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A NOTE ON DOMINATION IN BIPARTITE GRAPHS

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Abstract

DOMINATING SET remains NP-complete even when instances are restricted to bipartite graphs, however, in this case VERTEX COVER is solvable in polynomial time. Consequences to VECTOR DOMINATING SET as a generalization of both are discussed.

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For terminology and notation not defined here we refer to [2, 3]. Given a finite, simple, and undirected graph G without isolated vertices, $V(G) = \{1, \ldots, n\}$, $E(G)$, $G[V]$, and $d_i(G)$ denote its vertex set, its edge set, the subgraph of G induced by $V \subseteq V(G)$, and the degree of $i \in V(G)$ in G, respectively. Furthermore, let $\vec{1} = (1, \ldots, 1) \in R^n$ and $\vec{d}(G) = (d_1(G), \ldots, d_n(G))$. Given an integral vector $\vec{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i(G)$ for $i = 1, \ldots, n$, a set $D \subseteq V(G)$ is called $\vec{k}$*-dominating* if $d_i(G[V(G) \setminus D]) \leq d_i(G) - k_i$ for $i \in V(G) \setminus D$. Consider the corresponding decision problem, which was investigated in [4]:

VECTOR DOMINATING SET
Instance: A graph G on $V(G) = \{1, \ldots, n\}$, an integral vector $\vec{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i(G)$ for $i = 1, \ldots, n$, and a positive integer l.
Question: Does G contain a $\vec{k}$-dominating set D with $|D| \leq l$?

The restriction of VECTOR DOMINATING SET to $\vec{k} = \vec{1}$ is the decision problem DOMINATING SET, remaining NP-complete even when instances are restricted to bipartite graphs [1]. It is easy to see that $D \subseteq V(G)$ is $\vec{d}(G)$-dominating if and only if each edge of G has at least one endvertex in D.

The corresponding decision problem VERTEX COVER is NP-complete in general, however, is solvable in polynomial time for bipartite graphs [3]. The question arises in how many components $\vec{k}$ may differ from $\vec{1}$ or from $\vec{d}(G)$ such that the restriction of VECTOR DOMINATING SET to bipartite graphs remains NP-complete or is solvable in polynomial time, respectively. Theorem 1 and Theorem 2 give partial answers to this question.

Theorem 1. *Given $0 < c < \frac{1}{2}$, the following restriction of VECTOR DOMINATING SET remains NP-complete.*
Instance: A bipartite graph G on $V(G) = \{1, \ldots, n\}$, an integral vector $\vec{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i(G)$ for $i = 1, \ldots, n$ and $|\{i \in V(G) \mid k_i > 1\}| = \lceil cn \rceil$, and a positive integer l.
Question: Does G contain a $\vec{k}$-dominating set D with $|D| \leq l$?

Theorem 2. *Given $c > 0$, the following restriction of VECTOR DOMINATING SET is solvable in polynomial time.*
Instance: A bipartite graph G on $V(G) = \{1, \ldots, n\}$, an integral vector $\vec{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i(G)$ for $i = 1, \ldots, n$ and $|E(G[\{i \in V(G) \mid k_i < d_i(G)\}])| \leq c\ log_2 n$, and a positive integer l.
Question: Does G contain a $\vec{k}$-dominating set D with $|D| \leq l$?

***Proof of Theorem* 1.** Given $0 < c < \frac{1}{2}$, we shall transform DOMINATING SET to the restriction of VERTEX DOMINATING SET of Theorem 1. Let a graph H on $V(H) = \{1, \ldots, m\}$ and a positive integer l be an instance of DOMINATING SET, the positive integer r be choosen such that $\frac{2cm}{1-2c} \leq r < \frac{2cm}{1-2c} + 1$, hence, $c(2m + 2r) \leq r < c(2m + 2r) + (1 - 2c)$ implying $r = \lceil c(2m + 2r) \rceil \geq 1$, and G be constructed with $V(G) = \{1, \ldots, 2m + 2r\}$ and $E(G) = \{(i, m + j), (j, m + i) \mid (i, j) \in E(H)\} \cup \{(i, m+i), (m+i, 2m+j), (2m+j, 2m+r+j) \mid i = 1, \ldots, m,\ j = 1, \ldots, r\}$. Following the ideas in [1], it is easy to see that G is bipartite and that for $D(H) \subseteq V(H)$, $D(G) = \{i + m \mid i \in D(H)\} \cup \{2m + 1, \ldots, 2m + r\}$ is a $\vec{1}$-dominating set of G if and only if $D(H)$ is a $\vec{1}$-dominating set of H. Let $\vec{k} = (k_1, \ldots, k_{2m+2r})$ be an arbitrary integral vector with $1 \leq k_p \leq d_p(G)$ for $p = 1, \ldots, 2m+2r$, and $k_p = 1$ if and only if $p \in V(G) \setminus \{2m+1, \ldots, 2m+r\}$. Then $|\{i \in V(G) \mid k_i > 1\}| = r = \lceil c|V(G)| \rceil$, and $D(G)$ is even a $\vec{k}$-dominating set of G if $D(H)$ is a $\vec{1}$-dominating set of H. Hence, H contains a $\vec{1}$-dominating set of cardinality l if and only if G contains a $\vec{k}$-dominating set of cardinality $l+r$. With $|V(G)| = 2m+2r < 2m+2c(2m+2r)+2(1-2c) = 2|V(H)|+2c|V(G)|+2(1-2c)$, hence, $|V(G)| < \frac{2}{1-2c}|V(H)|+2$ we are done. ■

***Proof of Theorem* 2.** For $D \subseteq V(G)$, let $E(D)$ be the set of edges having no endvertex in D, and $H(D)$ be the graph arising from G by deleting the edges of $E(D)$. It is easy to see that D is a $\vec{k}$-dominating set of G if and only if $E(D) \subseteq E(G[\{i \in V(G) \mid k_i < d_i(G)\}])$, each endvertex i of an edge in $E(D)$ is endvertex of at most $d_i(G) - k_i$ edges in $E(D)$, and D is a $\vec{d}(H(D))$-dominating set of $H(D)$. Since VERTEX COVER is solvable in polynomial time and the number of sets $E(D)$ is at most $2^{|E(G[\{i \in V(G) \mid k_i < d_i(G)\}])|} \leq n^c$ we are done. ■

Remark. It remains open whether the bounds cn $(0 < c < \frac{1}{2})$ and $c\ log_2 n$ $(c > 0)$ of Theorem 1 and Theorem 2 can be made significantly greater such that VECTOR DOMINATING SET is still NP-complete or solvable in polynomial time, respectively.

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