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# DOMINANT-MATCHING GRAPHS

Igor' E. Zverovich and Olga I. Zverovich

*RUTCOR – Rutgers Center for Operations Research, Rutgers University of New Jersey*
*640 Bartholomew Rd, Piscataway, NJ 08854-8003, USA*

**e-mail:** igor@rutcor.rutgers.edu

**Abstract**

We introduce a new hereditary class of graphs, the dominant-matching graphs, and we characterize it in terms of forbidden induced subgraphs.

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## 1. Dominant-Covering Graphs

Let $G$ be a graph. The *neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = N(x)$ of all vertices in $G$ that adjacent to $x$. If vertices $x$ and $y$ of $G$ are adjacent (respectively, non-adjacent), we shall use notation $x \sim y$ (respectively, $x \not\sim y$). For disjoint sets $X, Y \subseteq V(G)$, we write $X \sim Y$ (respectively, $X \not\sim Y$) to indicate that each vertex of $X$ is adjacent to each vertex of $Y$ (respectively, no vertex of $X$ is adjacent to a vertex of $Y$).

A set $D \subseteq V(G)$ is called a *dominating set* in $G$ if $V(G) = N[D] = \bigcup_{d \in D} N[d]$, where $N[d] = N(d) \cup \{d\}$ is the closed neighborhood of $d$. A *minimum* dominating set in $G$ is a dominating set having the smallest cardinality. This cardinality is the *domination number* of $G$, denoted by $\gamma(G)$.

A set $C \subseteq V(G)$ is called a *vertex cover* in $G$ if every edge of $G$ is incident to at least one vertex in $C$. The minimum cardinality of a vertex cover in $G$ is the *vertex covering number* of $G$, denoted by $\tau(G)$.

**Definition 1.** A graph $G$ is called a *dominant-covering graph* if $\gamma(H) = \tau(H)$ for every isolate-free induced subgraph $H$ of $G$.

Many similarly defined classes were characterized in terms of forbidden induced subgraphs by Zverovich [3], Zverovich [4], Zverovich and Zverovich [5], and Zverovich and Zverovich [6]. We give such a characterization for dominant-covering graphs, and then we extend it to dominant-matching graphs.

**Theorem 1.** *A graph $G$ is a dominant-covering graph if and only if $G$ does not contain any of $G_1, G_2, \ldots, G_{10}$ shown in Figure 1 as an induced subgraph.*

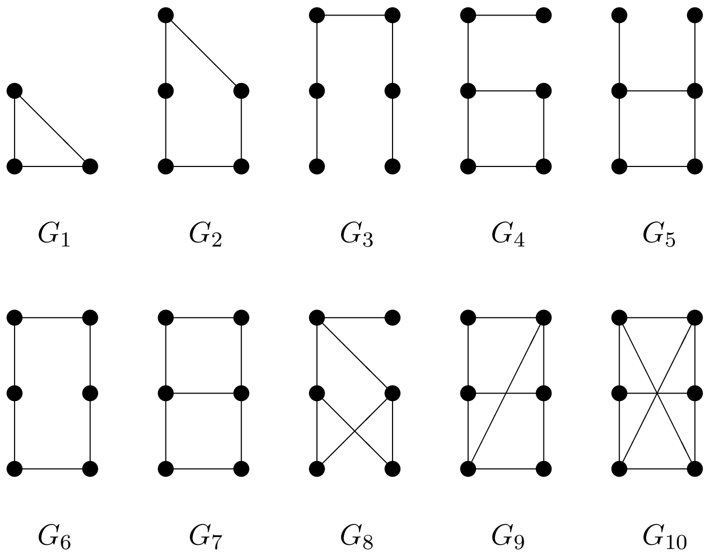


Figure 1. Forbidden induced subgraphs for dominant-covering graphs.

***Proof.*** Necessity. It is easy to check that the graphs $G_i \in \{G_1, G_2, \ldots, G_{10}\}$ (Figure 1) satisfies $2 = \gamma(G_i) < \tau(G_i)$, and therefore they are not dominant-covering. It follows that no one of them can be an induced subgraph of a dominant-covering graph.

Sufficiency. Let $G$ be a minimal forbidden induced subgraph for the class of all dominant-covering graphs. Suppose that $G \notin \{G_1, G_2, \ldots, G_{10}\}$. By minimality, $G$ does not contain any of $G_1, G_2, \ldots, G_{10}$ as an induced subgraph. Also, each proper induced subgraph of $G$ is a dominant-covering graph, therefore $\gamma(G) < \tau(G)$.

We consider a minimum dominating set $D$ of $G$ such that $D$ covers the maximum possible number of edges of $G$ [among all minimum dominating sets of $G$]. If $D$ covers all edges of $G$, then $\gamma(G) = \tau(G)$, a contradiction. Thus, we may assume that an edge $e = uv$ is not covered by $D$.

Since $D$ is a dominating set, there exist vertices $w$ and $x$ in $D$ which are adjacent to $u$ and $v$, respectively. If $w = x$ then $G(u, v, w) \cong G_1$, a contradiction. Therefore $w \neq x$. Moreover, $u$ is non-adjacent to $x$, and $v$ is non-adjacent to $w$.

Let $D_u = (D \setminus \{w\}) \cup \{u\}$. We have $|D_u| = |D|$, and $D_u$ covers the edges $uv$, $uw$ and $vx$.

*Case* 1. $D_u$ is not a dominating set.
Suppose that $D_u$ does not dominate a vertex $y$ of $G$. Since $D$ is a dominating set, $y$ is adjacent to $w$. Thus, the edge $f = yw$ is covered by $D$, and it is not covered by $D_u$.

*Case* 2. $D_u$ is a dominating set.
Clearly, $D_u$ is a minimum dominating set. The choice of $D$ implies that there exists an edge $f$ which is covered by $D$ and which is not covered by $D_u$. Obviously, $f$ is incident to the vertex $w$, i.e., we may assume that $f = yw$ for some vertex $y \notin \{u, v, x\}$.

In both cases, we have obtained that there exists some edge $yw$ covered by $D$ and not covered by $D_u$. If $y$ is adjacent to $u$ or $x$, then $G$ contains $G_1$ or $G_2$ as an induced subgraph, a contradiction. Hence edge-set of the induced subgraph $H = G(u, v, w, x, y)$ is one of the following:

**Variant 1H:** $E(H) = \{uv, uw, vx, wy\}$, or
**Variant 2H:** $E(H) = \{uv, uw, vx, wy, vy\}$, or
**Variant 3H:** $E(H) = \{uv, uw, vx, wy, wx\}$, or
**Variant 4H:** $E(H) = \{uv, uw, vx, wy, wx, vy\}$.

Now we consider the set $D_v = (D \setminus \{x\}) \cup \{v\}$. By symmetry, there exists an edge $g = zx$ which is covered by $D$ and which is not covered by $D_v$. Again, we have four variants for the induced subgraph $F = G(u, v, w, x, z)$:

**Variant 1F:** $E(H) = \{uv, uw, vx, xz\}$, or
**Variant 2F:** $E(H) = \{uv, uw, vx, xz, uz\}$, or
**Variant 3F:** $E(H) = \{uv, uw, vx, xz, wx\}$, or
**Variant 4F:** $E(H) = \{uv, uw, vx, xz, wx, uz\}$.

Note that the vertices $y$ and $z$ may or may not be adjacent. Combinations of Variants 1H, 2H, 3H, 4H and Variants 1F, 2F, 3F, 4F shows that the set $\{u, v, w, x, y, z\}$ induces one of $G_3, G_4, \ldots, G_{10}$, a contradiction. ■

## 2. Dominant-Matching Graphs

The *matching number* of a graph $G$ is denoted by $\mu(G)$, i.e., $\mu(G)$ is the maximum cardinality of a matching in $G$.

**Proposition 1** (see Lovász and Plummer [1]). *$\mu(G) \leq \tau(G)$ for every graph $G$.*

**Proposition 2** (Volkmann [2]). *$\gamma(G) \leq \mu(G)$ for every graph $G$ without isolated vertices.*

**Definition 2.** A graph $G$ is called a *dominant-matching graph* if $\gamma(H) = \mu(H)$ for every isolate-free induced subgraph $H$ of $G$.

Note that the class of all graphs such that $\mu(H) = \tau(H)$ for every induced subgraph $H$ of $G$ coincides with the class of all bipartite graphs, see e.g. Minimax König's Theorem in Lovász and Plummer [1]. Now we extend Theorem 1 by characterization of the dominant-matching graphs in terms of forbidden induced subgraphs.

**Theorem 2.** *A graph $G$ is a dominant-matching graph if and only if $G$ does not contain any of $G_3, G_4, \ldots, G_{10}$ (Figure 1) and $H_1, H_2, H_3, H_4, H_5$ (Figure 2) as an induced subgraph.*

***Proof.*** Necessity. It can be directly checked that

- $\gamma(H_i) = 1$ and $\mu(H_i) = 2$ for $i = 1, 2, 3$,
- $\gamma(H_j) = 2$ and $\mu(H_j) = 3$ for $j = 4, 5$, and
- $\gamma(G_k) = 2$ and $\mu(G_k) = 3$ for $k = 3, 4, \ldots, 10$.

Therefore none of $G_3, G_4, \ldots, G_{10}$ (Figure 1) and $H_1, H_2, H_3, H_4, H_5$ (Figure 2) can be an induced subgraph of a dominant-matching graph.

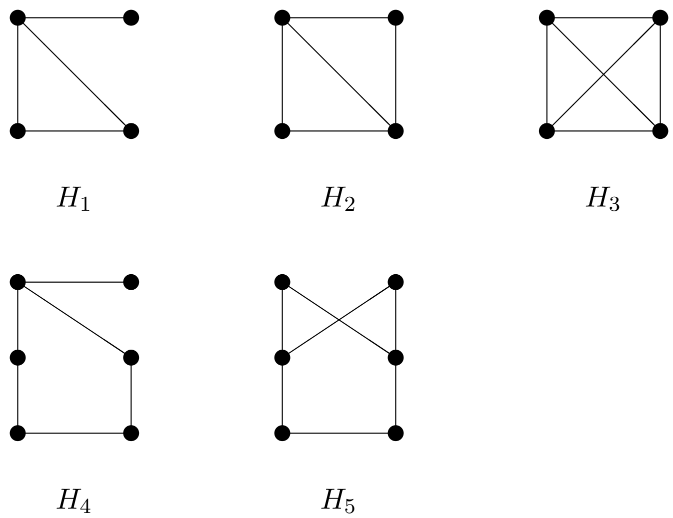


Figure 2. Some forbidden induced subgraphs for dominant-matching graphs.

Sufficiency. Suppose that the statement does not hold. We consider a minimal graph $G$ such that

- $G$ does not contain any of $G_3, G_4, \ldots, G_{10}$ (Figure 1) and $H_1, H_2, H_3, H_4, H_5$ (Figure 2) as an induced subgraph, and
- $G$ is not a dominant-matching graph.

The minimality of $G$ means that each proper induced subgraph of $G$ is a dominant-matching graph. If $G$ does not contain both $G_1$ and $G_2$ (Figure 1) induced subgraphs, then $G$ is a dominant-covering graph by Theorem 1. Hence $\gamma(G) = \tau(G)$. Proposition 1 and Proposition 2 imply that $\gamma(G) = \mu(G)$, a contradiction to the choice of $G$.

Thus, it is sufficient to consider two cases where either $G_1$ or $G_2$ is an induced subgraph of $G$. By minimality of $G$, $\gamma(G) < \mu(G)$, and $G$ is a connected graph.

*Case* 1. $G_1$ is an induced subgraph of $G$.
Since $\gamma(G) < \mu(G)$, $G \neq G_1$. By connectivity of $G$, there exists a vertex $u \in V(G) \setminus V(G_1)$ that is adjacent to at least one vertex of $G_1$. Clearly, the

set $V(G_1) \cup \{u\}$ induces one of $H_1, H_2$ or $H_3$ (Figure 2), a contradiction to the choice of $G$.

*Case* 2. $G_2$ is an induced subgraph of $G$.
As before, there exists a vertex $u \in V(G) \setminus V(G_2)$ that is adjacent to at least one vertex of $G_2$. We may assume that $G$ has no induced $G_1$ [see Case 1]. Hence the set $V(G_2) \cup \{u\}$ induces either $H_4$ or $H_5$ (Figure 2), a contradiction to the choice of $G$. ■

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