Discussiones Mathematicae
Graph Theory xx (xxxx) 1–23
https://doi.org/10.7151/dmgt.2610

DERANGED PERFECT MATCHINGS ON COMPLETE GRAPH AND BALANCED COMPLETE R-PARTITE GRAPH

Boqing Deng

Department of Mathematics and Computer Science
Wesleyan University
Middletown CT 06457
e-mail: tdeng@wesleyan.edu

Abstract

We proved that for any finite collection of sparse subgraphs $(D_m)_{m=1}^{\ell}$ of the complete graph K_{2n}, and a uniformly chosen perfect matching R in K_{2n}, the random vector $(|E(R \cap D_m)|)_{m=1}^{\ell}$ jointly converges to a vector of independent Poisson random variables with mean $|E(D_m)|/(2n)$. We also showed a similar result when K_{2n} is replaced by the balanced complete r-partite graph $K_{r\times 2n/r}$ for fixed r and determined the asymptotic joint distribution. The proofs rely on elementary tools of the Principle of Inclusion-Exclusion and generating functions. These results extend recent works of Johnston, Kayll and Palmer, Spiro and Surya, and Granet and Joos from the univariate to the multivariate setting.

Keywords: perfect matching, Poisson distribution, principle of Inclusion-Exclusion, generating function.

2020 Mathematics Subject Classification: 05C30,05C70.

1. Introduction

For a sequence simple graphs G_n where $\lim_{n\to\infty} V(G_n) = \infty$, let $\mathrm{pm}(G_n)$ denote the number of perfect matchings of G_n. If M_n is an arbitrary perfect matching of G_n, the problem of determining the asymptotic ratio

$$\frac{\mathrm{pm}(G_n - M_n)}{\mathrm{pm}(G_n)}$$

when $n \to \infty$ is of great combinatorial interest. For example, when $G_n = K_{n,n}$ is the balanced complete bipartite graphs, the limiting ratio equals the number of permutations with no fixed point and converges to the limit e^{-1}. A permutation without a fixed point is called a derangement, and hence a perfect matching of $G_n - M_n$ is called a deranged perfect matching. Counting the number of derangement was proposed by Montmort [10] in 1708, and this problem can be solved using the principle of Inclusion-Exclusion (See [12] for more details). When G_n is the complete graph K_{2n}, the limiting ratio equals $e^{-1/2}$. Brawner [1] conjectured this asymptotic ratio, and it was proved by Margolius [9]. For the rest of this paper, we will omit the index n and write G_n, M_n as G, M.

Let $r = r(n)$ be a integer valued function and $r|2n$, and let $K_{r\times 2n/r}$ denote the balanced complete r-partite graph, where $V(K_{r\times 2n/r}) = V_1 \uplus \cdots \uplus V_r$ is a partition of the vertex set, $|V_i| = 2n/r$ for each $i \in [r]$, and there is an edge between u and v if and only if u, v lies in different parts. As a generalization of both $K_{n,n}$ and $K_{2n/r}$, Johnston, Kayll and Palmer [5] conjectured that when $G = K_{r\times 2n/r}$, the limit ratio converges to $e^{-r/(2r-2)}$. They solved the conjecture when r is linearly proportional to n, $r = \Omega(n^\delta)$ for some $\delta > 0$, and a simplified variant when r is constant.

Spiro and Surya [11] fully solved the previous conjecture and proved that when R is a uniformly random perfect matching of $K_{r\times 2n/r}$, the number of edges in $R \cap M$ converges to the Poisson distribution with mean $r/(2r-2)$.

Granet and Joos [3] generalized G to regular robust expander graphs and M to any matchings or spanning regular subgraphs. Suppose R is a uniformly random perfect matching of G, and D is a matching or a regular spanning subgraph of G. In that case, they showed the number of edges R intersecting with D converges to the Poisson distribution with mean $|E(D)|/\deg(G)$.

In this paper, we generalize the result of [5, 11, 3] by extending the distribution into a multivariate joint Poisson distribution on the ℓ dimensional integer lattice for some fixed ℓ. We generalized the graph D to a collection of subgraphs $(D_m)_{m=1}^{\ell}$, and we proved that the asymptotic joint distribution of $(|E(R\cap D_m)|)_{m=1}^{\ell}$ is a multivariate Poisson distribution. Specificaly, if $(D_m)_{m=1}^{\ell}$ are disjoint, then the asymptotic distribution of $(|E(R \cap D_m)|)_{m=1}^{\ell}$ is independent. This phenomenon provides macroscopic evidence supporting the heuristic proposed by Granet and Joos, which we will outline in Section 2. We shall define the distance of total variation of two random vectors, taking values in the ℓ dimensional integer lattice as follows.

Definition 1.1. Let $\mathbf{X}, \mathbf{Y}$ be two random vectors taking values in $\mathbb{N}^\ell$. We denote the distance of total variation of $\mathbf{X}$ and $\mathbf{Y}$ by

$$d_{TV}(\mathbf{X}, \mathbf{Y}) = \sum_{\mathbf{k}\in\mathbb{N}^\ell} |\mathbb{P}(\mathbf{X} = \mathbf{k}) - \mathbb{P}(\mathbf{Y} = \mathbf{k})| .$$

Section 3 treats the complete graph K_{2n} as the parent graph and also examines $K_{2n} - N$, where N is a sparse subgraph. A specific example of $K_{2n} - N$ is when $G = K_{r\times 2n/r}$ where $r|2n$ and r is linearly proportional to n. In this case, $\deg(G) = n - n/r$ where n/r is a constant independent of n. The next theorem states the main result in Section 3.

Theorem 1.2. *Let ℓ, C be fixed constants independent of n, N be a subgraph of K_{2n} and $(D_m)_{m=1}^{\ell}$ be a collection of disjoint subgraphs of $K_{2n} - N$ such that $\Delta(N), \Delta(D_m) \le C$ for all m and some constant C. Let R be a uniformly random perfect matching of $K_{2n} - N$. Let $\mathbf{X} = (X_m)_{m=1}^{\ell}, \mathbf{Y} = (Y_m)_{m=1}^{\ell}$ be two random vectors such that $X_m = |E(R \cap D_m)|, Y_m$ indepdendently follows* $\mathrm{Po}(|E(D_m)|/2n)$. *Then,*

$$\lim_{n\to\infty} d_{TV}(\mathbf{X}, \mathbf{Y}) = 0.$$

As mentioned before, Johnston, Kayll and Palmer [5] proposed and solved a simplified varient of determining the asymptotic ratio $\mathrm{pm}(K_{r\times 2n/r} - M)/\mathrm{pm}(K_{r\times 2n/r})$. They defined a balanced perfect matching of $K_{r\times 2n/r}$ as a perfect matching such that the number of edges between V_i and V_j is the same for all $i \ne j \in [r]$. Let $\mathrm{bpm}(\cdot)$ denote the function that counts the number of balanced perfect matchings. If r is constant on n and M is a balanced perfect matching of $K_{r\times 2n/r}$, Johnston, Kayll and Palmer [5] proved the following result

$$\lim_{n\to\infty} \frac{\mathrm{bpm}(K_{r\times 2n/r} - M)}{\mathrm{bpm}(K_{r\times 2n/r})} = e^{-r/(2r-2)}. \tag{1}$$

In Section 4, we generalize (1) and derive an analogous theorem to Theorem 1.2. To ensure the existence of a balanced perfect matching in a balanced complete r-partite graph, we must have $r(r-1)|2n$. Therefore, for convenience we replace $2n$ by $r(r-1)n$. The next Theorem is the main result of Section 4.

Theorem 1.3. *Let ℓ, C, r be fixed, R be a uniformly random **balanced** perfect matching of $K_{r\times(r-1)n}$, $(D_m)_{m=1}^{\ell}$ be a collection of disjoint subgraphs of $K_{r\times(r-1)n}$ such that $\Delta(D_m) \le C$ for all m and for some constant C. Let $\mathbf{X} = (X_m)_{m=1}^{\ell}, \mathbf{Y} = (Y_m)_{m=1}^{\ell}$ be two random vectors such that $X_m = |E(R \cap D_m)|, Y_m$ independently follows* $\mathrm{Po}(E(D_m)/(r-1)^2 n)$. *Then,*

$$\lim_{n\to\infty} d_{TV}(\mathbf{X}, \mathbf{Y}) = 0.$$

1.1. Conventions and notations

Throughout the discussion in the paper, we assume that the dimensions ℓ, λ and the constants r, C are fixed, independent of n. Objects such as the graphs G, D_m and the random vectors $\mathbf{X}, \mathbf{Y}$, are defined with respect to n. We say a quantity is fixed or constant if it is independent of n and unless stated otherwise, a quantity depends on n.

We denote the natural number $\mathbb{N} = \{0, 1, 2, \ldots, \}$ and for each $k \geq 1, k \in \mathbb{N}$, we denote $[k] = \{1, 2, \ldots, k-1, k\}$. For $a, b \in \mathbb{N}$, we define the falling factorial $a_{(b)} = a(a-1)\cdots a(a-b+1)$. We will use the standard Landau notations $o(\cdot), O(\cdot)$ etc. We assume these asymptotic notations are defined with respect to n when $n \to \infty$.

We define a generating function as an analytic power series from $\mathbb{C}^\ell$ to $\mathbb{C}$. Specifically, we define a generating function $G : \mathbb{C}^\ell \to \mathbb{C}$ as a power series

$$G(\mathbf{s}) = \sum_{\mathbf{k} \in \mathbb{N}^\ell} \alpha_{\mathbf{x}} \mathbf{s}^{\mathbf{k}}$$

that converges absolutely for all $\mathbf{k} \in \mathbb{C}^\ell$.

For simplicity, in the rest of the paper we will use the multi-index notation to simplify the presentation. The object to consider are $\ell \times \lambda$ dimensional indices $\mathbf{x} = (x_{m,k})_{m \in [\ell], k \in [\lambda]} \in \mathbb{N}^{\ell \times \lambda}$ and ℓ dimensional variables $\mathbf{s} = (s_m)_{m=1}^{\ell} \in \mathbb{N}^\ell$. For any vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d, \gamma \in \mathbb{N}$ of dimension d, we define

$$\begin{aligned}
\boldsymbol{\alpha} + \boldsymbol{\beta} &= (\alpha_1 + \beta_1, \ldots, \alpha_d + \beta_d) \quad \text{and } \boldsymbol{\alpha} - \boldsymbol{\beta} \text{ similarly} \\
\gamma \boldsymbol{\alpha} &= (\gamma \alpha_1, \ldots, \gamma \alpha_d) \\
\boldsymbol{\alpha}^{\boldsymbol{\beta}} &= \alpha_1^{\beta_1} \cdots \alpha_d^{\beta_d} \\
\boldsymbol{\alpha}! &= \alpha_1! \cdots \alpha_d! \\
\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} &= \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} = \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!(\boldsymbol{\alpha} - \boldsymbol{\beta})!} \\
\binom{|\boldsymbol{\alpha}|_1}{\boldsymbol{\alpha}} &= \binom{|\boldsymbol{\alpha}|_1}{\alpha_1, \ldots, \alpha_d} = \frac{|\boldsymbol{\alpha}|_1!}{\alpha_1! \cdots \alpha_d!} = \frac{|\boldsymbol{\alpha}|_1!}{\boldsymbol{\alpha}!} \quad \text{this is the multinomial coefficient} \\
\boldsymbol{\alpha}_{(\boldsymbol{\beta})} &= \alpha_{1(\beta_1)} \cdots \alpha_{d(\beta_d)}.
\end{aligned}$$

We denote $\mathbf{1} = (1, 1, \ldots, 1)$ to be the all-one-vector where the dimension will be clear from the context. We also define $O(\gamma)\mathbf{1}$ to be the set of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)$ where $\beta_1, \ldots, \beta_d = O(\gamma)$. For $\mathbf{x} \in \mathbb{N}^{\ell \times \lambda}$, we define

$$\begin{aligned}
|\mathbf{x}|_1 &= \sum_{m \in [\ell], k \in \lambda} x_{m,k} \in \mathbb{N} \\
\mathbf{x}_m &= (x_{m,k})_{k=1}^{\lambda} \in \mathbb{N}^\lambda \\
|\mathbf{x}_m|_1 &= \sum_{k=1}^{\lambda} x_{m,k} \in \mathbb{N}.
\end{aligned}$$

We define the function $\psi_1 : \mathbb{N}^{\ell \times \lambda} \to \mathbb{N}^\ell$ by $\psi_1(\mathbf{x}) = (|\mathbf{x}_{\boldsymbol{m}}|_1)_{m=1}^{\ell}$.

1.2. Organization

The remainder of this paper will be organized as follows. In Section 2, we will give a heuristic reasoning of Theorem 1.2 and Theorem 1.3, a sketch of the proof, and a list of tools used in the proof. In Section 3, we will prove Theorem 1.2. In Section 4, we will prove Theorem 1.3. In Section 5, we will generalize Theorem 1.2 and Theorem 1.3 by dropping the restriction that $(D_m)_{m=1}^{\ell}$ are disjoint. In Section 6, we will suggest potential directions for future works on this problem.

2. Sketch and Tools of the Proof

2.1. Sketch of the proof

For simplicity, if $G = K_{2n}$ is a complete graph and R is a uniformly random perfect matching of G, then for any edge $e \in E(G)$, we have $\mathbb{P}(e \in R) = 1/(2n-1) = (1+o(1))/2n$. If $G = K_{r\times(r-1)n}$, then $\mathbb{P}(e \in R) = 1/(r-1)^2 n$. If we have k edges, $e_1, \dots, e_k$ in K_{2n} or $K_{r\times(r-1)n}$, R is chosen uniformly from K_{2n} or $K_{r\times(r-1)n}$, and n is much larger than k, for all $i \in [k]$, the probability that R contains all of e_i for when e_i are non incident equals

$$\mathbb{P}(\forall i \in [k],\ e_i \in R) = (1+o(1)) \prod_{i=1}^{k} \mathbb{P}(e_i \in R).$$

The event that R contains each e_i is roughly independent. Suppose there exists a pair e_i, e_j of incident edges, then we have $\mathbb{P}(\forall i \in [k],\ e_i \in R) = 0$. However, if we uniformly select k edges from a sparse subgraph of K_{2n} or $K_{r\times(r-1)n}$, it is rare that there exists an incidient pair.

Granet and Joos [3] suggests a heuristic that if G is a d-regular graph and D is a regular sparse subgraph of G, then the probability that each edge in D intersects R is roughly independent and identical, and $|E(R \cap D)|$ will approximately follow the binomial distribution $\mathrm{Binom}(|E(D)|, 1/d)$, which will converge to $\mathrm{Po}(|E(D)|/d)$ as $n \to \infty$. They proved this convergence given that G is a robust expander graph. Particularly, if $G = K_{2n}$, then $d = 2n-1 = (1+o(1))2n$ and if $G = K_{r\times(r-1)n}$, then $d = (r-1)^2 n$. If we have a collection of disjoint sparse graphs D_m, we expect that the joint distribution $|E(R \cap D_m)|$ should converge to the independent Poisson distribution.

Our proof strategy is as follows. We introduce the probability generating function $G(s_1, \dots, s_\ell)$ for which the coefficient for $s_1^{r_1} \cdots s_\ell^{r_\ell}$ is the probability that R intersect D_m in exactly r_m edges for all $m \in [\ell]$. We generalized the method of Johnston, Kayll, and Palmer [5], by using the principle of Inclusion-Exclusion to estimate the coefficient of this generating function. We will show that the coefficient in this probability generating functions gets close to the coefficient in the probability generating function of independent multivariate Poisson distribution as n gets large, thus showing the two random vectors converge in distance of total variation.

If $G = K_{2n}$, since X_m is roughly independent, we expect the conditional distribution $X_m|X_1 = 0$ to roughly equal the distribution of X_m. Therefore, given that R does not intersect D_1, the joint distribution $(|E(R\cap D_m)|)_{m=2}^{\ell}$ should still be independently Poisson. Specifically, the distribution of $(|E(R^*\cap D_m)|)_{m=2}^{\ell}$ when R^* is chosen uniformly from $K_{2n}-D_1$ should be the same as the distribution of $(|E(R\cap D_m)|)_{m=2}^{\ell}$ when R chosen uniformly in K_{2n}, conditioned on the event that R does not intersect D_1. We will use the notation N to denote this specific sparse subgraph D_1, and we hence generalize the base graph G from K_{2n} to $K_{2n} - N$. When r is constant and the parent graph is $G = K_{r\times(r-1)n}$, we will use a more sophisticated counting argument to determine the probability generating function, but the core idea and the structure of the proof are similar.

Finally, when D_m is not disjoint, the joint distribution X_m is no longer independent. We can break D_m into disjoint parts, where each part's intersection to R follows the Poisson distribution. Since a sum of independent random variables, each with a Poisson distribution still has a Poisson distribution, we can determine the limiting joint distribution as a not necessarily independent Poisson joint distribution. We will describe the process of decomposing graphs in Section 5.

2.2. Tools for the proof

The two main theorems we use are the Principle of Inclusion-Exclusion and Tannery's Theorem. Both are also used in the work of Johnston, Kayll, and Palmer [5]. We present the Principle of Inclusion-Exclusion in the form of generating functions. Interested readers could refer to [13] for reference. The following results presents the principle of Inclusion-Exclusion in the fullest generality needed for the proof.

The next theorem is a version of the Principle of Inclusion-Exclusion. For our application, $\mathcal{U}$ will be the set of perfect matchings of the parent graph G, $I_{m,k}$ will be a partition of the edges of each graph in $(D_m)_{m=1}^{\ell}$ in k parts, and the m,k th coordinate of $\mathbf{P}$ will be the set of edges of intersection between the perfect matching and $I_{m,k}$.

To better articulate the next theorem and arguments in the rest of the paper, we develop some notations here. For a class of sets $\mathbf{S} = (S_{m,k})_{m\in[\ell],k\in[\lambda]}$, we define function φ_1 such that

$$\varphi_1(\mathbf{S}) = \left(\sum_k |S_{m,k}|\right)_{m=1}^{\ell} \in \mathbb{N}^{\ell}.$$

For two class of sets $\mathbf{S} = (S_{m,k})_{m\in[\ell],k\in[\lambda]}$, $\mathbf{T} = (T_{m,k})_{m\in[\ell],k\in[\lambda]}$, we say $\mathbf{S} \sqsubset \mathbf{T}$ if $S_{m,k} \subset T_{m,k}$ for each $m \in [\ell], k \in [\lambda]$. Let the notation $\mathcal{P}(\cdot)$ denote the power set.

Theorem 2.1 (Principle of Inclusion-Exclusion). *Let $\mathcal{U}$ be a finite universal set, $\mathbf{I} = (I_{m,k})_{m\in[\ell],k\in[\lambda]}$ be a finite collection of finite index sets. Let $\mathbf{P} : \mathcal{U} \to \prod_{m,k} \mathcal{P}(I_{m,k})$ be a function. For each $\mathbf{S} \sqsubset \mathbf{I}$, we define the number*

$$N(\sqsupset \mathbf{S}) = |\{\omega \in \mathcal{U} : \mathbf{S} \sqsubset \mathbf{P}(\omega)\}|.$$

Then, for each $\mathbf{r} \in \mathbb{N}^\ell$, the coefficient of $\mathbf{s}^\mathbf{r}$ in the generating function

$$G(\mathbf{s}) = \sum_{\mathbf{S}} N(\sqsupset \mathbf{S})(\mathbf{s} - \mathbf{1})^{\varphi_1(\mathbf{S})}$$

is the number of ω such that $\varphi_1(\mathbf{P}(\omega)) = \mathbf{r}$.

Proof. We define the function

$$\varphi_2(\mathbf{S}) = (|S_{m,k}|)_{m\in[\ell],k\in[\lambda]} \in \mathbb{N}^{\ell\times\lambda}.$$

We know

$$\begin{aligned} G(\mathbf{s}+\mathbf{1}) &= \sum_{\mathbf{S}} N(\sqsupset \mathbf{S})\mathbf{s}^{\varphi_1(\mathbf{S})} = \sum_{\mathbf{S}} \sum_{\omega:\mathbf{S}\sqsubset\mathbf{P}(\omega)} \mathbf{s}^{\varphi_1(\mathbf{S})} = \sum_{\omega} \sum_{\mathbf{S}:\mathbf{S}\sqsubset\mathbf{P}(\omega)} \mathbf{s}^{\varphi_1(\mathbf{S})} \\ &= \sum_{\omega} \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \binom{\varphi_2(\mathbf{P}(\omega))}{\mathbf{x}} \mathbf{s}^{\psi_1(\mathbf{x})} = \sum_{\omega} (\mathbf{s}+\mathbf{1})^{\varphi_1(\mathbf{P}(\omega))}. \end{aligned}$$

The last equality is due to a version of the Binomial Theorem for vectors in multiple dimensions. It can be proved by applying the binomial theorem separately for each index m, k and multipling the result all together.

The coefficient of $(\mathbf{s}+\mathbf{1})^\mathbf{r}$ is the number of ω such that $\varphi_1(\mathbf{P}(\omega)) = \mathbf{r}$. Therefore, we obtain the original generating function by substituting $\mathbf{s}+\mathbf{1}$ with $\mathbf{s}$. ■

Theorem 2.1 has a straightforward extention in terms of probability-generating function. We shall state this as the next corollary and use this form of the Principle of Inclusion-Exclusion in Section 3 and Section 4. We shall formally define the probability generating function.

Definition 2.2. Let $\mathbf{X}$ be a random vector on $\mathbb{N}^\ell$. A probability generating function of $\mathbf{X}$ is a generating function defined as

$$G_\mathbf{X}(\mathbf{s}) = \sum_{\mathbf{k}} \mathbb{P}(\mathbf{X} = \mathbf{k})\mathbf{s}^\mathbf{k}.$$

Corollary 2.3. *Let Ω be a finite sample space where each sample is assigned a uniform probability measure. Let $\mathbf{I} = (I_{m,k})_{m\in[\ell],k\in[\lambda]}$ be a finite collection of finite index sets. Let $\mathbf{P} : \Omega \to \prod_{m,k} \mathcal{P}(I_{m,k})$ be a function. For each $\mathbf{S} \sqsubset \mathbf{I}$, define the event $A_{\mathbf{S}} = \{\omega \in \Omega : \mathbf{S} \sqsubset \mathbf{P}(\omega)\}$. Let $\mathbf{X}$ be the ℓ dimensional random vector such that $\mathbf{X}(\omega) = \varphi_1(\mathbf{P}(\omega))$. Then, the probability generating function of $\mathbf{X}$ is given by*

$$G_{\mathbf{X}}(\mathbf{s}) = \sum_{\mathbf{S}} \mathbb{P}(A_{\mathbf{S}})(\mathbf{s} - \mathbf{1})^{\varphi_1(\mathbf{S})}.$$

For our application, Ω will be the sample space of all perfect matchings of G, and the event $A_{\mathbf{S}}$ is the set of perfect matchings ω of G such that $\omega \cap I_{m,k}$ contains the m,k th coordinate of $\mathbf{S}$ for each m,k.

The next theorem is a special case of Lebesgue dominated convergence theorem and provides a sufficient condition for interchanging limits and infinite summation. Interested readers could refer to [8] for reference and proof using only elementary mathematical analysis.

Let I be a countably infinite set and $\{s_i\}_{i\in I}$ be a sequence of real numbers. If there exists a bijection $g : \mathbb{N} \to I$ such that $\sum_n s_{g(n)}$ converges absolutely, then we can define the sum $\sum_{i\in I} s_i$ as $\sum_n s_{g(n)}$. In this case, the choice of g will not alter the sum.

Theorem 2.4 (Tannery's Theorem)**.** *Let I be a countably infinite index set, $\{f_i(n)\}_{i\in I}$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{R}$, $\{\alpha_i\}_{i\in I}$, $\{M_i\}_{i\in I}$ be two sequences of real numbers. If for each $i \in I, n \in \mathbb{N}, |f_i(n)| \le M_i$, for each $i \in I$, $\lim_{n\to\infty} f_i(n) = \alpha_i$, and if $\sum_{i\in I} M_i < \infty$, then $\sum_{i\in I} f_i(n)$ is defined for each n and we have*

$$\lim_{n\to\infty} \sum_{i\in I} f_i(n) = \sum_{i\in I} \alpha_i.$$

For the remainder of this paper, we always choose the set I to be $\mathbb{N}^{\ell\times\lambda}$, the finite dimensional lattice on natural numbers. The next lemma gives an upper bound of the total variation distance of two random vectors in terms of their probability-generating function.

Lemma 2.5. *Let $\mathbf{X}, \mathbf{Y}$ be two random vectors on $\mathbb{N}^\ell$. Suppose there exists*

$$\{\alpha_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}}, \{\beta_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \subset \mathbb{R}$$

such that the probability generating function of $\mathbf{X}, \mathbf{Y}$ satisfies

$$G_{\mathbf{X}}(\mathbf{s}) = \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \alpha_{\mathbf{x}}(\mathbf{s} - \mathbf{1})^{\psi_1(\mathbf{x})},$$

$$G_{\mathbf{Y}}(\mathbf{s}) = \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \beta_{\mathbf{x}}(\mathbf{s} - \mathbf{1})^{\psi_1(\mathbf{x})}.$$

Then their total variation distance satisfies

$$d_{TV}(\mathbf{X},\mathbf{Y}) \le \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| 2^{|\mathbf{x}|_1}.$$

Proof. We can write

$$\begin{aligned} G_{\mathbf{X}}(\mathbf{s}) &= \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \alpha_{\mathbf{x}} (\mathbf{s}-\mathbf{1})^{\psi_1(\mathbf{x})} \\ &= \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \alpha_{\mathbf{x}} \sum_{\mathbf{k}\in\mathbb{N}^{\ell\times\lambda}} \binom{\mathbf{x}}{\mathbf{k}} (-\mathbf{1})^{\psi_1(\mathbf{x}-\mathbf{k})} \mathbf{s}^{\psi_1(\mathbf{k})} \\ &= \sum_{\mathbf{k}\in\mathbb{N}^{\ell\times\lambda}} \left(\sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \alpha_{\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} (-\mathbf{1})^{\psi_1(\mathbf{x}-\mathbf{k})} \right) \mathbf{s}^{\psi_1(\mathbf{k})}. \end{aligned}$$

Where the second equality is due to the multi-dimensional Binomial Theorem. Similarly, we can write

$$G_{\mathbf{Y}}(\mathbf{s}) = \sum_{\mathbf{k}\in\mathbb{N}^{\ell\times\lambda}} \left(\sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \beta_{\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} (-\mathbf{1})^{\psi_1(\mathbf{x}-\mathbf{k})} \right) \mathbf{s}^{\psi_1(\mathbf{k})}.$$

Hence, by Triangle Inequality

$$\begin{aligned} d_{TV}(\mathbf{X},\mathbf{Y}) &= \sum_{\mathbf{k}\in\mathbb{N}^{\ell\times\lambda}} \left| \left(\sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \alpha_{\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} (-\mathbf{1})^{\psi_1(\mathbf{x}-\mathbf{k})} \right) - \left(\sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} \beta_{\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}} (-\mathbf{1})^{\psi_1(\mathbf{x}-\mathbf{k})} \right) \right| \\ &\le \sum_{\mathbf{k}\in\mathbb{N}^{\ell\times\lambda}} \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| \binom{\mathbf{x}}{\mathbf{k}} \\ &= \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\lambda}} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| 2^{|\mathbf{x}|_1} \end{aligned}$$

where the last equality is due to the Binomial Theorem. ∎

The next technical lemma gives a bound of a difference of two products when the difference of each coordinates are small.

Lemma 2.6. *Let $\{a_i\}_{i\in[d]}, \{b_i\}_{i\in[d]}, K$ be real numbers such that $0 \le b_i \le a_i \le K$ for each i. Then*

$$\prod_1^d a_i - \prod_1^d (a_i - b_i) \le K^{d-1} \sum_i b_i.$$

Proof. We have

$$\prod_{1}^{d} a_i - \prod_{1}^{d}(a_i - b_i) = \sum_{j=1}^{d}\left(\prod_{i<j}(a_i - b_i)\right)\left(\prod_{i\geq j} a_i\right) - \left(\prod_{i\leq j}(a_i - b_i)\right)\left(\prod_{i>j} a_i\right)$$
$$= \sum_{j=1}^{d} b_i \left(\prod_{i<j}(a_i - b_i)\right)\left(\prod_{i>j} a_i\right) \leq \sum_{i} b_i K^{d-1}.$$

The second inequality is due to the fact that both a_i and $a_i - b_i$ are bounded by K, and so the lemma follows. ■

3. Case of K_{2n}

In this section, we assume $\lambda = 1$, ℓ to be finite and independent of n. Therefore, we denote $\mathbf{x} = (x_m)_{m=1}^{\ell}$, and we denote $|\mathbf{x}|_1 = \sum_{m=1}^{\ell} x_m$. We begin from the case where $G = K_{2n}$ and D_m are disjoint. The next theorem is the main result in this section, and we will prove it by proving a series of lemmas.

Theorem 3.1. *Let $(D_m)_{m=1}^{\ell}$ be a collection of disjoint subgraphs of K_{2n}, where $\Delta(D_m) \leq C$ for each m. Let R be a uniformly random perfect matching of K_{2n}. Define $\mathbf{X} = (X_m), \mathbf{Y} = (Y_m), X_m = |E(R \cap D_m)|$, $Y_m \sim \mathrm{Po}(|E(D_m)|/2n)$ independently. Then $\lim_{n\to\infty} d_{TV}(\mathbf{X}, \mathbf{Y}) = 0$.*

We shall use the following notation to help with counting.

Definition 3.2. Let $\mathbf{x} \in \mathbb{N}^{\ell}$. We define an $\mathbf{x}$-matching of K_{2n} as a $|\mathbf{x}|_1$-matching M where $|E(M \cap D_m)| = x_m$ for each m. We define $\mu_{\mathbf{x}}$ as the number of $\mathbf{x}$-matchings on K_{2n}.

We prove the following lemma by applying the principle of Inclusion-Exclusion. In this case, we apply Corollary 2.3. Since $\lambda = 1$, for a set $\mathbf{S} = (S_m)_{m=1}^{\ell}$, we define $\varphi_1(\mathbf{S}) = (|S_m|)_{m=1}^{\ell}$ for convenience.

Lemma 3.3. *Let $\mathbf{X}$ be defined as Theorem* 3.1. *The probability-generating function of $\mathbf{X}$ is given by*

$$G_{\mathbf{X}}(\mathbf{s}) = \sum_{\mathbf{x}:|\mathbf{x}|_1\leq n} \mu_{\mathbf{x}} \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} (\mathbf{s} - \mathbf{1})^{\mathbf{x}}.$$

Proof. Let Ω be the set of all perfect matchings of K_{2n} and let $\mathbf{I} = (|E(D_m)|)_{m=1}^{\ell}$. For $\mathbf{S} \sqsubset \mathbf{I}$, let $\hat{\mathbf{S}} = \bigcup_{m=1}^{\ell} S_m$ be the natural identification of $\mathbf{S}$ into a sets of edges in K_{2n}. If $\hat{\mathbf{S}}$ is a matching, i.e., the edges in $\hat{\mathbf{S}}$ are non-incident, it is an $\mathbf{x}$-matching where $\mathbf{x} = \varphi_1(\mathbf{S})$. Then we must have $|\mathbf{x}|_1 \leq n$ and

$$\mathbb{P}(A_{\mathbf{S}}) = \frac{\mathrm{pm}(K_{2n-2|\mathbf{x}|_1})}{\mathrm{pm}(K_{2n})} = \frac{(2n-2\,|\mathbf{x}|_1)!}{(n-|\mathbf{x}|_1)!2^{n-|\mathbf{x}|_1}} \Bigg/ \frac{(2n)!}{n!2^n} = \frac{(n)_{|\mathbf{x}|_1} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}},$$

where $(2n-2\,|\mathbf{x}|_1)!/(n-|\mathbf{x}|_1)!2^{n-|\mathbf{x}|_1}$ is the number of ways to extend $\hat{\mathbf{S}}$ into a perfect matching of K_{2n}. If $\hat{\mathbf{S}}$ is not a matching, then $\mathbb{P}(A_{\mathbf{S}}) = 0$. Since there are $\mu_{\mathbf{x}}$ $\mathbf{x}$-matchings, we know

$$\begin{aligned}\sum_{\mathbf{S}} \mathbb{P}(A_{\mathbf{S}})(\mathbf{s}-\mathbf{1})^{\varphi_1(\mathbf{S})} &= \sum_{\mathbf{x}:|\mathbf{x}|_1\leq n} \sum_{\mathbf{S}:\varphi(\mathbf{S})=\mathbf{x}} \mathbb{P}(A_{\mathbf{S}})(\mathbf{s}-\mathbf{1})^{\mathbf{x}} \\ &= \sum_{\mathbf{x}:|\mathbf{x}|_1\leq n} \mu_{\mathbf{x}} \frac{(n)_{|\mathbf{x}|_1} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} (\mathbf{s}-\mathbf{1})^{\mathbf{x}}\end{aligned}$$

and the lemma follows. ■

For convenience, we write $|E(D_m)| = d_m$ and $\mathbf{d} = (d_m)_{m=1}^{\ell}$. The next lemma estimates the quantity $\mu_{\mathbf{x}}$.

Lemma 3.4. *Let $\mu_{\mathbf{x}}, C$ be defined as above. Then*

$$\mu_{\mathbf{x}} = \frac{(\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}}{\mathbf{x}!}.$$

Proof. We choose an $\mathbf{x}$-matching $M_{\mathbf{x}}$ one edge at a time. For each $1 \leq m \leq \ell, 1 \leq k \leq x_m$, suppose that we have chosen the edges in $M_{\mathbf{x}} \cap D_{m'}$ for each $m' < m$ and we have chosen $k-1$ edges in D_m. Then we have chosen at most $|\mathbf{x}|_1$-edges. The chosen edges are adjacent to at most $2C\,|\mathbf{x}|_1$ edges, so we can choose the kth edge in at least $d_m - 2C\,|\mathbf{x}|_1$ ways and at most d_m ways. Applying the product rule for counting and dividing $x_m!$ for each m as we choose the edges unorderly, we know there are $(d_m - O(|\mathbf{x}|_1))/x_m!$ ways to choose the x_m edges in D_m to form the $\mathbf{x}$-matching. Applying the product rule again, we have

$$\mu_{\mathbf{x}} = \prod_m \frac{(d_m - O(|\mathbf{x}|_1))^{x_m}}{x_m!} = \frac{(\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}}{\mathbf{x}!}.$$

■

Recall that $\mathbf{Y} = (Y_m)_{m=1}^{\ell}$ and Y_m follows independently to $\mathrm{Po}(d_m/2n)$. Let $\bullet : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ denote the vector dot product. We can write the probability generating function of $\mathbf{Y}$ as

$$G_{\mathbf{Y}}(\mathbf{s}) = e^{(1/2n)\mathbf{d}\cdot(\mathbf{s}-1)} = \sum_{\mathbf{x}\in\mathbb{N}^\ell} \frac{(\mathbf{d}/2n)^{\mathbf{x}}}{\mathbf{x}!}(\mathbf{s}-1)^{\mathbf{x}}.$$

Our last step is to apply Tannery's theorem. Set

$$\alpha_{\mathbf{x}} = \begin{cases} \mu_{\mathbf{x}} \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} & |\mathbf{x}|_1 \le n, \\ 0 & \text{otherwise}, \end{cases} \qquad \beta_{\mathbf{x}} = \left(\prod_m \frac{(|E(D_m)|/2n)^{x_m}}{x_m!} \right).$$

Then, by Theorem 2.5, we know

$$d_{TV}(\mathbf{X}, \mathbf{Y}) \le \sum_{\mathbf{x} \in \mathbb{N}^\ell} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| 2^{|\mathbf{x}|_1}.$$

We want to show

$$\lim_{n \to \infty} \sum_{\mathbf{x} \in \mathbb{N}^\ell} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| 2^{|\mathbf{x}|_1} = 0.$$

In order to apply Tannery's theorem to switch the order of limit and infinite summation, we want to show that there exists $\gamma_{\mathbf{x}}$, independent of n, such that

$$|\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| = |\alpha_{\mathbf{x}}(n) - \beta_{\mathbf{x}}(n)| \le \gamma_{\mathbf{x}} \qquad \text{for all } n \in \mathbb{N}$$

and $\sum_{\mathbf{x}} \gamma_{\mathbf{x}} 2^{|\mathbf{x}|_1} < \infty$. Then, we want to show, $|\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| \to 0$ for all $\mathbf{x}$ fixed when $n \to \infty$. If so, by Tannery's theorem, we have

$$\lim_{n \to \infty} \sum_{\mathbf{x} \in \mathbb{N}^\ell} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| 2^{|\mathbf{x}|_1} = \sum_{\mathbf{x} \in \mathbb{N}^\ell} \lim_{n \to \infty} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| 2^{|\mathbf{x}|_1} = 0.$$

This proves $d_{TV}(\mathbf{X}, \mathbf{Y}) \to 0$.

***Proof of Theorem* 3.1.** We will first find $\gamma_{\mathbf{x}}$ such that $\sum_{\mathbf{x}} \gamma_{\mathbf{x}} 2^{|\mathbf{x}|_1} < \infty$. If $|\mathbf{x}|_1 \le n$, we have

$$\begin{aligned} \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} &= \frac{2^{|\mathbf{x}|_1}}{n^{|\mathbf{x}|_1}} \frac{(n)_{(|\mathbf{x}|_1)} n^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} \\ &= \frac{2^{|\mathbf{x}|_1}}{n^{|\mathbf{x}|_1}} \left(\prod_{k=1}^{|\mathbf{x}|_1} \frac{n}{2n-k+1} \right) \left(\prod_{k=1}^{|\mathbf{x}|_1} \frac{n-k+1}{2n-|\mathbf{x}|_1-k+1} \right) \le \frac{2^{|\mathbf{x}|_1}}{n^{|\mathbf{x}|_1}} \end{aligned} \tag{2}$$

as $n \le 2n-k+1$ and $n-k+1 \le 2n-|\mathbf{x}|_1-k+1$. By Lemma 3.4, $\mu_{\mathbf{x}} \le \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!}$, therefore, we have

$$0 \le \alpha_{\mathbf{x}}, \beta_{\mathbf{x}} \le \frac{(2\mathbf{d}/n)^{\mathbf{x}}}{\mathbf{x}!} \le \frac{(2C)^{|\mathbf{x}|_1}}{\mathbf{x}!}.$$

Hence $\gamma_{\mathbf{x}} = (2C)^{|\mathbf{x}|_1}/\mathbf{x}!$ will suffice, and $\gamma_{\mathbf{x}}$ is independent of n. We have

$$\sum_{\mathbf{x}} \frac{(2C)^{|\mathbf{x}|_1}}{\mathbf{x}!} 2^{|\mathbf{x}|_1} = e^{4\ell C} < \infty$$

and we have completed the first part. Then, we want to show that for all fixed $\mathbf{x}$, $\lim_{n\to\infty} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| = 0$. We know given n sufficiently large, holding $\mathbf{x}$ fixed,

$$|\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| \le \left| \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} - \frac{(\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}}{\mathbf{x}!} \right| \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} + \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} \left| \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} - (2n)^{-|\mathbf{x}|_1} \right|$$

by triangle inequality. We can view $\mathbf{d}^{\mathbf{x}} - (\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}$ as a product of $|\mathbf{x}|_1$ terms since $d_m \le |\mathbf{d}|_1 \le Cn$, we can apply Lemma 2.6 and (2) to obtain the following bound

$$\left| \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} - \frac{(\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}}{\mathbf{x}!} \right| \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} \le |\mathbf{x}|_1 O(|\mathbf{x}|_1)(Cn)^{|\mathbf{x}|_1 - 1} \frac{2^{|\mathbf{x}|_1}}{n^{|\mathbf{x}|_1}}. \tag{3}$$

Moreover,

$$\begin{aligned} &\frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} \left| \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1}}{(2n)_{(2|\mathbf{x}|_1)}} - (2n)^{-|\mathbf{x}|_1} \right| \\ &\le (Cn)^{|\mathbf{x}|_1} (2n)^{-|\mathbf{x}|_1} \frac{(n)_{(|\mathbf{x}|_1)} 2^{|\mathbf{x}|_1} (2n)^{|\mathbf{x}|_1} - (2n)_{(2|\mathbf{x}|_1)}}{(2n)_{(2|\mathbf{x}|_1)}}. \end{aligned} \tag{4}$$

Holding $|\mathbf{x}|_1$ fixed, the numerator of the fraction in (4) is a polynomial of n of degree at most $2|\mathbf{x}|_1 - 1$, so the fraction can be bounded by $o(1)$. Therefore, we can view both (3) and (4) as bounded by $o(1)$, and we have $\lim_{n\to\infty} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| = 0$. Hence, we have

$$\lim_{n\to\infty} d_{TV}(\mathbf{X}, \mathbf{Y}) = 0.$$

■

If we let $\ell = 1$, we obtain the next corollary. It can also be shown using a result of Godsil [2] and Zaslavsky [14].

Corollary 3.5. *Let D be a subgraph of K_{2n} such that $\Delta(D) \le C$ for some constant C, and R be a uniformly random perfect matching of K_{2n}. Then*

$$\lim_{n\to\infty} \mathbb{P}(|E(R \cap D)| = 0) = \lim_{n\to\infty} e^{-|E(D)|/2n}.$$

Note that this ratio is bounded below by $e^{-(C/2)(1+o(1))}$ and is asymptotically nonzero. In particular, if D is d-regular for some constant d, then the limiting probability equals $e^{-d/2}$.

Now we want to generalize K_{2n} to $K_{2n} - N$ for some graphs N where $\Delta(N) \le C$. We can define $X^* = |E(R \cap N)|$. Applying previous results, we show that the random vector $\mathbf{X}, X^*$ jointly converges to some independent Poisson distribution. Therefore, we expect the conditional distribution $\mathbf{X}|X^*$ to be close to $\mathbf{X}$ asymptotically. In particular, the distribution $\mathbf{X}|X^* = 0$ should also be independently Poisson. The next result is a corollary of Theorem 3.1 and is a restatement of Theorem 1.2.

Corollary 3.6. *Let $N, (D_m)_{m=1}^{\ell}$ be disjoint subgraphs of K_{2n} such that $\Delta(N)$, $\Delta(D_m) \leq C$ for some constant C. Let R be a uniformly random perfect matching of $K_{2n} - N$. Define $\mathbf{X} = (X_m)_{m=1}^{\ell}$, $X_m = |E(R \cap D_m)|$, $\mathbf{Y} = (Y_m)_{m=1}^{\ell}$ where Y_m follows independently to* $\mathrm{Po}(|E(D_m)|/2n)$*. Then*

$$\lim_{n\to\infty} d_{TV}(\mathbf{X}, \mathbf{Y}) = 0.$$

Proof. Let R^* be a uniformly random perfect matching from K_{2n}. Consider the $\ell + 1$-dimensional random vectors $(\mathbf{X}^*, X^*) = (X_1^*, \ldots, X_\ell^*, X^*), (\mathbf{Y}, Y^*) = (Y_1, \ldots, Y_\ell, Y^*)$ where $\mathbf{X}^* = (|E(R^* \cap D_m)|)_{m=1}^{\ell}$, and $X^* = |E(R^* \cap N)|$, $Y^* \sim \mathrm{Po}(|E(N)|/2n)$, Y^* independent to $\mathbf{Y}$. We want to show the distribution $d_{TV}(\mathbf{X}^*|X^* = 0, \mathbf{Y}) \to 0$. We have

$$\begin{aligned}
&d_{TV}(\mathbf{X}^*|X^* = 0, \mathbf{Y}) \\
&= \sum_{\mathbf{k}} |\mathbb{P}(\mathbf{X}^* = \mathbf{k}|X^* = 0) - \mathbb{P}(\mathbf{Y} = \mathbf{k})| \\
&= \sum_{\mathbf{k}} \left| \frac{\mathbb{P}(\mathbf{X}^* = \mathbf{k}, X^* = 0)}{\mathbb{P}(X^* = 0)} - \frac{\mathbb{P}(\mathbf{Y} = \mathbf{k})\mathbb{P}(Y^* = 0)}{\mathbb{P}(Y^* = 0)} \right| \\
&= \sum_{\mathbf{k}} \left| \frac{\mathbb{P}(\mathbf{X}^* = \mathbf{k}, X^* = 0)}{\mathbb{P}(X^* = 0)} - \frac{\mathbb{P}(\mathbf{Y} = \mathbf{k}, Y^* = 0)}{\mathbb{P}(Y^* = 0)} \right| \quad \text{by independence} \\
&\leq \frac{1}{\mathbb{P}(X^* = 0)} \left(\sum_{\mathbf{k}} |\mathbb{P}(\mathbf{X}^* = \mathbf{k}, X^* = 0) - \mathbb{P}(\mathbf{Y} = \mathbf{k}, Y^* = 0)| \right) \\
&+ \left| \frac{1}{\mathbb{P}(X^* = 0)} - \frac{1}{\mathbb{P}(Y^* = 0)} \right| \left(\sum_{\mathbf{k}} |\mathbb{P}(\mathbf{Y} = \mathbf{k}, Y^* = 0)| \right) \\
&\leq \frac{1}{\mathbb{P}(X^* = 0)} d_{TV}((\mathbf{X}^*, X^*), (\mathbf{Y}, Y^*)) + \left| \frac{\mathbb{P}(Y^* = 0)}{\mathbb{P}(X^* = 0)} - 1 \right|.
\end{aligned}$$

We can use Theorem 3.1 to show $d_{TV}((\mathbf{X}^*, X^*), (\mathbf{Y}, Y^*)) \to 0$ and use Corollary 3.5 to show $\lim_{n\to\infty} d_{TV}(\mathbf{X}^*|X^* = 0, \mathbf{Y}) = 0$. ∎

4. Case of $K_{r\times(r-1)n}$

In this section, we assume r is fixed, $\lambda = \binom{r}{2}$. We will develop analogous results for Theorem 3.1 in Section 3, where $G = K_{r\times(r-1)n}$. The following theorem is a restatement of Theorem 1.3 and is the main result of this section.

Theorem 4.1. *Let $K_{r\times(r-1)n}$ be the balanced complete r-partite graph, let $(D_m)_{m=1}^{\ell}$ be a collection of disjoint subgraphs of $K_{r\times(r-1)n}$ with maximum degree $\Delta(D_m) \leq C$ for all m, where C is independent of n. Let R be a uniformly random **balanced** perfect matchings of $K_{r\times(r-1)n}$. Define random vector*

$\mathbf{X} = (|E(R \cap D_m)|)_{m=1}^{\ell}$, $\mathbf{Y} = (Y_m)_{m=1}^{\ell}$ *where* $Y_m \sim \mathrm{Po}(|E(D_m)|/((r-1)^2 n))$ *independently. Then*

$$\lim_{n\to\infty} d_{TV}(\mathbf{X}, \mathbf{Y}) = 0.$$

We will first develop some notation for this section. Let $\mathbf{x} = (x_{m,i,j})_{m\in[\ell],i<j\in[r]} \in \mathbb{N}^{\ell\times\binom{r}{2}}$. We define by convention that $x_{m,i,j} = x_{m,j,i}$ if $i > j$. For fixed m, we define $\mathbf{x}_m = (x_{m,i,j})_{i<j\in[r]} \in \mathbb{N}^{\binom{r}{2}}$, for fixed i, j, we define $\mathbf{x}_{i,j} = (x_{m,i,j})_{m\in[\ell]} \in \mathbb{N}^{\ell}$, and for fixed i, we define $\mathbf{x}_i = (x_{m,i,j})_{m\in[\ell],j\neq i\in[r]} \in \mathbb{N}^{\ell\times(r-1)}$. We define $|\cdot|_1$ as the sum of the coordinates as usual.

We define $\psi_1 : \mathbb{N}^{\ell\times\binom{r}{2}} \to \mathbb{N}^{\ell}, \psi_2 : \mathbb{N}^{\ell\times\binom{r}{2}} \to \mathbb{N}^{r}, \psi_3 : \mathbb{N}^{\ell\times\binom{r}{2}} \to \mathbb{N}^{\binom{r}{2}}$ as follows.

$$\psi_1(\mathbf{x}) = (|\mathbf{x}_m|_1)_{m=1}^{\ell} \qquad \psi_2(\mathbf{x}) = (|\mathbf{x}_i|_1)_{i=1}^{r} \qquad \psi_3(\mathbf{x}) = (|\mathbf{x}_{i,j}|_1)_{i<j\in[r]}.$$

Let $V(K_{r\times(r-1)n}) = V_1\uplus, \dots, \uplus V_r$ be the vertex partition. We define the vector

$$\mathbf{d} = (d_{m,i<j})_{m\in[\ell],i<j\in[r]} \in \mathbb{N}^{\ell\times\binom{r}{2}}$$

where $d_{m,i,j}$ is the number of edges of D_m between V_i and V_j. Let $\mathbf{d}_m = (d_{m,i,j})_{i<j\in[r]} \in \mathbb{N}^{\binom{r}{2}}$. Then $|\boldsymbol{d_m}|_1 = |E(D_m)|$. The following definition will be analogous to Definition 3.2.

Definition 4.2. Let $\mathbf{x} \in \mathbb{N}^{\ell\times\binom{r}{2}}$. We define an $\mathbf{x}$-matching of $K_{r\times(r-1)n}$ as a $|\mathbf{x}|_1$-matching where the number of edges in D_m between V_i and V_j is $x_{m,i,j}$. We define $\mu_{\mathbf{x}}$ as the number of $\mathbf{x}$-matchings on $K_{r\times(r-1)n}$.

The next result is due to Johnston, Kayll and Palmer [5].

Lemma 4.3. *The number of balanced perfect matching of* $K_{r\times(r-1)n}$ *is*

$$\mathrm{bpm}(K_{r\times(r-1)n}) = \left(\frac{((r-1)n)!}{n!^{r-1}}\right)^{r} (n!)^{\binom{r}{2}}.$$

The following lemma is analogous to Lemma 3.4.

Lemma 4.4. *Let* $\mathbf{X}$ *be defined as Theorem* 4.1. *The probability generating function of* $\mathbf{X}$ *is given by*

$$G_{\mathbf{X}}(\mathbf{s}) = \sum_{\substack{\mathbf{x}\in\mathbb{N}^{\ell\times\binom{r}{2}} \\ \forall i\neq j, \left|\boldsymbol{x_{i,j}}\right|_1 \leq n}} \mu_{\mathbf{x}} \frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} (\mathbf{s}-\mathbf{1})^{\psi_1(\mathbf{x})}.$$

Proof. Let Ω be the set of all balanced perfect matching with a uniform probability measure, and let $\mathbf{I} = (I_{m,i,j})_{m\in[\ell],i<j\in[r]}$ where $I_{m,i,j}$ is the set of edges of D_m between V_i and V_j. For $\mathbf{S} \sqsubset \mathbf{I}$, let $\hat{\mathbf{S}} = \bigcup_{m\in[\ell],i<j\in[r]} S_{m,i,j}$ be the natural identification of $\mathbf{S}$ to a collection of edges in $K_{r\times(r-1)n}$. Suppose $\hat{\mathbf{S}}$ is a submatching of a balanced perfect matching of $K_{r\times(r-1)n}$. Then let $\mathbf{x} \in \mathbb{N}^{\ell\times\binom{r}{2}}$ be such that $\hat{\mathbf{S}}$ is an $\mathbf{x}$-matching. We must have $|\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1 \le n$ for all $i \neq j \in [r]$. We want to count the number of ways to extend $\hat{\mathbf{S}}$ to a balanced perfect matching. Denote $V_i' = V_i \setminus V(\hat{\mathbf{S}})$, and $|V_i'| = ((r-1)n - |\mathbf{x}_{\boldsymbol{i}}|_1)$. We want to partition V_i' into $\{V_{i,j}\}_{j\neq i\in[r]}$ with $|V_{i,j}| = n - |\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1$. There are

$$\frac{((r-1)n - |\mathbf{x}_{\boldsymbol{i}}|_1)!}{\prod_{j\neq i}(n - |\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1)!}$$

partitions. Then, we match the vertices in $V_{i,j}$ with the vertices in $V_{j,i}$ in $(n - |\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1)!$ ways. Therefore, the probability equals

$$\begin{aligned}
\mathbb{P}(\hat{\mathbf{S}} \subset R) &= \mathbb{P}(A_{\mathbf{S}}) \\
&= \prod_i \frac{((r-1)n - |\mathbf{x}_{\boldsymbol{i}}|_1)!}{\prod_{j\neq i}(n - |\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1)!} \prod_{i<j}(n - |\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1)! \Big/ \left(\frac{((r-1)n)!}{n!^{r-1}}\right)^r (n!)^{\binom{r}{2}} \\
&= \frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}}.
\end{aligned}$$

If $\hat{\mathbf{S}}$ is not a submatching of a balanced perfect matching, then $\mathbb{P}(A_{\mathbf{S}}) = 0$. Since there are $\mu_{\mathbf{x}}$ $\mathbf{x}$-matchings, by Corollary 2.3, we know

$$G_{\mathbf{X}}(\mathbf{s}) \sum_{\substack{\mathbf{x}\in\mathbb{N}^{\ell\times\binom{r}{2}} \\ \forall i\neq j, |\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1 \le n}} \mu_{\mathbf{x}} \frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} (\mathbf{s} - \mathbf{1})^{\psi_1(\mathbf{x})}.$$

■

The next lemma is analogous to Lemma 3.4 and estimates the quantity $\mu_{\mathbf{x}}$.

Lemma 4.5. *Let $\mu_{\mathbf{x}}, C$ be defined as Theorem 4.1. Then*

$$\mu_{\mathbf{x}} = \frac{(\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}}{\mathbf{x}!}.$$

Proof. We want to count the number of ways to choose the $\mathbf{x}$-matching. We assign an order to (m, i, j) where $m \in [\ell], i < j \in [r]$ and choose the edges of D_m between V_i and V_j following that order. Given m_0, i_0, j_0, suppose for all m, i, j that precedes m_0, i_0, j_0 in that order, we have chosen the edges in the $\mathbf{x}$-matching that intersect D_m and are between V_i and V_j. Then, we want to select x_{m_0,i_0,j_0}

non-incident edges among the d_{m_0,i_0,j_0} available edges. For $1 \leq k \leq x_{m_0,i_0,j_0}$, suppose we have already selected $k-1$ edges. Then the total number of previously selected edges is no more than $|\mathbf{x}|_1$, and hence they are incident to no more than $2C\,|\mathbf{x}|_1 = O(|\mathbf{x}|_1)$ edges because the maximum degree of D_m is bounded by C. Therefore, the number of ways to select edges in D_{m_0} between V_{i_0} and V_{j_0} is

$$\frac{(d_{m_0,i_0,j_0} - O(|\mathbf{x}|_1))}{x_{m_0,i_0,j_0}!}$$

where we divide by $x_{m_0,i_0,j_0}!$ because we do not distinguish orders among the x_{m_0,i_0,j_0} edges. Finally, we take the product of all m_0, i_0, j_0 and by product rule, we obtained the expression by replacing the index with m, i, j. ■

The next lemma will be used to bound the coefficients of the generating function.

Lemma 4.6. *Given $n \geq |\mathbf{x}_{i,j}|_1$ for each $i \neq j \in [r]$, we have*

$$\frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} \leq \frac{2^{2|\mathbf{x}|_1}}{n^{|\mathbf{x}|_1}}.$$

Proof. The lemma is equivalent to

$$\frac{\prod_{i<j} n^{|\mathbf{x}_{i,j}|_1} n_{(|\mathbf{x}_{i,j}|_1)}}{\prod_i ((r-1)n)_{(|\mathbf{x}_i|_1)}} \leq 2^{2|\mathbf{x}|_1}.$$

Squaring on both sides, it suffices to show

$$\prod_i \frac{\prod_{j\neq i} n^{|\mathbf{x}_{i,j}|_1} n_{(|\mathbf{x}_{i,j}|_1)}}{((r-1)n)^2_{(|\mathbf{x}_i|_1)}} \leq 2^{4|\mathbf{x}|_1}.$$

Since for each $a \in \mathbb{N}, n^a n_{(a)} \leq (2n)_{(2a)}$, it suffices to show that for each i,

$$\frac{\prod_{j\neq i} (2n)_{(2|\mathbf{x}_{i,j}|_1)}}{((r-1)n)^2_{(|\mathbf{x}_i|_1)}} \leq 2^{2|\mathbf{x}_i|_1}.$$

Since

$$\frac{(2n)_{(2a)}}{n^2_{(a)}} \leq 2^{2a}$$

we know

$$\frac{\prod_{j\neq i} (2n)_{(2|\mathbf{x}_{i,j}|_1)}}{((r-1)n)^2_{(|\mathbf{x}_i|_1)}} \leq \prod_{j<i} \frac{(2n)_{(2|\mathbf{x}_{i,j}|_1)}}{((r-j)n)^2_{(|\mathbf{x}_{i,j}|_1)}} \prod_{j>i} \frac{(2n)_{(2|\mathbf{x}_{i,j}|_1)}}{((r+1-j)n)^2_{(|\mathbf{x}_{i,j}|_1)}}$$

$$\leq \prod_j \frac{(2n)_{(2|\mathbf{x}_{i,j}|_1)}}{n^2_{(|\mathbf{x}_{i,j}|_1)}} \leq 2^{2|\mathbf{x}_i|_1}$$

and the proof follows. ∎

Since $\mathbf{Y} = (Y_m)_{m=1}^{\ell}$ and $Y_m \sim \mathrm{Po}(|\mathbf{d}_{\boldsymbol{m}}|_1/(r-1)^2 n)$, we can express the generating function of $\mathbf{Y}$ as

$$\begin{aligned}G_{\mathbf{Y}}(\mathbf{s}) &= \exp\left(\frac{1}{(r-1)^2 n}\psi_1(\mathbf{d})\cdot(\mathbf{s}-\mathbf{1})\right)\\ &= \sum_{\mathbf{y}\in\mathbb{N}^{\ell}} \frac{(1/((r-1)^2 n))\psi_1(\mathbf{d})^{\mathbf{y}}}{\mathbf{y}!}(\mathbf{s}-\mathbf{1})^{\mathbf{y}}\\ &= \sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\binom{r}{2}}} \frac{(1/((r-1)^2 n))\mathbf{d})^{\mathbf{x}}}{\mathbf{x}!}(\mathbf{s}-\mathbf{1})^{\psi_1(\mathbf{x})}\end{aligned}$$

where in the last step we used the multinomial theorem.

Set

$$\alpha_{\mathbf{x}} = \begin{cases}\mu_{\mathbf{x}}\frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} & \text{for all } i\neq j\in[r], |\mathbf{x}_{\boldsymbol{i},\boldsymbol{j}}|_1\leq n,\\ 0 & \text{otherwise,}\end{cases}$$

$$\beta_{\mathbf{x}} = \frac{(1/(r-1)^2 n)\mathbf{d})^{\mathbf{x}}}{\mathbf{x}!}.$$

By the same argument in Section 3, we need to find $\gamma_{\mathbf{x}}$, independent of n, such that $0\leq\alpha_{\mathbf{x}}(n),\beta_{\mathbf{x}}(n)\leq\gamma_{\mathbf{x}}$ and $\sum_{\mathbf{x}}\gamma_{\mathbf{x}}2^{|\mathbf{x}|_1}<\infty$. We also need to show

$$\lim_{n\to\infty}|\alpha_{\mathbf{x}}-\beta_{\mathbf{x}}| = 0$$

holding $\mathbf{x}$ fixed. Then, by Tannery's Theorem, we have

$$\lim_{n\to\infty} d_{TV}(\mathbf{X},\mathbf{Y}) \leq \lim_{n\to\infty}\sum_{\mathbf{x}}|\alpha_{\mathbf{x}}-\beta_{\mathbf{x}}|2^{\mathbf{x}} = \sum_{\mathbf{x}}\lim_{n\to\infty}|\alpha_{\mathbf{x}}-\beta_{\mathbf{x}}|2^{\mathbf{x}} = 0.$$

***Proof of Theorem* 4.1.** We first want to find $\gamma_{\mathbf{x}}$ such that $\sum_{\mathbf{x}}\gamma_{\mathbf{x}}2^{|\mathbf{x}|_1}<\infty$. By Lemma 4.6

$$\alpha_{\mathbf{x}},\beta_{\mathbf{x}} \leq \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!}\frac{2^{2|\mathbf{x}|_1}}{n^{|\mathbf{x}|_1}} = \frac{((4/n)\mathbf{d})^{\mathbf{x}}}{\mathbf{x}!}.$$

Since $d_{m,i,j}/n\leq(r-1)C$, we know

$$\frac{((4/n)\mathbf{d})^{\mathbf{x}}}{\mathbf{x}!} \leq \frac{(4(r-1)C)^{|\mathbf{x}|_1}}{\mathbf{x}!}$$

and

$$\sum_{\mathbf{x}\in\mathbb{N}^{\ell\times\binom{r}{2}}}\frac{(4(r-1)C)^{|\mathbf{x}|_1}}{\mathbf{x}!} = \exp\left(4(r-1)C\binom{r}{2}\right)<\infty.$$

Therefore, $\gamma_{\mathbf{x}} = (4(r-1)C)^{|\mathbf{x}|_1}/\mathbf{x}!$ will suffice. Then, we want to show $\lim_{n\to\infty} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| = 0$ for all fixed $\mathbf{x}$. By the triangle inequality, we can bound $|\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}|$ as

$$\begin{aligned} |\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| = &\left| \frac{(\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}}{\mathbf{x}!} - \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} \right| \frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} \\ &+ \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} \left| \frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} - \frac{1}{((r-1)^2 n)^{|\mathbf{x}|_1}} \right|. \end{aligned}$$

Applying Lemma 2.6 and Lemma 4.6, we know

$$\begin{aligned} &\left| \frac{(\mathbf{d} - O(|\mathbf{x}|_1)\mathbf{1})^{\mathbf{x}}}{\mathbf{x}!} - \frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} \right| \frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} \\ &\le \ell \binom{r}{2} O(|\mathbf{x}|_1)((r-1)Cn)^{|\mathbf{x}|_1 - 1} \frac{2^{2|\mathbf{x}|_1}}{n^{|\mathbf{x}|_1}} = o(1) \end{aligned}$$

and

$$\begin{aligned} &\frac{\mathbf{d}^{\mathbf{x}}}{\mathbf{x}!} \left| \frac{(n\mathbf{1})_{(\psi_3(\mathbf{x}))}}{((r-1)n\mathbf{1})_{(\psi_2(\mathbf{x}))}} - \frac{1}{((r-1)^2 n)^{|\mathbf{x}|_1}} \right| \\ &\le ((r-1)Cn)^{|\mathbf{x}|_1} o\left(n^{-|\mathbf{x}|_1}\right) = o(1) \end{aligned}$$

holding $\mathbf{x}$ fixed. Therefore, $|\alpha_{\mathbf{x}} - \beta_{\mathbf{x}}| \to 0$ and the proof follows. ∎

5. Graph Decomposition and Case of Non-Disjont D_m

In this Section, we generalize D_m to a collection of not necessarily disjoint subgraphs of G, where $G = K_{2n} - N$ or $K_{r\times(r-1)n}$. The idea is to decompose the ℓ subgraphs into $2^\ell - 1$ disjoint pieces and apply the previous results to these pieces to get a $2^\ell - 1$ independent Poisson joint distribution, and then appropriately sum the random variables with the corresponding distribution. We shall formalize this notion of graph decomposition using the following definition.

Definition 5.1. Let $(D_m)_{m=1}^{\ell}$ be a collection of subgraphs of G. Let $\mathcal{P}^*([\ell])$ be the collection of non-empty subsets of $[\ell]$. For each $S \in \mathcal{P}^*([\ell])$, define

$$D_S = \bigcap_{m\in S} D_m \Big/ \bigcup_{m\in S^c} D_m^c.$$

Then, $(D_S)_{S\in\mathcal{P}^*([\ell])}$ is a collection of disjoint subgraphs.

If we define the random variables $\bar{X}_S = |E(R \cap D_S)|$, then by applying our previous results, $\bar{X}_S$ approaches an independent Poisson joint distribution. Since

$X_m = \sum_{S:m\in S} \bar{X}_S$ and the sum of a Poisson distribution is also Poisson, the random vector $(X_m)_{m=1}^{\ell}$ should approach a joint Poisson distribution. The next lemma characterizes this phenomenon.

Lemma 5.2. *Consider four random vectors* $\mathbf{X} = (X_m)_{m=1}^{\ell}, \mathbf{Y} = (Y_m)_{m=1}^{\ell}, \bar{\mathbf{X}} = (\bar{X}_S)_{S\in\mathcal{P}^*([\ell])}$, $(\bar{Y}_S)_{S\in\mathcal{P}^*([\ell])}$ *such that* $X_m = \sum_{S:m\in S} \bar{X}_S, Y_m = \sum_{S:m\in S} \bar{Y}_S$ *for each* m. *Then*

$$d_{TV}(\mathbf{X},\mathbf{Y}) \le d_{TV}(\bar{\mathbf{X}},\bar{\mathbf{Y}}).$$

Proof. For vector $\bar{\mathbf{k}} = (\bar{k}_S)_{S\in\mathcal{P}^*([\ell])}$, let $\mathbf{k} = (k_m)_{m=1}^{\ell}$ be defined as $k_m = \sum_{S:m\in S} \bar{k}_S$. We define a function $T : \mathbb{N}^{\mathcal{P}^*([\ell])} \to \mathbb{N}^{\ell}$ such that $T(\bar{\mathbf{k}}) = \mathbf{k}$. The function T is surjective, because if we let $\bar{k}_S = k_m$ if $S = \{m\}$ and $\bar{k}_S = 0$ otherwise, and let $\bar{\mathbf{k}} = (\bar{k}_S)_{S\in\mathcal{P}^*([\ell])}$, then $T(\bar{\mathbf{k}}) = (k_m)_{m=1}^{\ell}$. Then we have

$$\begin{aligned}
d_{TV}(\mathbf{X},\mathbf{Y}) &= \sum_{\mathbf{k}\in\mathbb{N}^{\ell}} |\mathbb{P}(\mathbf{X}=\mathbf{k}) - \mathbb{P}(\mathbf{Y}=\mathbf{k})| \\
&= \sum_{\mathbf{k}\in\mathbb{N}^{\ell}} \left| \sum_{\bar{\mathbf{k}}\in T^{-1}(\mathbf{k})} \mathbb{P}(\bar{\mathbf{X}}=\bar{\mathbf{k}}) - \mathbb{P}(\bar{\mathbf{X}}=\bar{\mathbf{k}}) \right| \\
&\le \sum_{\mathbf{k}\in\mathbb{N}^{\ell}} \sum_{\bar{\mathbf{k}}\in T^{-1}(\mathbf{k})} \left|\mathbb{P}(\bar{\mathbf{X}}=\bar{\mathbf{k}}) - \mathbb{P}(\bar{\mathbf{X}}=\bar{\mathbf{k}})\right| \\
&= \sum_{\bar{\mathbf{k}}\in\mathbb{N}^{\mathcal{P}^*([\ell])}} \left|\mathbb{P}(\bar{\mathbf{X}}=\bar{\mathbf{k}}) - \mathbb{P}(\bar{\mathbf{X}}=\bar{\mathbf{k}})\right| = d_{TV}(\bar{\mathbf{X}},\bar{\mathbf{Y}}).
\end{aligned}$$

■

The next corollary generalizes Theorem 1.2, and the proof follows naturally from this lemma and Theorem 1.2.

Corollary 5.3. *Let* N *be a subgraph of* K_{2n}, *and let* $(D_m)_{m=1}^{\ell}$ *be a collection of subgraphs of* $K_{2n} - N$ *such that* $\Delta(D_m), \Delta(N) \le C$ *for some constant* C. *Let* $(D_S)_{S\in\mathcal{P}^*([\ell])}$ *be the graph decomposition of* $(D_m)_{m=1}^{\ell}$. *Let* R *be a uniformly chosen random perfect matching of* $K_{2n} - N$ *and define* $\mathbf{X} = (X_m)_{m=1}^{\ell}, X_m = |E(R\cap D_m)|$. *For each* $S \in \mathcal{P}^*([\ell])$, *let* Y_S *follow independently to* $\mathrm{Po}(|E(D_S)|/2n)$ *and let* $Y_m = \sum_{S:m\in S} Y_S$, *and* $\mathbf{Y} = (Y_m)_{m=1}^{\ell}$. *Then*

$$\lim_{n\to\infty} d_{TV}(\mathbf{X},\mathbf{Y}) = 0.$$

Proof. We know $(D_S)_{S\in\mathcal{P}^*([\ell])}$ is a collection of disjoint subgraphs of $K_{2n} - N$ where $\Delta(D_S) \le C$ for all S. Define $\bar{\mathbf{X}} = (|E(R\cap D_S)|)_{S\in\mathcal{P}^*([\ell])}$, $\bar{\mathbf{Y}} = (Y_S)_{S\in\mathcal{P}^*([\ell])}$. By Theorem 1.2, we know

$$d_{TV}(\bar{\mathbf{X}},\bar{\mathbf{Y}}) \to 0.$$

Therefore, by Lemma 5.2, $d_{TV}(\mathbf{X},\mathbf{Y}) \to 0$. ■

The next corollary generalizes Theorem 1.3 and is proven by the exact same method. We merely replace $K_{2n} - N$ by $K_{r\times(r-1)n}$ and use Theorem 1.3 instead of Theorem 1.2.

Corollary 5.4. *Let $(D_m)_{m=1}^{\ell}$ be a collection of subgraphs of $K_{r\times(r-1)n}$ such that $\Delta(D_m) \leq C$ for some constant C. Let $(D_S)_{S\in\mathcal{P}^*([\ell])}$ be the graph decomposition of $(D_m)_{m=1}^{\ell}$. Let R be a uniformly chosen random perfect matching of $K_{r\times(r-1)n}$ and define $\mathbf{X} = (|E(R\cap D_m|)_{m=1}^{\ell}$. For each $S \in \mathcal{P}^*([\ell])$, let Y_S follow independently to $\mathrm{Po}(|E(D_S)|/(r-1)^2 n)$ and let $Y_m = \sum_{S:m\in S}(Y_S)$, and $\mathbf{Y} = (Y_m)_{m=1}^{\ell}$. Then*

$$\lim_{n\to\infty} d_{TV}(\mathbf{X}, \mathbf{Y}) = 0.$$

6. Potential Directions of Future Work

A promising direction for future research is to extend the analysis from the parent graph G to a broader class of graphs, such as regular robust expander graphs (see [3, 4, 6, 7] for details). The current method of counting via the Principle of Inclusion-Exclusion may not be well-suited for this generalization. Instead, adopting techniques similar to those employed in [3] could provide a more effective approach.

Another potential direction for future research involves relaxing the constraints on the constants ℓ, C, and r, allowing them to grow, possibly slowly, as a function of n. If $\ell \to \infty$ as $n \to \infty$, the current approach of applying Tannery's theorem to interchange the limit and infinite summation becomes inapplicable, as the theorem requires a countable index set I, whereas this scenario involves the uncountable set $\mathbb{N}^{\mathbb{N}}$. Furthermore, the graph decomposition technique outlined in Section 5 may no longer be effective, as it increases the number of decomposed graphs D from ℓ to $2^{\ell} - 1$, where the latter grows significantly faster than the former.

A third direction for future research is to investigate the rate of convergence of $d_{TV}(\mathbf{X}, \mathbf{Y})$. Current results establish only that $d_{TV}(\mathbf{X}, \mathbf{Y}) = o(1)$. We conjecture that, depending on the specific properties of the graph sequence $(D_m)_{m=1}^{\ell}$, it may be possible to identify a constant $K = K((D_m)_{m=1}^{\ell})$ such that $d_{TV}(\mathbf{X}, \mathbf{Y}) = Kn^{-1} + o(n^{-1})$.

Acknowledgement

The author thanks his advisor, Professor Karen L. Collins, in the Department of Mathematics and Computer Science at Wesleyan University, for advising this research process.

References

[1] J.N. Brawner, *Dinner, dancing, and tennis, anyone*?, Math. Mag. **73** (2000) 29–36. https://doi.org/10.1080/0025570X.2000.11996795

[2] C.D. Godsil, *Hermite polynomials and a duality relation for matchings polynomials*, Combinatorica **1** (1981) 257-262. https://doi.org/10.1007/BF02579331

[3] B. Granet and F. Joos, *Random perfect matchings in regular graphs*, Random Structures Algorithms **64** (2024) 3–14. https://doi.org/10.1002/rsa.21172

[4] V. Gruslys and S. Letzter, *Cycle partitions of regular graphs*, Combin. Probab. Comput. **30** (2021) 526–549. https://doi.org/10.1017/s0963548320000553

[5] D. Johnston, P.M. Kayll and C. Palmer, *Deranged matchings: proofs and conjectures*, Amer. Math. Monthly **131** (2024) 95–111. https://doi.org/10.1080/00029890.2023.2274239

[6] D. Kühn, A. Lo, D. Osthus and K. Staden, *The robust component structure of dense regular graphs and applications*, Proc. Lond. Math. Soc. (3) **110** (2015) 19–56. https://doi.org/10.1112/plms/pdu039

[7] D. Kühn and D. Osthus, *Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments*, Adv. Math. **237** (2013) 62–146. https://doi.org/10.1016/j.aim.2013.01.005

[8] P. Loya, Amazing and Aesthetic Aspects of Analysis (Springer, New York, 2017). https://doi.org/10.1007/978-1-4939-6795-7

[9] B.H. Margolius, *Avoiding your spouse at a bridge party*, Math. Mag. **74** (2001) 33–41. https://doi.org/10.2307/2691151

[10] P. Rémond de Montmort, Essay d'Analyse Sur les Jeux de Hazard, Jacque Quillau, Paris (1708) Seconde Edition, Revûe & augmentée de plusieurs Lettres. Jacque Quillau, Paris (1713). Photographic reprint, Chelsea Publishing Co., New York (1980). https://doi.org/10.1002/0471725161.ch18

[11] S. Spiro and E. Surya, *Counting deranged matchings*, European J. Combin. **120** (2024) 103980. https://doi.org/10.1016/j.ejc.2024.103980

[12] R.P. Stanley, Enumerative Combinatorics. Volume 1 (Cambridge University Press, Cambridge, 2012). https://doi.org/10.1017/CBO9781139058520

[13] H.S. Wilf, Generatingfunctionology (AK Peters, Ltd., Wellesley, MA, 2006).

[14] T. Zaslavsky, *Complementary matching vectors and the uniform matching extension property*, European J. Combin. **2** (1981) 91–103.
https://doi.org/10.1016/S0195-6698(81)80025-X

Received 18 May 2025
Revised 15 October 2025
Accepted 15 October 2025
Available online 11 November 2025