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γ-CYCLES IN ARC-COLORED DIGRAPHS

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Abstract

We call a digraph D an m-colored digraph if the arcs of D are colored with m colors. A directed path (or a directed cycle) is called monochromatic if all of its arcs are colored alike. A subdigraph H in D is called rainbow if all of its arcs have different colors. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of D if it satisfies the two following conditions:

(i) for every pair of different vertices $u, v \in N$ there is no monochromatic path in D between them, and

(ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic path in D.

A γ-cycle in D is a sequence of different vertices $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ such that for every $i \in \{0, 1, \ldots, n\}$:

(i) there is a $u_i u_{i+1}$-monochromatic path, and

(ii) there is no $u_{i+1} u_i$-monochromatic path.

The addition over the indices of the vertices of γ is taken modulo $(n+1)$. If D is an m-colored digraph, then the closure of D, denoted by $\mathfrak{C}(D)$, is the m-colored multidigraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) =$

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$A(D) \cup \{(u, v)$ with color $i \mid$ there exists a uv-monochromatic path colored i contained in $D\}$.

In this work, we prove the following result. Let D be a finite m-colored digraph which satisfies that there is a partition $C = C_1 \cup C_2$ of the set of colors of D such that:

(1) $D[\widehat{C}_i]$ (the subdigraph spanned by the arcs with colors in C_i) contains no γ-cycles for $i \in \{1, 2\}$;

(2) If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of C_1 and C_2, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;

(3) If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of C_1 and C_2, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: (u, w), (w, u), (x_0, u), (u, x_0), (x_0, w), (z, u), (z, x_0).

Then D has a kernel by monochromatic paths.

This theorem can be applied to all those digraphs that contain no γ-cycles. Generalizations of many previous results are obtained as a direct consequence of this theorem.

Keywords: digraph, kernel, kernel by monochromatic paths, γ-cycle.

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