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ON 3-SIMPLICIAL VERTICES IN PLANAR GRAPHS

Endre Boros

RUTCOR, Rutgers
The State University of New Jersey
640 Bartholomew Road, Piscataway, NJ 08854-8003, USA

e-mail: boros@rutcor.rutgers.edu

Robert E. Jamison

Department of Mathematical Sciences
Clemson University
Clemson, SC 29634-0975, USA

e-mail: rejam@clemson.edu

Renu Laskar

Department of Mathematical Sciences
Clemson University
Clemson, SC 29634-0975, USA

e-mail: rclsk@clemson.edu

and

Henry Martyn Mulder

Econometrisch Instituut, Erasmus Universiteit
P.O. Box 1738, NL-3000 DR Rotterdam, The Netherlands

e-mail: hmmulder@few.eur.nl

Abstract

A vertex v in a graph $G = (V, E)$ is *k-simplicial* if the neighborhood $N(v)$ of v can be vertex-covered by k or fewer complete graphs. The main result of the paper states that a planar graph of order at least four has at least four 3-simplicial vertices of degree at most five. This result is a strengthening of the classical corollary of Euler's Formula that a planar graph of order at least four contains at least four vertices of degree at most five.

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1. Introduction

A simple consequence of the classical Euler Formula for planar graphs is that any planar graph of order at least four has at least four vertices of degree at most five. Grünbaum and Motzkin [4] showed that this result is best possible.

In this paper we strengthen the result on the number of vertices of degree at most five in the following sense. A vertex is 3-*simplicial* if its neighborhood can be vertex-covered by at most three cliques. Clearly any vertex of degree at most three is 3-simplicial. But a vertex of degree four needs at least one edge in its neighborhood to make it 3-simplicial. And a vertex of degree five needs a triangle or two independent edges in its neighborhood to make it 3-simplicial. Our main result is that each planar graph of order at least four has at least four 3-simplicial vertices of degree at most five. We also exhibit an infinite class of planar graphs that contain exactly four 3-simplicial vertices. So our result is in a sense best possible. But in our example the four 3-simplicial vertices are all of degree two. Hence it is natural to ask, (if the order is large enough) whether excluding vertices of degree two forces *more* than four 3-simplicial vertices. The icosahedron is a particularly intriguing example because all twelve of its vertices have degree five and all twelve are 3-simplicial.

In Section 3 we prove our main result. In Section 4 we prove the analog for outerplanar graphs: an outerplanar graph of order at least four has at least four 2-simplicial vertices of degree at most three unless it is the 3-sun or $K_{1,3}$. This result is best possible.

2. Preliminaries

Let $G = (V, E)$ be a graph. The *neighborhood* $N(v)$ of a vertex v of G consists of all vertices adjacent to v. A vertex v in G is k-*simplicial* if the neighborhood $N(v)$ of v can be vertex-covered by k or fewer complete graphs. That is, there are (not necessarily distinct) cliques $C_1, C_2, \ldots, C_k$ in G such that each *vertex* in $N(v)$ lies in at least one C_i. The *simpliciality* $\sigma(v)$ of a vertex v is the smallest k such that v is k-simplicial. Clearly, we have $\sigma(v) \leq d(v)$, where $d(v)$ is the degree of v. By definition v is k-simplicial for all $k \geq \sigma(v)$. A *proper clique cover* of $N(v)$ is a covering of $N(v)$ by mutually disjoint non-empty cliques. So, if the simpliciality of v is m, then any proper clique cover of $N(v)$ consists of m or more cliques. Of course the number of cliques in a proper clique cover of $N(v)$ is at most $d(v)$.

Notice that our notion of a 1-simplicial vertex coincides with the now classical notion of "simplicial vertex" (a vertex of which the neighborhood is a single clique). This concept plays a central role in the algorithmic theory of chordal graphs [2]. The notion of a k-simplicial vertex was introduced by Jamison and Mulder [3], who showed that 3-simplicial vertices always exist in a certain class of graphs representable by sufficiently overlapping subtrees of a binary tree. This is analogous to the representation of chordal graphs by intersecting subtrees of a tree [1].

3. Planar Graphs

We now embark on the proof of the main result. Recall that the *order* of a graph is the number of its vertices and the *size* is the number of its edges.

Theorem 1. *Every planar graph $G = (V, E)$ of order at least four has at least four vertices that are both 3-simplicial and of degree at most 5.*

Proof. For brevity, a vertex z in a planar graph that is both 3-simplicial and has $d(z) \leq 5$ will be called a *good vertex* of that graph.

First we assume that G is 2-connected. Consider a fixed plane drawing of G. We follow the ideas introduced by Lebesgue [5] to extend Euler's formula on planar graphs. Let F be the set of faces of G. For any face f in F, we denote the number of edges on f by $l(f)$. We write $v \in f$ if v is a vertex incident with f. Let v be any vertex of G, and set

$$w(v) = \sum_{f \ni v} \frac{1}{l(f)}.$$

Note that the number of terms in this sum is precisely the degree $d(v)$ of v. Then we have

$$\sum_{v \in V} w(v) = \sum_{v \in V} \sum_{f \ni v} \frac{1}{l(f)} = \sum_{f \in F} \sum_{v \in f} \frac{1}{l(f)} = \sum_{f} \frac{1}{l(f)} \sum_{v \in f} 1 = \sum_{f} 1 = |F|.$$

Now we use Euler's formula $|V|-|E|+|F|=2$ and the basic edge-counting formula $2|E|=\sum_{u\in V} d(u)$. Then we get

$$2=|V|-|E|+|F|=\sum_{v\in V}1-\frac{1}{2}\sum_{v\in V}d(v)+\sum_{v\in V}w(v)$$

$$=\sum_{v\in V}(1-\frac{1}{2}d(v)+w(v)).$$

For v in V, we write $\rho(v)=1-\frac{1}{2}d(v)+w(v)$. Then the above formula reads as follows:

$$\sum_{v\in V}\rho(v)=2. \tag{1}$$

Note that this equation implies that there must be enough vertices v with a positive $\rho(v)$, which sum up to at least 2. Now we prove two claims for $\rho(v)$.

Claim 1. *If $\rho(v)>0$, then v is a good vertex.*

Since G is 2-connected, we have $d(v)\geq 2$. If $d(v)\leq 3$, then v is trivially a good vertex. If $d(v)=4$, then either v is incident with a triangular face, in which case v is good, or every face incident with v has length at least 4. In the latter case we would have $w(v)\leq 4\times\frac{1}{4}=1$, so that $\rho(v)\leq 1-\frac{1}{2}4+1=0$.

If $d(v)=5$, then either v is incident with at least three triangular faces, in which case v is good, or v is incident with three or more faces of length at least 4. In the latter case we would have $w(v)\leq\frac{1}{3}+\frac{1}{3}+3\times\frac{1}{4}=1\frac{5}{12}$, so that $\rho(v)\leq 1-\frac{1}{2}\times 5+1\frac{5}{12}<0$.

Finally, if $d(v)\geq 6$, then we would have $\rho(v)\leq 1-\frac{1}{2}d(v)+\frac{1}{3}d(v)=1-\frac{1}{6}d(v)\leq 0$. This proves Claim 1.

Claim 2. $\rho(v)\leq\frac{7}{12}$, *for any vertex v.*

Since G is 2-connected, we have $d(v)\geq 2$. Note that we have $\rho(v)\leq 1-\frac{1}{6}d(v)$. Hence, if $d(v)\geq 3$, then $\rho(v)\leq\frac{1}{2}$. If $d(v)=2$, then v is incident with two faces. Since G has at least five vertices, at least one of the faces incident with v is not triangular. Therefore we have $w(v)\leq\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$, which settles Claim 2.

Note that equation (1) and Claim 2 imply that there are at least four vertices with $\rho(v) > 0$, so that, by Claim 1, the theorem is proved in the case G is 2-connected.

We prove the general case by induction on the number of vertices of G. In a planar graph with four vertices all vertices are good. So let G have at least five vertices. By the above argument, we may assume that G is not 2-connected. If G is not connected, then, by induction, each component of more than three vertices has at least four good vertices. All smaller components have only good vertices. So G itself must have at least four good vertices.

Now assume that G is connected but has a cut-vertex z. Let $H_1, H_2, \ldots,$ H_m denote the components of $G - z$, and let B_i denote the subgraph induced by $H_i \cup \{z\}$, for $i = 1, 2, \ldots m$. Note that $m \geq 2$. Now we say that B_i is a *small branch* (respectively, *big branch*) if B_i has order at most three (respectively, four or more). By induction, each big branch has at least four good vertices. Of course, in each small branch, all vertices are good. When the branches are glued back together at z, the only neighborhood that changes is that of z. Hence z is the only vertex whose simpliciality and degree can differ in G from what they are in the branches. Therefore each big branch will contribute at least three good vertices to G itself, and each small branch will contribute at least one. Since G has at least five vertices, it follows that G has at least four good vertices, which proves the theorem. ■

In search of an extremal graph satisfying the conditions of Theorem 1, the following Theorem is obtained leading to a class of a planar graphs with exactly four 3-simplicial vertices. See Figure 1 for an instance from the class.

Theorem 2. *For each composite, positive integer m, there is a plane graph G of order $4m + 2$ with all faces bounded by 4-cycles, exactly four vertices of degree 2, and the other $4m - 2$ vertices of degree 4.*

Proof. Let $m = (a + 1)(b + 1)$ be a composite number with $a \geq 1$ and $b \geq 1$. Now form a $(2a + 1) \times (2b + 1)$ rectangular grid which can be viewed as a plane drawing of the Cartesian product of the two paths P_{2a+1} and P_{2b+1}. Let $v_1, v_2, \ldots, v_{2a+1}$ be the vertices along the top row of this grid. As illustrated in the figure, place a vertices $w_1, w_2, \ldots, w_a$ in a vertical row above v_{a+1}, with w_1 at the top and w_a closest to v_{a+1}. Introduce another vertex u between v_{a+1} and w_a. Join the vertices $v_{a+1}, u, w_a, w_{a-1}, \ldots, w_1$

into a path. Now join v_i and v_{2a+2-i} to w_i for $1 \leq i \leq a$, to form "tents" over the top row of the grid.

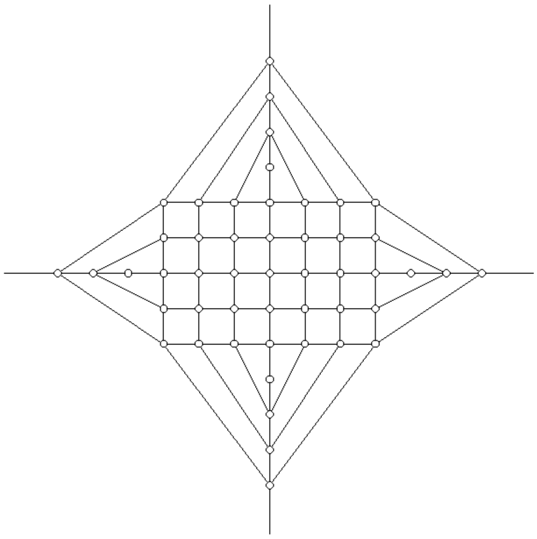

Figure 1. A planar graph with exactly four 3-simplicial vertices. (One of the vertices is at infinity.)

Repeat this for each side of the grid. Now adjoin a vertex at infinity which is joined to the top tent vertex over each side. Figure 1 illustrates this construction for $a = 2$ and $b = 3$. Note that the vertex at infinity is not drawn. The four u-vertices are the vertices of degree 2. All other vertices are of degree 4, and all faces are 4-gons. It remains to count the number of vertices. There are

$(2a+1)(2b+1)$	vertices in the grid,
$2a$	w-vertices extending out from the top and bottom,
$2b$	w-vertices extending out from the sides,
4	u-vertices of degree 2,
1	vertex at infinity.

Thus the number of vertices is $4ab + 2a + 2b + 1 + 2a + 2b + 4 + 1 = 4(ab + a + b + 1) + 2 = 2m + 2$ as claimed. ■

4. Outerplanar Graphs

In this section we study the outerplanar case. It turns out that now we can find at least four 2-simplicial vertices of degree at most three, whenever the order is at least four, unless the graph is one of two exceptional graphs.

The 3-*sun*, or the *triangle of triangles* is the graph on six vertices consisting of a central triangle and three extra vertices each adjacent to a different pair of vertices of the central triangle. The 3-sun is a chordal graph and a maximal outerplanar graph as well. The 3-sun has three vertices of degree two and three vertices of degree four. The vertices of degree four have simpliciality 3. The vertices of degree two have simpliciality 1, so that they are 2-simplicial as well. They are mutually non-adjacent. The star $K_{1,3}$ also has three mutually non-adjacent 2-simplicial vertices.

Theorem 3. *Let G be an outerplanar graph of order at least four. Then G contains at least four 2-simplicial vertices of degree at most three unless G is the 3-sun or $K_{1,3}$.*

Proof. Assume the contrary, and let G be a counterexample of minimum order, and amongst the counterexamples of minimum order one of maximum size. Now we call a vertex *good* if it is 2-simplicial and has degree at most three. Let G be embedded in the plane with an outerplanar embedding.

Claim. *G is 2-connected.*

If G is disconnected, then we can join two components by an edge. Thus we obtain an outerplanar graph which still has less then four good vertices but has more edges than G. Since this contradicts the maximality of the size of G, it follows that G is connected.

Suppose G has a cutvertex z. Let $H_1, H_2, \ldots, H_m$ denote the components of $G - z$, and let B_i denote the subgraph induced by $H_i \cup \{z\}$, for $i = 1, 2, \ldots m$. Note that $m \geq 2$. Again say B_i is a *small branch* (respectively, *big branch*) if B_i has order at most three (respectively, four or more). By the choice of G as a counterexample of minimal order, each big branch has at least four good vertices — unless it happens to be a 3-sun or $K_{1,3}$. As before all vertices are good in each small branch. When the branches are glued back together at z, the only neighborhood that changes is that of z. Hence z is the only vertex whose simpliciality and degree can differ in G from what they are in the branches. Therefore each big branch (except a 3-sun or $K_{1,3}$) will contribute at least three good vertices to G itself, and each small branch will contribute at least one. Thus, since G is a counterexample, it has at most three branches and no big branch (except possibly a 3-sun or $K_{1,3}$). It follows that G must consist of either (1) two branches of order two, or (2) three branches of order two, (3) one branch of order three and one of order two, or (4) a 3-sun or $K_{1,3}$ branch and one branch of order two. The $K_{1,2}$ which arises in case (1) has only three vertices, and the $K_{1,3}$, which arises from case (2), is excluded by the theorem. Since the five graphs that arise from cases (3) and (4) all have four good vertices, this settles the proof of the Claim.

Let C be the cycle that is the boundary of the outerface. Then C is of length at least four. If $G = C$, then all vertices of G are of degree two, whence all are good. This is impossible, so C must have chords. Take any chord xy. Then $\{x, y\}$ is a cutset in G that cuts G into two components H_1 and H_2. Let G_i be the subgraph of G induced by xy and H_i, for $i = 1, 2$. If both G_1 and G_2 are of order at least four, then they both contain at least two good vertices distinct from x and y (note that in the 3-sun the three good vertices are mutually non-adjacent.) So G is not a counterexample after all. Hence at least one of G_1 and G_2 is of order three, say it consist of x, y and a third vertex z. Then z is necessarily a vertex of degree two in G with x and y as its neighbors. We call such a vertex z of degree two on the outerface with its neighbors x and y joined by a short chord a *cap* on the chord xy.

Thus we have shown that every chord of C implies the existence of a cap on that chord. The only way that a chord can have two caps is that G is a 4-cycle with exactly one chord. But this is not a counterexample. So each chord gives rise to exactly one cap. Clearly caps are good vertices. So G has at most three caps, whence has at most three chords. Let D be the cycle in G obtained after removing all the caps of G. If D is of length at least four, then there is an edge uv of D on the outerface. If u is of degree three then one of its neighbors is a cap, say w. Then both u and w are good vertices. If u is a vertex of degree two, then let w be the neighbor of u distinct from v. Then again u and w are two good vertices. Similarly, we find two good vertices amongst v and its neighbors distinct from u. Hence G has four good vertices. To avoid this D must be a triangle. If D has no caps, then G has only the three vertices of D. If D has one cap, then all four vertices of G are good. If D has two caps, then G is a 5-cycle with two chords and again has four good vertices. Finally, if every edge of D is capped, then D is the 3-sun, which was excluded in theorem as an exceptional case. This concludes the proof of the Theorem. ■

Consider $K_2 \square P_n$, the cartesian product of an edge with the path on n vertices. This is a triangle-free outerplanar graph with exactly four vertices of degree two, $2n-4$ vertices of degree three, and all (inner) faces 4-cycles. This yields an infinite family of graphs where Theorem 3 is best possible.

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