Discussiones Mathematicae
Graph Theory 24 (2004) 431–441

EVEN $[a, b]$-FACTORS IN GRAPHS

Mekkia Kouider

Laboratoire de Recherche en Informatique
UMR 8623 Bât. 490, Université Paris Sud
91405 Orsay, France

e-mail: km@lri.fr

and

Preben Dahl Vestergaard

Department of Mathematics
Aalborg University, Fredrik Bajers Vej 7G
DK-9220 Aalborg Øst, Denmark

e-mail: pdv@math.aau.dk

Abstract

Let a and b be integers $4 \leq a \leq b$. We give simple, sufficient conditions for graphs to contain an even $[a, b]$-factor. The conditions are on the order and on the minimum degree, or on the edge-connectivity of the graph.

Keywords: even factor, eulerian, spanning subgraph.

2000 Mathematics Subject Classification: 05C70.

1. Introduction

We denote by G a graph of order $n = |V(G)|$. For a vertex x in $V(G)$ let $d_G(x)$ denote its degree. By $\delta = \delta(G) = \min\{d_G(x)|x \in V(G)\}$ we denote the minimum degree in G. Let X, Y be an ordered pair of disjoint subsets of $V(G)$, and f, g be mappings from $V(G)$ into $\mathbb{N}$. By $e(X, Y)$ we denote the number of edges with one endvertex in X and the other in Y. By $h(X, Y)$, we denote the number of odd components in $G - (X \cup Y)$. A component C

of $G-(X\cup Y)$ is called odd if $e(C,Y)+\sum_{c\in V(C)} f(c)$ is an odd number. An even factor of G is a spanning subgraph all of whose degrees are even. If $g(x)\le f(x)$ for all x in $V(G)$, by a $[g,f]$-factor we understand a spanning subgraph F of G satisfying $g(x)\le d_F(x)\le f(x)$, for all $x\in V(G)$.

Theorem 1 (Lovász' parity $[g,f]$-factor theorem [13], [3]). *Let G be a graph and let g, f be maps from $V(G)$ into the nonnegative integers such that for each $v\in V(G)$, $g(v)\le f(v)$ and $g(v)\equiv f(v) \pmod 2$. Then G contains a $[g,f]$-factor F such that $d_F(v)\equiv f(v) \pmod 2$, for each $v\in V(G)$, if and only if, for every ordered pair X,Y of disjoint subsets of $V(G)$*

$$(*)\qquad h(X,Y)-\sum_{x\in X} f(x)+\sum_{y\in Y} g(y)-\sum_{y\in Y} d_G(y)+e(X,Y)\le 0.$$

Tutte's f-factor theorem is surveyed in [1]. Let us recall other results on $[a,b]$-factors. In [7], Kano and Saito proved that, for any nonnegative integers k,r,s,t satisfying $k\le r, 1\le r, ks\le rt$, every graph with degrees in the interval $[r,r+s]$ has a $[k,k+t]$-factor. Berge, Las Vergnas, and independently Amahashi and Kano, proved for any integer $b\ge 2$, that a graph has a $[1,b]$-factor if and only if $b|N(X)|\ge |X|$ for all independent vertex sets X of the graph. Kano proved a sufficient condition for a graph to have an $[a,b]$-factor giving a condition on the sizes $|N(X)|$ for subsets X of $V(G)$ [8]. Cui and Kano generalized Tutte's 1-factor theorem. They consider a map $f: V(G)\to\{1,3,5,\ldots\}$ and call F an odd $[1,f]$-factor of G if F is a factor of G with $d_F(v)$ odd and $d_F(v)\in[1,f(v)]$ for all vertices v in G. They prove that G has an odd $[1,f]$-factor if and only if $G-X$ has at most $\sum_{x\in X} f(x)$ components of odd cardinality for any subset $X\subseteq V(G)$ [5]. Then,Topp and Vestergaard restrict the number of subsets to be considered above, and, as a consequence, proved that a graph of even order n in which no vertex v is the center of an induced $K_{1,nf(v)+1}$ -star has an odd $[1,f]$-factor [15]. In [9, 10], Kouider and Maheo prove the existence of connected $[a,b]$ factors in graphs of high degree. For even factors with degrees between 2 and b we establish a sufficent condition in [11].

Theorem 2. *Let $b\ge 2$ be an even integer and let G be a 2-edge connected graph with n vertices and with minimum degree $\delta(G)\ge \min\{3,\frac{2n}{b+2}\}$. Then G contains an even $[2,b]$-factor.*

We shall now generalize this to even factors with degrees between a and b, where a is an even integer ≥ 4.

2. Results

Let a, b, $a \leq b$, be even, positive integers. In the inequality $(*)$, we substitute $e(X,Y)$ by $|X||Y|$, and derive a sufficient condition for existence of an even $[a,b]$-factor in G:

$$(**) \qquad h(X,Y) - b|X| + a|Y| - \delta|Y| + |X||Y| \leq 0.$$

We shall prove the following results.

Theorem 3. *Let a, b be two even integers satisfying $4 \leq a \leq b$. Let G be a 2-edge connected graph of order n at least $\max\{\frac{(a+b)^2}{b}, \frac{3(a+b)}{2}\}$, and of minimum degree δ at least $\frac{an}{a+b}$. Then G has an even $[a,b]$-factor.*

Example 1. Take even integers a, b such that $a \geq 12$, $b = 2a^2$, let $\delta = \frac{3a}{2} + 4$ and let G be the graph which consists of $2a-2$ disjoint copies of a complete graph $K_{\delta+1}$, each copy joined by one edge to a common vertex y. The order of G is $n = (\frac{3a}{2}+5)(2a-2)+1 = 3a^2+7a-9$, and it is easy to see that $n \geq \max\{\frac{(a+b)^2}{b}, \frac{3}{2}(a+b)\}$. The minimum degree of G is $\delta = \frac{3a}{2}+4$ and the inequality $\delta \geq \frac{an}{a+b}$ follows from $\frac{an}{a+b} = \frac{a(3a^2+7a-9)}{a+2a^2} = \frac{3a^2+7a-9}{2a+1} \leq \frac{3a^2+7a}{2a} \leq \frac{3}{2}a + \frac{7}{2}$. So G is not 2-edge connected but satisfies all other conditions of Theorem 3. The graph G has no even $[a,b]$-factor F, because F must contain an edge from y to K, one of the complete graphs $K_{\delta+1}$, and the rectriction of F to K should contain exactly one odd vertex, which is impossible.

Example 2. For a positive integer $k \geq 5$, let $a = 2k+2$ and $b = ka$. Let $n = k(3k+2)+1$. We consider a graph G of order n, composed of k vertex disjoint copies of the complete graph K_{3k+2}, and an external vertex x_0 joined to each copy by 3 edges. This graph is 2-edge connected, its minimum degree is $\delta = 3k \geq \frac{an}{a+b}$, and $n \geq \frac{(a+b)^2}{b}, n \geq \frac{3b}{2}$. In an even $[a,b]$-factor F of G the vertex x_0 must be joined to at least $2k+2$ other vertices, so in F at least one of the K_{3k+2}'s, say K, is joined to x_0 by exactly 3 edges. Thus the graph K should have a subgraph, namely $K \cap F$, with an odd number of odd vertices. Hence G has no even $[a,b]$-factor.

This example shows that even if G is 3-edge connected the conditions $\delta \geq \frac{an}{a+b}$ and $n \geq max\{\frac{(a+b)^2}{b}, 3b/2\}$ are not sufficient for existence of an even $[a,b]$-factor, even if a is much more smaller than b.

Theorem 4. *Let $a \geq 4$ and $b \geq a$ be two even integers. Let G be a 2-edge connected graph of order $n \geq \frac{(a+b)^2}{b}$ and of minimum degree at least $\frac{an}{a+b} + \frac{a}{2}$. Then G has an even $[a,b]$-factor.*

In the following result, we have a weaker condition on the order, but a stronger one on the edge-connectivity.

Theorem 5. *Let $a \geq 4$ and $b \geq a$ be two even integers, and let $k \geq a + \min\{\sqrt{a}, \frac{b}{a}\}$. Let G be a k-edge-connected graph of order $n \geq \frac{(a+b)^2}{b}$ and of minimum degree at least $\frac{an}{a+b}$. Then G has an even $[a,b]$-factor.*

Example 3. Let a, b, k be integers such that $b > 3a^2$, and $k \leq a - 1$; furthermore a, b are even and k is odd. We define a k-connected graph G as follows.

Let Y be a set k independant vertices, and consider a family of $k+2$ complete graphs H_i for $1 \leq i \leq k+2$ such that $H_i = K_{a+2}$ for $i \leq k+1$, and $H_{k+2} = K_{b+3a-(k+1)(a+3)+1}$. Each $y \in Y$ is joined to exactly $a+1$ vertices, one in H_i for each i, $1 \leq i \leq k+1$, and $a-k$ vertices in H_{k+2} so that no two vertices of Y have a common neighbour. So $d_H(y) = a+1$, for each $y \in Y$. The order n of G is $3a+b$. As $b > 3a^2$, one can verify that $\delta \geq \frac{an}{a+b}$. Thus G satisfies all conditions in Theorem 5, except the one on k. Suppose that G has an even $[a,b]$-factor F. Now, let y be any vertex in Y. As $d_G(y) = a+1$ and $a+1$ is odd, it follows that $d_F(y) = a$. Then necessarily, there exists a copy H_t for some $t \leq k$ such that $e_G(Y, H_t) = e_F(Y, H_t)$. It follows that the restriction of the factor F to H_t has k odd vertices; as k is odd, that is impossible. So, the graph G has no even $[a,b]$-factor.

3. Proofs

We shall use Claims 1–4 below for the proof of Theorem 3. First we establish the truth of $(*)$ for a large class of ordered pairs X, Y.

Let $\tau(X,Y) = h(X,Y) - b|X| + a|Y| - \sum_{y \in Y} d_G(y) + e(X,Y)$.

The hypotheses of Theorem 3 imply that $\delta \geq \max\{\frac{3a}{2}, a + \frac{a^2}{b}\}$.

Claim 1. Inequality $(*)$ holds if $-b|X| + a|Y| \leq 0$.

Proof. Recall, that for any odd component C, $b|V(C)| + e(C, Y)$ is odd; as b is even, that implies $e(C, Y) \geq 1$. Hence, between Y and each odd component of $G - (X \cup Y)$ there is at least one edge, therefore $h(X, Y) + e(X, Y) \leq \sum_{y \in Y} d_G(y)$, and $(*)$ follows as $-b|X| + a|Y| \leq 0$. ■

Claim 2. Inequality $(*)$ holds if $|Y| \geq a + b$.

Proof. Let $-b|X| + a|Y| = p$. By Claim 1, we may assume $p > 0$. By definition of $h(X, Y)$, we have $|X| + |Y| + h(X, Y) \leq n$. Then we obtain

$$|X| = \frac{a|Y| - p}{b} \leq \frac{a(n - h(X, Y) - |X|) - p}{b},$$

and thus

$$|X| \leq \frac{a(n - h(X, Y)) - p}{a + b}.$$

So

$$e(X, Y) \leq |X||Y| \leq \frac{a(n - h(X, Y)) - p}{a + b}|Y|.$$

By hypothesis on δ we have

$$-\sum_{y \in Y} d_G(y) \leq -\delta|Y| \leq -\frac{an}{a + b}|Y|.$$

That yields the inequality

$$\tau(X, Y) \leq h(X, Y) + p - \frac{an}{a + b}|Y| + \frac{a(n - h(X, Y)) - p}{a + b}|Y|.$$

So now, since $|Y| \geq a + b$, we get

$$\tau(X, Y) \leq h(X, Y) + p - \frac{a(h(X, Y) + p)}{a + b}|Y| \leq (1 - a)(h(X, Y) + p).$$

As $a \geq 4$ and $p > 0$, we conclude that $\tau(X, Y) \leq 0$ and $(*)$ is proven. ■

By Claims 1 and 2 we may henceforth assume $0 \leq \frac{b}{a}|X| < |Y| \leq a + b - 1$.

***Proof of Theorem* 3.** We assume $0 \leq \frac{b}{a}|X| < |Y| \leq a+b-1$ and, following the different values of $|Y|$, we proceed to prove that $\tau(X,Y) \leq 0$. As $h(X,Y) \leq n - |X| - |Y|$, $\tau(X,Y)$ is bounded as follows:

$$\tau(X,Y) \leq h(X,Y) - b|X| + a|Y| - \delta|Y| + |X||Y|$$

$$\leq n - (\delta - a + 1)|Y| + |X|(|Y| - b - 1),$$

and therefore, to prove $\tau(X,Y) \leq 0$ it suffices to prove that

$$(***) \qquad n - (\delta - a + 1)|Y| + |X|(|Y| - b - 1) \leq 0.$$

Case $|Y| \geq b+1$.
Let us set

$$\phi(|Y|) = n - (\delta - a + 1)|Y| + \frac{a}{b}|Y|(|Y| - b - 1).$$

As $|X| < \frac{a}{b}|Y|$, we see that $(***)$ will follow if $\phi(|Y|) \leq 0$.

Claim 3. $\phi(|Y|) \leq 0$.

Proof. For $|Y|$ varying in the interval of integers, $[b+1, a+b-1]$, the maximum value of the parabola ϕ is attained at an endpoint of the interval. In both ends we shall show that $\phi(|Y|) \leq 0$.

$$\phi(b+1) = n - (\delta - a + 1)(b+1);$$

and as $-\delta \leq -\frac{an}{a+b}$, we get

$$\phi(b+1) \leq n\frac{b - ab}{a+b} + ab + a - b - 1.$$

As $-n \leq -\frac{(a+b)^2}{b}$, we obtain

$$\phi(b+1) \leq (a+b)(1-a) - (b+1)(1-a) = -(1-a)^2 \leq 0.$$

At the other endpoint,

$$\phi(a+b-1) = n + \left(-(\delta - a + 1) + \frac{a}{b}(a-2)\right)(a+b-1).$$

As $\delta \geq \frac{an}{a+b}$, we get

$$\phi(a+b-1) \leq n\frac{2a+b-a^2-ab}{a+b} + (a+b-1)(a^2-2a+ab-b)\frac{1}{b}.$$

Now the inequalities $n \geq \frac{(a+b)^2}{b}$ and $2a-a^2+b-ab = -a(a-2)-b(a-1) \leq 0$ imply

$$\phi(a+b-1) \leq \frac{2a+b-a^2-ab}{b}(a+b-a-b+1)$$

$$\phi(a+b-1) \leq \frac{-a(a-2)-b(a-1)}{b} \leq 0.$$

This proves Claim 3. ■

Henceforth we may assume $|Y| \leq b$ and $|X| \leq a-1$, as $|X| < \frac{a}{b}|Y|$.

Let H be the set of odd components C of $G-(X \cup Y)$. Then, $H = H_1 \cup H_2$ where H_1 is the set of the odd components C having $e(C,Y)=1$, and H_2 is the set of those for which $e(C,Y) \geq 3$. Let us set $h = h(X,Y) = |H|$ and $h_i = h_i(X,Y) = |H_i|$, $i = 1,2$. So $h = h_1 + h_2$.

Claim 4. $h_1 \leq \frac{n-|Y|}{\delta+1-|X|}$.

***Proof of Claim* 4.** A component C in H_1 has at least two vertices. Otherwise $C = \{c\}$ and, the degree of the vertex c could be at most $|X|+1$; and, as $|X| \leq a-1$, then $d_G(c) \leq a$; that contradicts $d_G(c) \geq \delta \geq \frac{3a}{2}$. So the component C contains a vertex c' not joined to any vertex in Y, and hence having at least $\delta - |X|$ neighbours in C, therefore $|C| \geq \delta - |X| + 1$ and we obtain $h_1 \leq \frac{n-|Y|}{\delta+1-|X|}$. ■

We continue with the proof of Theorem 3.

Case $|Y| \leq b$ and $|X| = 0$.
To prove that $\tau(X,Y) \leq 0$ we shall show that

$$h(X,Y) + a|Y| - \sum_{y \in Y} d(y) \leq 0.$$

As G has no bridge, and $|X| = 0$ necessarily $h_1 = 0$, $h = h_2$ and $h \leq \frac{1}{3}\sum_{y\in Y} d(y)$. Then

$$\tau(X,Y) \leq -\frac{2}{3}\sum_{y\in Y} d(y) + a|Y| \leq |Y|\left(a - 2\frac{\delta}{3}\right).$$

As $\delta \geq \frac{3a}{2}$, we conclude $\tau(X,Y) \leq 0$.

From now, $|Y| \leq b$ and $|X| \geq 1$.

Case $|Y| \leq b$ and $1 \leq |X| \leq a-1$.

We note that $\sum_{y\in Y} d(y) \geq e(Y,H) + e(X,Y)$, and $e(Y,H) \geq h_1 + 3h_2 = 3h - 2h_1$, so

$$\begin{aligned} 3h &\leq \sum_{y\in Y} d(y) - e(X,Y) + 2h_1; \\ h &\leq \frac{\sum_{y\in Y} d(y) - e(X,Y) + 2h_1}{3}. \end{aligned}$$

By Claim 4, then

$$h \leq \frac{\sum_{y\in Y} d(y) - e(X,Y)}{3} + \frac{2n}{3(\delta + 1 - |X|)}.$$

Recalling $\tau(X,Y) = h - b|X| + a|Y| - \sum_{y\in Y} d(y) + e(X,Y)$, we obtain the following upper bound for $\tau(X,Y)$.

$$\tau(X,Y) \leq -2\frac{\sum_{y\in Y} d(y) - e(X,Y)}{3} + \frac{2n}{3(\delta + 1 - |X|)} - b|X| + a|Y|.$$

From $e(X,Y) \leq |X||Y|$ and $\sum_{y\in Y} d(y) \geq \delta|Y|$ we obtain

$$\tau(X,Y) \leq -2\frac{|Y|\delta}{3} + |X|(\frac{2|Y|}{3} - b) + a|Y| + \frac{2n}{3(\delta + 1 - |X|)}.$$

As $\delta \geq \frac{3a}{2}$, this gives

$$\tau(X,Y) \leq |X|\left(\frac{2|Y|}{3} - b\right) + \frac{2n}{3(\delta+1-|X|)}.$$

Inserting $|Y| \leq b$ yields

$$\tau(X,Y) \leq -\frac{b|X|}{3} + \frac{2n}{3(\delta+1-|X|)}.$$

Then τ is strictly positive if and only if

$$b|X| < \frac{2n}{\delta+1-|X|};$$

in other words if

$$(****) \qquad |X|(\delta+1-|X|) < \frac{2n}{b}.$$

Let us consider the left side of this inequality as a function $f(|X|)$ of $|X|$. We have assumed $1 \leq |X| \leq a-1 < \delta$.

For $|X|$ varying in the interval $[1,\delta]$ the function f has its minimum for $|X| = 1$ and $|X| = \delta$, namely $f(1) = f(\delta) = \delta$. Hence inequality $(****)$ implies that $\delta < \frac{2n}{b}$. As $\delta \geq \frac{an}{a+b}$, we should have $b(a-2) < 2a$. But this does not hold for $b \geq a \geq 4$. So we conclude that τ is nonpositive, and Theorem 3 is proven. ■

***Proof of Theorem* 4.** $\delta \geq \frac{a}{b}(a+b) + \frac{a}{2}$ implies $\delta \geq \frac{a^2}{b} + \frac{3a}{2} \geq \max\left\{\frac{3a}{2}, a + \frac{a^2}{b}\right\}$, and all arguments, including the argument for the case $|Y| \leq b$, can be carried through. ■

***Proof of Theorem* 5.** Claims 1, 2 and 3 still hold with the hypotheses of Theorem 5, so the proof of Theorem 5 begins analogously to that of Theorem 3, and we reach the assumption $0 \leq \frac{b}{a}|X| < |Y| \leq b$. Now, we examine the missing case.

Case $|Y| \leq b$.
We know that $0 \leq |X| \leq a-1$ (as $|X| < \frac{a}{b}|Y|$). Since G has edge-connectivity at least k, each component of $G-(X \cup Y)$ sends at least $k-|X|$ edges to Y, so $h(X,Y) \leq \frac{\sum_{y\in Y} d(y) - e(X,Y)}{k-|X|}$.

It follows that

$$\tau(X,Y) \le \frac{\sum_{y\in Y} d(y) - e(X,Y)}{k-|X|} - b|X| + a|Y| + e(X,Y) - \sum_{y\in Y} d(y),$$

$$\tau(X,Y) \le \frac{k-|X|-1}{k-|X|}(e(X,Y) - \sum_{y\in Y} d(y)) - b|X| + a|Y|.$$

As $0 \le |X| \le a-1$ and $k > a$ we have $\frac{k-|X|-1}{k-|X|} > 0$ and inserting $e(X,Y) - \sum_{y\in Y} d(y) \le |X||Y| - \delta|Y|$ we obtain

$$\tau(X,Y) \le \frac{k-|X|-1}{k-|X|}(|X||Y| - \delta|Y|) - b|X| + a|Y|,$$

$$\tau(X,Y) \le |Y|(a - \frac{k-|X|-1}{k-|X|}\delta) + (\frac{k-|X|-1}{k-|X|}|Y| - b)|X|.$$

The last term is nonpositive, since $|Y| \le b$; so to have $\tau(X,Y) \le 0$ it will suffice that

(i) $$\delta \ge a\frac{k-|X|}{k-|X|-1}.$$

On one hand, as $\delta \ge k$, it is sufficient that $k \ge a\frac{k-|X|}{k-|X|-1}$; as $|X| \le a-1$, we see that the inequality (i) is satisfied if $k \ge a + \sqrt{a}$, because $a\frac{k-|X|}{k-|X|-1} = a(1 + \frac{1}{k-|X|-1}) \le a(1 + \frac{1}{\sqrt{a}}) = a + \sqrt{a} \le k$.
On the other hand, we have

$$\delta \ge \frac{an}{a+b} \ge a(1 + \frac{a}{b}).$$

If $k \ge a + \frac{b}{a}$, and as $|X| \le a-1$, it follows that $k - |X| - 1 \ge \frac{b}{a}$ and $a\frac{k-|X|}{k-|X|-1} = a(1 + \frac{1}{k-|X|-1}) \le a(1 + \frac{a}{b}) \le \delta$ and hence, also in this case, the inequality (i) is satified; and $\tau(X,Y) \le 0$. This proves Theorem 5. ■

References

[1] J. Akiyama and M. Kano, *Factors and factorizations of graphs — a survey*, J. Graph Theory **9** (1985) 1–42.

[2] A. Amahashi, *On factors with all degrees odd*, Graphs and Combin. **1** (1985) 111–114.

[3] Mao-Cheng Cai, *On some factor theorems of graphs*, Discrete Math. **98** (1991) 223–229.

[4] G. Chartrand and O.R. Oellermann, Applied and Algorithmic Graph Theory (McGraw-Hill, Inc., 1993).

[5] Y. Cui and M. Kano, *Some results on odd factors of graphs*, J. Graph Theory **12** (1988) 327–333.

[6] M. Kano, $[a, b]$*-factorization of a graph*, J. Graph Theory **9** (1985) 129–146.

[7] M. Kano and A. Saito, $[a, b]$*-factors of a graph*, Discrete Math. **47** (1983) 113–116.

[8] M. Kano, *A sufficient condition for a graph to have* $[a, b]$*-factors*, Graphs Combin. **6** (1990) 245–251.

[9] M. Kouider and M. Maheo, *Connected* (a, b)*-factors in graphs*, 1998. Research report no. 1151, LRI, (Paris Sud, Centre d'Orsay). Accepted for publication in Combinatorica.

[10] M. Kouider and M. Maheo, 2 *edge-connected* $[2, k]$*-factors in graphs*, JCMCC **35** (2000) 75–89.

[11] M. Kouider and P.D. Vestergaard, *On even* $[2, b]$*-factors in graphs*, Australasian J. Combin. **27** (2003) 139–147.

[12] Y. Li and M. Cai, *A degree condition for a graph to have* $[a, b]$*-factors*, J. Graph Theory **27** (1998) 1–6.

[13] L. Lovász, *Subgraphs with prescribed valencies*, J. Comb. Theory **8** (1970) 391–416.

[14] L. Lovász, *The factorization of graphs* II, Acta Math. Acad. Sci. Hungar. **23** (1972) 223–246.

[15] J. Topp and P.D. Vestergaard, *Odd factors of a graph*, Graphs and Combin. **9** (1993) 371–381.

Received 22 April 2003
Revised 9 October 2003