

## 1-RESTRICTED OPTIMAL RUBBLING ON GRAPHS

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### Abstract

Let  $G$  be a graph with vertex set  $V$  and a distribution of pebbles on the vertices of  $V$ . A pebbling move consists of removing two pebbles from a vertex and placing one pebble on a neighboring vertex, and a rubbling move consists of removing a pebble from each of two neighbors of a vertex  $v$  and placing a pebble on  $v$ . We seek an initial placement of a minimum total number of pebbles on the vertices in  $V$ , so that no vertex receives more than one pebble and for any given vertex  $v \in V$ , it is possible, by a sequence of pebbling and rubbling moves, to move at least one pebble to  $v$ . This minimum number of pebbles is the 1-restricted optimal rubbling number. We determine the 1-restricted optimal rubbling numbers for Cartesian products. We also present bounds on the 1-restricted optimal rubbling number.

**Keywords:** graph pebbling, graph rubbing, optimal rubbing,  $t$ -restricted optimal pebbling.

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