

CYCLE-PANCYCLISM IN BIPARTITE TOURNAMENTS II

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Abstract

Let T be a hamiltonian bipartite tournament with n vertices, γ a hamiltonian directed cycle of T , and k an even number. In this paper the following question is studied: What is the maximum intersection with γ of a directed cycle of length k contained in $T[V(\gamma)]$? It is proved that for an even k in the range $\frac{n+6}{2} \leq k \leq n-2$, there exists a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k)$, $h(k) \in \{k, k-2\}$ with $|A(\mathcal{C}_{h(k)}) \cap A(\gamma)| \geq h(k) - 4$ and the result is best possible. In a previous paper a similar result for $4 \leq k \leq \frac{n+4}{2}$ was proved.

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1. Introduction

The subject of pancyclism has been studied by several authors (e.g. [1, 2, 3, 5, 13, 15]). Three types of pancyclism have been considered. A digraph D is: *pancyclic* if it has directed cycles of all the possible lengths; *vertex-pancyclic* if given any vertex v there are directed cycles of every length containing v ; and *arc-pancyclic* if given any arc e there are directed cycles of every length containing e .

It is well known that a hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (with only very few exceptions) but not necessarily

arc-pancyclic (see for example [3, 12, 14]). The concept of cycle-pancyclicism studies the following question: Given a directed cycle γ of a digraph D , find the maximum number of arcs which a directed cycle of length k , (if such a directed cycle exists) contained in $D[V(\gamma)]$ (the subdigraph of D induced by $V(\gamma)$) has in common with γ . Cycle-pancyclicism in tournaments has been studied in [6, 7, 8, 9]. In a previous paper [10] it was attempted to study cycle-pancyclicism in bipartite tournaments; in fact it was proved that for an even k , $4 \leq k \leq \frac{n+4}{2}$ there exists a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k)$, $h(k) \in \{k, k-2\}$ with $|A(\mathcal{C}_{h(k)}) \cap A(\gamma)| \geq h(k) - 3$ and the result is best possible. In this paper, the study of cycle-pancyclicism in bipartite tournaments is completed. To study this question it is sufficient to consider a hamiltonian bipartite tournament where γ is a hamiltonian directed cycle (because we are looking for directed cycles of length k contained in $D[V(\gamma)]$ whose arcs intersect the arcs of γ the most possible). We will assume (without saying it explicitly in Lemmas, Theorems or Corollaries) that we are working in a hamiltonian bipartite tournament with a vertex set $V = \{0, 1, \dots, n-1\}$ and arc set A . Also we assume without loss of generality that $\gamma = (0, 1, \dots, n-1, 0)$ is a hamiltonian directed cycle of T ; k will be an even number; $\mathcal{C}_{h(k)}$ will denote a directed cycle of length $h(k)$ with $h(k) \in \{k, k-2\}$ and $\mathcal{J}(\mathcal{C}_{h(k)}) = |A(\mathcal{C}_{h(k)}) \cap A(\gamma)|$. This paper is the second part of the study on the existence of a directed cycle $\mathcal{C}_{h(k)}$ where $\mathcal{J}(\mathcal{C}_{h(k)})$ is the maximum. For general concepts we refer the reader to [4].

2. Preliminaries

A *chord* of a cycle C is an arc not in C with both terminal vertices in C . The *length* of a chord $g = (u, v)$ of C , denoted $\ell(g)$, is equal to the length of $\langle u, C, v \rangle$ where $\langle u, C, v \rangle$ denotes the uv -directed path contained in C . We say that g is a *c-chord* if $\ell(g) = c$ and $g = (u, v)$ is a *-c-chord* if $\ell\langle v, C, u \rangle = c$. Observe that if g is a c -chord then it is also a $-(n-c)$ -chord. All the chords considered in this paper are chords of γ , also observe that since T is bipartite all the the chords of γ have odd lengths. We will denote by \mathcal{C}_k a directed cycle of length k . In what follows all notation is taken modulo n . In what follows we assume $k \geq 10$ (In [10] it was proved that for $k = 4, 6, 8$ there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3$).

Observation 2.1. *If $n = 2k - 6$, then there exists a directed cycle \mathcal{C}_{k-2} with $\mathcal{J}(\mathcal{C}_{k-2}) = k - 3$.*

Proof. Consider the arc between 0 and $k - 3$; when $(0, k - 3) \in A$ we have $\mathcal{C}_{k-2} = (0, k - 3) \cup \langle k - 3, \gamma, 0 \rangle$ a directed cycle with $\mathcal{J}(\mathcal{C}_{k-2}) = k - 3$; when $(k - 3, 0) \in A$ we obtain $\mathcal{C}_{k-2} = \langle 0, \gamma, k - 3 \rangle \cup (k - 3, 0)$ a directed cycle with $\mathcal{J}(\mathcal{C}_{k-2}) = k - 3$. ■

In view of Observation 2.1 we will assume in what follows that $k + 2 \leq n \leq 2k - 8$.

Lemma 2.2. *At least one of the following properties holds:*

- (i) *There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 1$ ($h(k) \in \{k, k - 2\}$).*
- (ii) *All the following arcs are in A : (a) Every $(k - 1)$ -chord; (b) Every $(k - 3)$ -chord.*

Proof. Suppose that (i) is not true. (a) If $(k - 1, 0)$ is a $-(k - 1)$ -chord, then $\mathcal{C}_k = \langle 0, \gamma, k - 1 \rangle \cup (k - 1, 0)$ is a directed cycle with $\mathcal{J}(\mathcal{C}_k) = k - 1$. (b) If $(k - 3, 0)$ is a $-(k - 3)$ -chord, then $\mathcal{C}_{k-2} = \langle 0, \gamma, k - 3 \rangle \cup (k - 3, 0)$ is a directed cycle with $\mathcal{J}(\mathcal{C}_{k-2}) = k - 3$. ■

Lemma 2.3. *Let $P = (x, x + 1, \dots, \ell)$, x, ℓ even, $\ell \geq x + 4$, be a directed path contained in γ , let z be odd, $x \in V - V(P)$, and $\{(x, z), (x + 2, z), (z, \ell), (z, \ell - 2), \dots, (z, \ell - (a - 1))\} \subseteq A$ with a odd, $1 \leq a \leq \ell - x - 3$. Then there exists an index i , $x + 2 \leq i \leq \ell - (a + 1)$, such that $\{(i, z), (z, i + a + 1)\} \subseteq A$.*

Proof. Let $i \in V(P)$ be the maximum vertex in P such that $(i, z) \in A$ clearly $x + 2 \leq i \leq \ell - (a + 1)$ and $\{(i, z), (z, i + a + 1)\} \subseteq A$. ■

Lemma 2.4. *If all the $(k - 3), (k - 1), \dots, p$ -chords, p is odd, $k - 1 \leq p < n - 3$ are in T , then at least one of the two following properties holds.*

- (i) *There exists a directed cycle \mathcal{C}_k with $\mathcal{J}(\mathcal{C}_k) \geq k - 3$.*
- (ii) *Every $(p + 2)$ -chord is in T .*

Proof. We show that if (ii) is false, then (i) holds. Let (s_1, s_2) be a $-(p + 2)$ -chord and let z be odd in $\langle s_1, \gamma, s_2 \rangle - \{s_1, s_2\}$. Assume w.l.o.g. that $s_2 = 0$. Let $x = z + n - p \pmod{n}$. Observe that

$$(1) \quad \{(x, z), (x + 2, z), \dots, (x + p - (k - 3), z)\} \subseteq A.$$

since these are the $p, p - 2, \dots, (k - 3)$ -chords of γ ending in z . Similarly

$$(2) \quad \{(z, z + p), (z, z + p - 2), \dots, (z, z + k - 3)\} \subseteq A.$$

Observe that the start points of the arcs in set (1) are consecutive in-neighbors of z in γ and less than the endpoints of the arcs in set (2), which are consecutive out-neighbors of z in γ . This is because the largest start point of an arc in (1) is $z + n - (k - 3)$ and the last endpoint of an arc in (2) is $z + (k - 3)$ and $z + (k - 3) > z + n - (k - 3)$ (as $n \leq 2k - 8$).

Now, consider the directed path $\langle x, \gamma, z + p \rangle$. Since $x = z + n - p \pmod{n}$ and $2p > n$ it is obvious that $z \notin V(\langle x, \gamma, z + p \rangle)$. Note that the cardinality of (1) is at least 2 and the cardinality of (2) is $\frac{p-k+5}{2}$. Thus letting $a = p - k + 4$ and $\ell = z + p$ it follows from Lemma 2.3 that there exists j , $x \leq j < z + (k - 3)$ such that $\{(j, z), (z, j + a + 1)\} \subseteq A$. And then $C = (s_1, s_2) \cup \langle s_2, \gamma, j \rangle \cup (j, z, j + a + 1) \cup \langle j + a + 1, \gamma, s_1 \rangle$ is a directed cycle. In order to see that $\ell(C) = k$ note that $\ell\langle s_1, \gamma, s_2 \rangle = n - (p + 2)$, and thus $\ell\langle s_2, \gamma, s_1 \rangle = p + 2$. Clearly, $\ell\langle j, \gamma, j + a + 1 \rangle = a + 1$, therefore $\ell(C) = p + 2 - (a + 1) + 3 = k$ and $\mathcal{J}(C) = k - 3$. ■

It follows directly from Lemmas 2.2 and 2.4 the following

Theorem 2.5. *At least one of the following conditions holds.*

- (i) *There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*
- (ii) *For each p odd, $k - 3 \leq p \leq n - 3$, every p -chord of γ is in T .*

3. The Main Result

In this section we prove the following

Theorem 3.1. *For every k such that $\frac{n+6}{2} \leq k \leq n-2$, there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 4$.*

Proof. In view of Observation 2.1 we will assume $k \geq \frac{n+8}{2}$. It follows from Theorem 2.5 that we can assume that for each odd p , $k - 3 \leq p \leq n - 3$, every p -chord is in T ; i.e., for each odd q $3 \leq q \leq n - (k - 3)$, every $(-q)$ -chord is in T . This assumption will be maintained in the whole proof. Let s be

the minimum integer such that γ has an s -chord. Note that $s \geq n - k + 5$ (s is odd), and thus $-s \leq k - 5$. This is because for each odd p , $k - 3 \leq p \leq n - 3$, every p -chord is in T , and therefore for each $3 \leq n - p \leq n - k + 3$, every $(n - p)$ -chord is not in T . Let $g = (u, v)$ be an s -chord of γ .

Denote by w the last vertex of $\langle v + 1, \gamma, u - 1 \rangle$ such that there exists an arc (w, z) with $z \in \langle u + 1, \gamma, v - 1 \rangle$. Notice that $\ell\langle u, \gamma, v \rangle \geq 7$ (because $\ell\langle u, \gamma, v \rangle = s \geq n - k + 5$ and $n \geq k + 2$), and thus $\langle u + 1, \gamma, v - 1 \rangle$ has at least one vertex. Also, the vertex w is well defined because $(v + 2, v - 1)$ is a (-3) -chord and hence it is in A . Hence for every vertex $x \in \langle w + 1, \gamma, u - 1 \rangle$, every (x', x) arc with $x' \in \langle u + 1, \gamma, v - 1 \rangle$ and $x \not\equiv x' \pmod{2}$, is in A ; by Definition of w . Also for any x, x' such that: $x \not\equiv x' \pmod{2}$, $2 \leq \ell\langle x, \gamma, x' \rangle < s$, and $\{x, x'\} \subseteq V(\langle u, \gamma, v - 1 \rangle)$ the arc (x', x) is in A because of the definition of s . Therefore we have the following Claims:

Claim 1. (a) For every $z_1 \in \langle z, \gamma, v - 1 \rangle$ and $u_1 \in \langle u, \gamma, z - 1 \rangle$ such that $u_1 \not\equiv z_1 \pmod{2}$ and $\ell\langle u_1, \gamma, z_1 \rangle \geq 2$, it holds $(z_1, u_1) \in A$.

(b) For every $u_2 \in \langle u + 1, \gamma, v - 1 \rangle$ and $w_1 \in \langle w + 1, \gamma, u \rangle$ such that $w_1 \not\equiv u_2 \pmod{2}$ and $\ell\langle w_1, \gamma, u_2 \rangle \geq 2$, it holds $(u_2, w_1) \in A$.

As a direct consequence we have the following Claim.

Claim 2. (a) If $z_1 \in \langle z, \gamma, v - 1 \rangle$ and $w_1 \in \langle w + 1, \gamma, u \rangle$ such that $z_1 \not\equiv w_1 \pmod{2}$ and $\ell\langle w_1, \gamma, z_1 \rangle \geq 2$, then

$$\mathcal{C}^1 = (z_1, w_1) \cup \langle w_1, \gamma, u \rangle \cup (u, v) \cup \langle v, \gamma, w \rangle \cup (w, z) \cup \langle z, \gamma, z_1 \rangle$$

is a directed cycle of T of length m with $J(\mathcal{C}^1) = m - 3$.

(b) If $z_1 \in \langle z, \gamma, v - 1 \rangle$, $w_1 \in \langle w + 1, \gamma, u \rangle$, $u_1 \in \langle u + 1, \gamma, z - 1 \rangle$ and $u_2 \in \langle u_1, \gamma, z - 1 \rangle$ such that: $u_1 \not\equiv z_1 \pmod{2}$; $\ell\langle u_1, \gamma, z_1 \rangle \geq 2$, $w_1 \not\equiv u_2 \pmod{2}$ and $\ell\langle w_1, \gamma, u_2 \rangle \geq 2$, then

$$\begin{aligned} \mathcal{C}^2 = & (z_1, u_1) \cup \langle u_1, \gamma, u_2 \rangle \cup (u_2, w_1) \cup \langle w_1, \gamma, u \rangle \\ & \cup (u, v) \cup \langle v, \gamma, w \rangle \cup (w, z) \cup \langle z, \gamma, z_1 \rangle \end{aligned}$$

is a directed cycle of length q with $J(\mathcal{C}^2) = q - 4$.

Observe that \mathcal{C}^2 is a directed cycle because $\langle u + 1, \gamma, v - 1 \rangle$ is non-empty, and because $z \neq u_2$, $u \neq u_1$, $w \neq w_1$ and $v \neq z_1$. A similar observation holds for \mathcal{C}^1 .

We proceed to prove the existence of a directed cycle of length k intersecting γ in at least $k - 4$ arcs. We split the problem into several cases according to the position of z in $\langle u + 1, \gamma, v - 1 \rangle$ and according to $\ell\langle u + 1, \gamma, v - 1 \rangle$. We are able to use constructions equal or similar to \mathcal{C}^1 or \mathcal{C}^2 . Consider the path $\alpha = (u, v) \cup \langle v, \gamma, w \rangle \cup (w, z)$, and let $r = k - \ell(\alpha)$. We now extend α to a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{J}(\mathcal{C}_{h(k)}) \geq h(k) - 4$. Observe that since $\ell\langle v, \gamma, u \rangle = n - s \leq k - 5$ (because $s \geq n - k + 5$) it follows that $\ell\langle v, \gamma, u - 1 \rangle \leq k - 6$ and $\ell(\alpha) \leq k - 4$; hence $r \geq 4$.

Case 1. $\ell\langle w, \gamma, u \rangle - 1 + \ell\langle z, \gamma, v \rangle - 1 \geq r - 1 > 0$.

Let r_1 and r_2 be such that $r_1 + r_2 = r - 1$, $0 \leq r_1 \leq \ell\langle w, \gamma, u \rangle - 1$, $0 \leq r_2 \leq \ell\langle z, \gamma, v \rangle - 1$, $w_1 = u - r_1$ and $z_1 = z + r_2$.

The proof that $(z_1, w_1) \in A$ is as follows:

First, we prove that $\ell\langle w_1, \gamma, z_1 \rangle \geq 2$. We have that $\ell\langle w_1, \gamma, z_1 \rangle = z + r_2 - (u - r_1) = z - u + r - 1$, since $r_1 + r_2 = r - 1$; the definition of z implies $z - u \geq 1$, and therefore $z - u + r - 1 \geq r \geq 4$.

Now we prove that $z_1 \not\equiv w_1 \pmod{2}$ by considering two possible cases: When $\ell(\alpha)$ is odd we have: r is odd (because $r = k - \ell(\alpha)$, and k even), $r - 1$ is even, $r_1 = r_2 \pmod{2}$ (because $r_1 + r_2 = r - 1$), and consequently $r_2 = -r_1 \pmod{2}$ (Notice $r_1 \equiv -r_1 \pmod{2}$); moreover, $\ell\langle v, \gamma, w \rangle$ is odd which implies $\ell\langle z, \gamma, v \rangle$ is even (Notice that since (w, z) is a chord we have $\ell\langle w, \gamma, z \rangle$ is odd and $\ell\langle z, \gamma, w \rangle$ is odd and therefore $\ell\langle u, \gamma, z \rangle$ is odd (because $\ell\langle u, \gamma, v \rangle$ is odd); so we have $u \not\equiv z \pmod{2}$, we conclude $u - r_1 \not\equiv z + r_2 \pmod{2}$ (because $u - r_1 \equiv z + r_2 \pmod{2}$ implies $u \equiv z \pmod{2}$ as $r_2 \equiv -r_1 \pmod{2}$). When $\ell(\alpha)$ is even we have: r is even, $r - 1$ is odd, $r_1 \not\equiv r_2 \pmod{2}$, $r_2 \not\equiv -r_1 \pmod{2}$; moreover, $\ell\langle v, \gamma, w \rangle$ is even, $\ell\langle z, \gamma, v \rangle$ is odd, $\ell\langle u, \gamma, z \rangle$ is even; so we have $u \equiv z \pmod{2}$ and since $r_2 \not\equiv -r_1$ we conclude $u - r_1 \not\equiv z + r_2 \pmod{2}$. So in any case we have $u - r_1 \not\equiv z + r_2 \pmod{2}$.

It follows from Claim 2 that \mathcal{C}^1 is a directed cycle of length $\ell(\alpha) + r_1 + r_2 + 1 = \ell(\alpha) + r = k$ with $\mathcal{J}(\mathcal{C}^1) = k - 3$.

Case 2. $\ell\langle w, \gamma, u \rangle - 1 + \ell\langle z, \gamma, v \rangle - 1 < r - 1$.

Observe that $\ell\langle w, \gamma, u \rangle - 1 + \ell\langle z, \gamma, v \rangle - 1 + \ell\langle u, \gamma, z \rangle - 1 = n - \ell(\alpha) - 1 > k - \ell(\alpha) = r$. Thus $\ell\langle u, \gamma, z \rangle - 1 \geq r - (\ell\langle w, \gamma, u \rangle - 1 + \ell\langle z, \gamma, v \rangle - 1) + 1$

From the hypothesis of Case 2, $\ell\langle u, \gamma, z \rangle \geq 4$.

Let $r_3 = r - (\ell\langle w, \gamma, u \rangle - 1 + \ell\langle z, \gamma, v \rangle - 1) - 2$. It follows from the hypothesis of Case 2 that $0 \leq r_3 < \ell\langle u, \gamma, z \rangle - 1$.

Denote: $w_1 = w + 1$, $u_1 = u + 1$, $u_2 = u + r_3 + 1 = u_1 + r_3$ and $z_1 = v - 1$.

We have that $\ell\langle u_1, \gamma, z_1 \rangle = v - 1 - u - 1 = v - u - 2 \geq 2$. The last equality follows because $\ell\langle u, \gamma, v \rangle = s \geq n - k + 5$ and $k \leq n - 2$, and hence $\ell\langle u, \gamma, v \rangle \geq 7$. So $\ell\langle u_1, \gamma, z_1 \rangle \geq 2$ and the fact $u_1 \not\equiv z_1 \pmod{2}$ (because $(u, v) \in A$ implies $u + 1 \not\equiv v - 1 \pmod{2}$ (as $1 \equiv -1 \pmod{2}$)).

Hence $(z_1, u_1) \in A$.

Now $\ell\langle w_1, \gamma, u_2 \rangle = u + r_3 + 1 - w - 1 = u - w + r_3$. Since by Definition $u - w \geq 1$ and $r_3 \geq 0$; it is sufficient to consider two possibilities: $u_2 - w_1 \geq 2$ and $u_2 - w_1 = 1$.

Case 2.a. $\ell\langle w_1, \gamma, u_2 \rangle \geq 2$.

In this case we only need to prove that $w_1 \not\equiv u_2 \pmod{2}$ in order to have $(u_2, w_1) \in A$. Since $w \not\equiv z \pmod{2}$ (because $(w, z) \in A$), and $w_1 = w + 1$ it suffices to prove $\ell\langle u_2, \gamma, z \rangle$ is odd. We proceed by contradiction; suppose $\ell\langle u_2, \gamma, z \rangle$ is even and consider the two possible cases: When $\ell(\alpha)$ is odd we have r is odd, $r - 2$ is odd, and $\ell\langle v, \gamma, w \rangle$ is odd. Moreover, since $\ell\langle v, \gamma, w \rangle$ is odd we have $\ell\langle z, \gamma, z_1 \rangle$ is odd (because $z_1 = v - 1$ and $\ell\langle z, \gamma, w \rangle$ is odd as $(w, z) \in A$); now $\ell\langle u_1, \gamma, u_2 \rangle$ is even (Notice $\ell\langle u_1, \gamma, z_1 \rangle$ is odd because $(z_1, u_1) \in A$) and $\ell\langle w_1, \gamma, u \rangle$ is odd (Notice $\ell\langle w, \gamma, z \rangle$ is odd and $\ell\langle u_1, \gamma, z \rangle$ is even), so we obtain $\ell\langle w_1, \gamma, u \rangle + \ell\langle u_1, \gamma, u_2 \rangle + \ell\langle z, \gamma, z_1 \rangle$ is even. Also we have $\ell\langle w_1, \gamma, u \rangle + \ell\langle u_1, \gamma, u_2 \rangle + \ell\langle z, \gamma, z_1 \rangle = \ell\langle w, \gamma, u \rangle - 1 + r_3 + \ell\langle z, \gamma, v \rangle - 1 = r - 2$ is odd; a contradiction. The case when $\ell(\alpha)$ is even is completely analogous (by interchanging even with odd).

We conclude $(u_2, w_1) \in A$ and by Claim 2 $\mathcal{C}_q = \mathcal{C}^2$ is a directed cycle with $\mathcal{J}(\mathcal{C}^2) = q - 4$. Furthermore, $q = k$ since by construction $\ell(\mathcal{C}^2) = \ell(\alpha) + r = k$.

We now divide the remaining case of $u_2 - w_1 = 1$ into two subcases. In both the following holds: $u - w + r_3 = 1$ (because $u_2 - w_1 = 1$), $w = u - 1$, $r_3 = 0$, $u = w_1$ and $u_2 = u_1 = u + 1$. Also, by Definition of r_3 , $r = \ell\langle w, \gamma, u \rangle + \ell\langle z, \gamma, v \rangle$.

Case 2.b.1. $u_2 - w_1 = 1$ and $z \leq z_1 - 1$.

Notice that $\ell\langle u, \gamma, z_1 - 1 \rangle = \ell\langle u, \gamma, v \rangle - 2 \geq 3$. (because $s \geq 5$). Moreover, since $u \not\equiv v \pmod{2}$ (as $(u, v) \in A$) we have $u \not\equiv z_1 - 1 = v - 2 \pmod{2}$. Thus $(z_1 - 1, u) \in A$, and the fact $z \leq z_1 - 1$ implies $C_q = \alpha \cup \langle z, \gamma, z_1 - 1 \rangle \cup (z_1 - 1, u)$ is a directed cycle of length q with $\mathcal{J}(C_q) = q - 3$.

Now we prove $q = k - 2$; observe that $r = u - w + v - z$, and since $w = u - 1$ and $v = z_1 + 1$ we obtain $r = z_1 - z + 2$ and $z_1 - 1 - z = r - 3$. Thus $\ell(C_q) = \ell(\alpha) + r - 3 + 1 = \ell(\alpha) + r - 2 = k - 2$.

We conclude the proof of Theorem with the next subcase.

Case 2.b.2. $u_2 - w_1 = 1$ and $z = z_1 = v - 1$.

From $r_3 = 0$, $w = u - 1$, $z = v - 1$ and $r = \ell\langle w, \gamma, u \rangle + \ell\langle z, \gamma, v \rangle$ it follows that $r = 2$, thus $\ell(\alpha) = k - r = k - 2$. Since $\ell\langle v, \gamma, w \rangle = \ell(\alpha) - 2 = k - 4$ and $w = u - 1$ we have $\ell\langle v, \gamma, u \rangle = k - 3$ and hence $\mathcal{C}_{k-2} = \langle v, \gamma, u \rangle \cup \langle u, v \rangle$ is a directed cycle of length $k - 2$ with $J(\mathcal{C}_{k-2}) = k - 3$. ■

4. Remarks

In this section it is proved that the hypothesis of Theorem 3.1 are tight, and the result is best possible.

Definition 4.1 [10]. A digraph D with vertex set V is called cyclically p -partite complete ($p \geq 3$) provided one can partition $V = V_0 \cup V_1 \cup \dots \cup V_{p-1}$ so that (u, v) is an arc of D if and only if $u \in V_i, v \in V_{i+1}$ (notation modulo p).

Remark 4.2 [10]. The cyclically 4-partite complete digraph T_4 is a bipartite tournament and clearly, every directed cycle of T_4 has length $\equiv 0 \pmod{4}$. So for $k = 4m + 2$, T_4 has no directed cycles of length k and for $k = 4m$, T_4 has no directed cycles of length $k - 2$.

Remark 4.3. For $n \geq 6, k \geq 6$, such that $n \leq 2k - 8$, there exists a bipartite hamiltonian tournament T_n with no directed cycles $\mathcal{C}_{h(k)}$ with $J(\mathcal{C}_{h(k)}) \geq h(k) - 3$. (recall $h(k) \in \{k, k - 2\}$).

Proof. Define T_n as follows:

$$A(T_n) = \{(i, i + 1) \mid i \in \{0, 1, \dots, n - 1\}\} \cup \left\{ (i, i + j) \mid j \in \left\{ \frac{n}{2} + 1, \frac{n}{2} + 3, \dots, n - 3 \right\} \right\}.$$

Notice that $n \equiv 0 \pmod{4}$, otherwise the arc $(i, i + j)$ is not defined.

Consider a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k), h(k) \in \{k, k - 2\}$. Observe that the definition of T_n and the fact $n \leq 2k - 8$ imply $J(\mathcal{C}_{h(k)}) < h(k) - 2$. We prove that $J(\mathcal{C}_{h(k)}) < h(k) - 3$ by showing that for any directed cycle \mathcal{C} with $J(\mathcal{C}) = k - 3$, it holds $\ell(\mathcal{C}) \leq k - 4$.

Let $f_1 = (x_1, x_2), f_2 = (x_3, x_4)$, and $f_3 = (x_5, x_6)$ be the three arcs of \mathcal{C} not in γ . Hence without loss of generality

$$\mathcal{C} = (x_1, x_2) \cup \langle x_2, \gamma, x_3 \rangle \cup (x_3, x_4) \cup \langle x_4, \gamma, x_5 \rangle \cup (x_5, x_6) \cup \langle x_6, \gamma, x_1 \rangle.$$

By the definition of T_n it follows that $\ell(f_i) \geq \frac{n}{2} + 1$, for each $i \in \{1, 2, 3\}$. On the other hand,

$$\begin{aligned} \ell(\mathcal{C}) &= \ell\langle x_2, \gamma, x_1 \rangle + \ell\langle x_6, \gamma, x_5 \rangle - \ell\langle x_3, \gamma, x_4 \rangle + 3 \\ &= n - \ell(f_1) + n - \ell(f_3) - \ell(f_2) + 3 \\ &\leq \frac{n}{2} - 1 + \frac{n}{2} - 1 - \frac{n}{2} - 1 + 3 \\ &= \frac{n}{2} \end{aligned}$$

Therefore $\ell(\mathcal{C}) \leq k - 4$, since $n \leq 2k - 8$. ■

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