2$ . But then in the tree  $T$ ,  $w \in S - S(T)$  and  $|\text{pn}(w, S)| = 1$ , contradicting Theorem 13. Hence,  $u_2$  has status  $B$  in  $T_2$ . Thus,  $T$  can be obtained from  $T_1 \cup T_2$  by Operation  $\mathcal{T}_3$ . ■

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