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CONNECTIVITY AND MATCHING EXTENDABILITY OF OPTIMAL 1-EMBEDDED GRAPHS ON THE TORUS

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Abstract

In this paper, we discuss optimal 1-toroidal graphs (abbreviated as O1TG), which are drawn on the torus so that every edge crosses another edge at most once, and has n vertices and exactly $4n$ edges. We first consider connectivity of O1TGs, and give the characterization of O1TGs having connectivity exactly k for each $k \in \{4, 5, 6, 8\}$. In our argument, we also show that there exists no O1TG having connectivity exactly 7. Furthermore, using the result above, we discuss extendability of matchings, and give the characterization of 1-, 2- and 3-extendable O1TGs in turn.

Keywords: 1-embeddable graph, torus, connectivity, perfect matching.

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1. Introduction

All graphs considered in this paper are finite, simple and connected. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *order* of G means the number of vertices of G. A path P of length k is a *k-path*, which has k edges. In particular, if $k = 0$, then P is a trivial graph. A cycle of length k is a k-*cycle*. For a cycle C in a graph G, an edge $e \in E(G)$ such that $V(e) \subset V(C)$ and $e \notin E(C)$ is called a *chord* of C. A cycle C in G is *separating* if $G - V(C)$ is a disconnected graph. We denote the induced subgraph of G by $S \subset V(G)$ by $G[S]$.

A graph G is 1-*embeddable* on a closed surface F^2 if it can be drawn on F^2 so that every edge of G crosses another edge at most once. The drawn image of G on F^2 is a 1-*embedded* graph on F^2. (We implicitly consider *good drawings*, that is, (i) vertices are on different points on the surface, (ii) no adjacent edges cross, (iii) no three edges cross at the same point, and (iv) any non-adjacent edges do not touch tangently.) The study of 1-*planar* graphs, which are 1-embeddable graphs on the plane or the sphere, was first introduced by Ringel [16], and recently developed in various points of view (see e.g., [6, 19]); the drawn image is called a 1-*plane graph*. It is known that if G is a 1-embedded graph on F^2 with at least three vertices, then $|E(G)| \leq 4|V(G)| - 4\chi(F^2)$ holds, where $\chi(F^2)$ stands for the Euler characteristic of F^2 (see [8] for example). In particular, a 1-embedded graph G on F^2 that satisfies the equality, that is $|E(G)| = 4|V(G)| - 4\chi(F^2)$, is *optimal*. An edge in a 1-embedded graph G is *crossing* if it crosses another edge, and *non-crossing* otherwise.

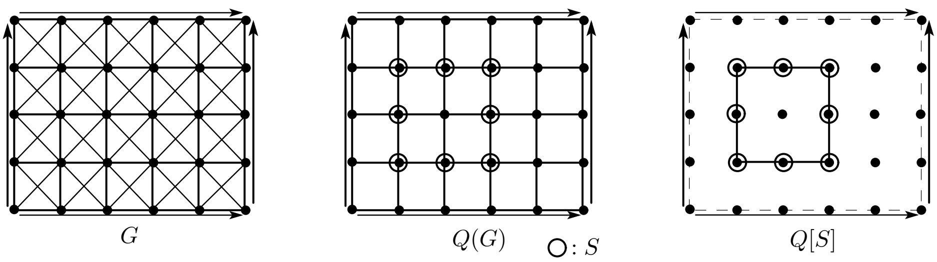

Figure 1. 8-regular O1TG G, quadrangular subgraph $Q(G)$ and $Q[S]$ induced by S.

It was shown in [8] that every simple optimal 1-embedded graph G on F^2 is obtained from a polyhedral quadrangulation H by adding a pair of crossing edges in each face of H. We call the quadrangulation H, which consists of all the non-crossing edges of G, the *quadrangular subgraph* of G, and denote it by $Q(G)(= H)$ (see the left-hand side and the center of Figure 1). By the property above, every vertex of G has even degree, that is, G is Eulerian. For example, "optimal 1-plane graph" is abbreviated as "O1PG" in past research; see e.g., [17, 20]. Similarly, "optimal 1-embedded graph on the torus" (respectively, "the projective plane)" is also said like "*optimal* 1-*toroidal graph*" (respectively, "*optimal* 1-*projective plane graph*", and abbreviated as "O1TG" (respectively, "O1PPG").

In the first half of the paper, we discuss connectivity of O1TGs. In [4], Fujisawa *et al.* discussed connectivity of O1PGs, and show that every O1PG has connectivity either 4 or 6; they actually gave the characterization of O1PGs having connectivity k for each $k \in \{4, 6\}$. Furthermore in [7], connectivity of O1PPGs was discussed, and it was shown that there exist O1PPGs that have connectivity exactly 5. That is, every O1PPG has connectivity 4, 5 or 6, and the authors actually characterized such O1PPGs with connectivity k for each $k \in \{4, 5, 6\}$. Since every O1PG or O1PPG G has a vertex of degree exactly 6, and hence $\kappa(G) \leq 6$, where $\kappa(G)$ represents the connectivity of G. However for O1TGs, the situation is different, since there are infinitely many 8-regular O1TGs whose quadrangular subgraphs are 4-regular. The first main result in this paper is the characterization of 8-connected O1TGs as follows, where $Q(p, q, r)$ is a well-known representation of 4-regular quadrangulations in topological graph theory, which will be introduced in Section 3.

Theorem 1. *Let G be an 8-regular* O1TG. *Then, we have $\kappa(G) \in \{6, 8\}$. Furthermore, G is 8-connected if and only if $Q(G)$ is not isomorphic to $Q(p, r, 3)$ ($p \geq 4$ and $r \geq 0$).*

Actually, it is easy to show that every optimal 1-embedded graph on a closed surface is 4-connected; this fact is proved as Theorem 18 in this paper. Hence, we characterize O1TGs, which are not 8-regular, having connectivity exactly $k \in \{4, 5\}$ as follows. (As a result, O1TGs other than those have connectivity exactly 6.)

Theorem 2. *Let G be an* O1TG. *Then the following statements hold.*

(i) *$\kappa(G) = 4$ if and only if $Q(G)$ has a trivial 4-cycle that does not bound a face of $Q(G)$.*

(ii) *Assume that G is 5-connected. Then $\kappa(G) = 5$ if and only if $Q(G)$ has two homotopic cycles xy_1z_1x and xy_2z_2x, where x, y_1, y_2, z_1 and z_2 are distinct vertices.*

In the latter half of the paper, we discuss matching extendability of O1TGs using the result related to connectivity of those graphs discussed above. A matching $M \subset E(G)$ of G is *extendable* if G has a perfect matching containing M. Moreover, a graph G with at least $2m + 2$ vertices is *m-extendable* if any matching M in G with $|M| = m$ is extendable. Matching extendability has been widely studied in literature (e.g., see [15]). In particular, matching extendability of graphs embedded on closed surfaces was investigated in [1, 2, 5, 13]; for example, it was proven as a basic result that no planar graph is 3-extendable.

The matching extendability of 1-embedded graphs on F^2 was first addressed in [4], and the authors proved that every O1PG of even order is 1-extendable. Further in the same paper, they discussed 2-extendability of O1PGs, and proved that an O1PG G of even order is 2-extendable unless G contains a barrier 4-cycle, where a *barrier k-cycle* is a k-cycle of $Q(G)$ that bounds a 2-cell containing an odd number of vertices; it is precisely defined in Section 5. As mentioned above, every O1PG has a vertex v of degree exactly 6, and it is clear that every O1PG is not 3-extendable; consider three independent edges covering $N_G(v)$. In the same paper, they actually characterize three independent edges that are extendable. Moreover in [7], the argument was extended to O1PPGs.

We first give the characterization of 1-extendable O1TGs as follows, where T^2 represents the torus throughout the paper.

Theorem 3. *Let G be an* O1TG *of even order. Then G is* 1*-extendable if and only if $Q(G)$ does not contain a subgraph H satisfying the following conditions:*

(i) *H is a quadrangulation of T^2, and*

(ii) *every facial cycle of H corresponds to a barrier* 4*-cycle of G.*

The following statement concerning the 2-extendability of O1TGs is similar to those for O1PGs and O1PPGs.

Theorem 4. *An* O1TG *G of even order is* 2*-extendable if and only if G has no barrier* 4*-cycle.*

Unlike in the case of O1PGs and O1PPGs, there exist 3-extendable O1TGs, and those graphs are characterized as follows.

Theorem 5. *An* O1TG *G of even order is* 3*-extendable if and only if G is* 8*-regular.*

This paper is organized as follows. In the next section, we first define terminology used in the paper, and introduce the fundamental results holding for optimal 1-embedded graphs on general closed surfaces. Next, we discuss connectivity of O1TGs and separating short cycles in quadrangular subgraphs. In Section 3, we provide the characterization of 8-regular O1TGs having connectivity exactly 8. Furthermore in Section 4, we characterize not 8-regular O1TGs having connectivity k for each $k \in \{4, 5, 6\}$. In Section 5, we discuss matching extendability of O1TGs, and show the characterization of m-extendable O1TGs for each $m \in \{1, 2, 3\}$.

2. Preliminaries and Basic Results

A vertex set S of a connected graph G is a *cut* if $G - S$ has at least two connected components. A cut S of G is *minimal* if any proper subset of S is not a cut of G. For a cut S of G, if $|S| = k$, then we call S a *k-cut* of G. We denote the number of odd components of $G - S$ for $S \subset V(G)$ by $C_o(G - S)$; that is, a connected component of odd order.

Let G be a graph embedded on a closed surface F^2. Then a connected component of $F^2 - G$, which is as a topological space, is a *face* of G, and we denote the face set of G by $F(G)$; that is, "a face" in this paper is not necessarily homeomorphic to an (open) 2-cell. In particular, if every face of G is homeomorphic to a 2-cell, then G is a 2-*cell embedding* or 2-*cell embedded graph* on F^2.

In general, each boundary component of a face f forms a closed walk of G. That is, the boundary of f, which is denoted by ∂f, is the union of closed walks of G. In particular, if f has the unique boundary component, then it is said to be the *boundary closed walk* of f. Let f be a face of G embedded on F^2, and assume that $\partial f = W_1 \cup \cdots \cup W_l$, where W_i is a closed walk corresponding to a boundary component of f for each $i \in \{1, \ldots, l\}$. Then, the invariant $\deg(f)$, which is called the *size* of f, is defined as follows where $|W_i|$ is the length of a closed walk W_i: $\deg(f) = \sum_{i=1}^{l} |W_i|$. Furthermore, we put $V(\partial f) = V(W_1) \cup \cdots \cup V(W_l)$. A k-*face* f of G is a face with $\deg(f) = k$. In particular, if a k-face f is bounded by a closed walk $W_1 = v_0v_1 \cdots v_{k-1}v_0$ of length k, that is, $\partial f = W_1$, then f is a k-*gonal face* of G. In this case, we simply denote $f = v_0v_1 \cdots v_{k-1}$ in our latter arguments. Furthermore, if f is a 2-cell face, we say that f is a k-*gonal* 2-*cell face*. Observe that every 4-face is a 4-gonal face since it cannot have at least two boundary components, except for a very special case where the boundary consists of two independent edges such that the 4-gonal face is incident to those edges on both sides.

A simple closed curve γ on a closed surface F^2 is *trivial* if γ bounds a 2-cell on F^2, and *essential* otherwise. We apply these definitions to cycles of graphs embedded on F^2, regarding them as simple closed curves. A simple closed curve γ on a closed surface F^2 is *surface separating* if $F^2 - \gamma$ is disconnected as a topological space. We also apply the definition to cycles of graphs on F^2. It is well-known that every surface separating simple closed curve on the sphere, the projective plane or the torus is trivial. The following proposition is known in topological graph theory, and is commonly used.

Proposition 6 [9]**.** *Let G be a graph 2-cell embedded on a closed surface F^2 so that each face is bounded by a closed walk of even length. Then the lengths of two cycles in G have the same parity if they are homotopic to each other on F^2. Furthermore, there is no surface separating odd cycle in G.*

We often consider the induced subgraph of $Q(G)$ by a cut S of G in our argument, which is $Q(G)[S]$ under our definition. However, when the underlying graph G is clear, we use $Q[S]$ in place of $Q(G)[S]$, to simplify the notation (see the right-hand side of Figure 1). In the following five lemmas, we assume that G is an optimal 1-embedded graph on F^2, and $S \subset V(G)$ is a cut of G. First of all, we show the following one, which is related to Proposition 6.

Lemma 7. *For every face f of $Q[S]$, the size of f is even.*

Proof. Consider the graph $H = Q(G) \cap (f \cup \partial f)$ embedded on the surface F_0 with the boundary components $W_1 \cup \cdots \cup W_l$. We glue a disc D_i to W_i so that the boundary of D_i coincides with W_i for each $i \in \{1, \ldots, l\}$, to obtain a closed surface $\tilde{F}_0$. Now, take the dual H^* of H on $\tilde{F}_0$. If the size of f is odd, then it immediately contradicts the odd point theorem. ∎

The remaining four lemmas are proved in [7].

Lemma 8 [7]**.** *Every face of $Q[S]$ contains at most one connected component of $G - S$.*

Lemma 9 [7]**.** *If S is minimal, then the minimum degree of $Q[S]$ is at least 2.*

Lemma 10 [7]**.** *If $Q[S]$ has p faces each of which has the size at least $2q \geq 6$, then the following inequalities hold:*

(i) $|E(Q[S])| \geq 2|F(Q[S])| + (q-2)p$,

(ii) $|S| - \chi(F^2) + (2-q)p \geq |F(Q[S])|$.

Lemma 11 [7]**.** *If $|S| \leq C_o(G-S) + 2m$ holds for some integer m, then we have the following*

$$2|F(Q[S])| + 2m - \chi(F^2) \geq |E(Q[S])|.$$

3. Connectivity of 8-Regular O1TGs

In this section, we discuss the connectivity of 8-regular O1TGs. First of all, we show that every O1TG is 6-connected.

Lemma 12. *Let G be an 8-regular* O1TG. *Then $6 \leq \kappa(G) \leq 8$ holds.*

Proof. An 8-regular O1TG G has a 6-regular triangulation T as a spanning subgraph, which is obtained from $Q(G)$ by adding a diagonal edge in each face; note that $Q(G)$ is a 4-regular quadrangulation of T^2. Then T is 6-connected by the result in [11], and $\kappa(G) \geq 6$ holds. In addition, we clearly have $\kappa(G) \leq 8$, since every vertex of G has degree exactly 8. ∎

The following lemma describes the inner structures of 2-cell regions of quadrangulations of closed surfaces bounded by closed walks of length 4, 6 and 8. Note that a *region* in this paper might contain vertices and edges of the graph.

Lemma 13 ([18])**.** *Let G be a quadrangulation of a closed surface F^2 and let D be a 2-cell region bounded by a closed walk C of length 4, 6 or 8 such that*

(i) *there is at least one vertex inside D,*

(ii) *all vertices inside D have degree at least 3, and*

(iii) *D does not have a unique vertex x of degree 4 such that $N_G(x) \subset V(C)$ and D contains exactly four faces incident to x (when $|C| = 8$).*

Then, there exists a vertex of degree 3 inside D.

Next, we discuss $Q[S]$ for a k-cut S in 8-regular O1TGs, and present the following three lemmas, which are keys to prove our main result.

Lemma 14. *Let G be an 8-regular* O1TG, *and let S be a minimal k-cut of G with $k \in \{6, 7\}$. Then $Q[S]$ has at least two faces that are not homeomorphic to 2-cells.*

Proof. Suppose, for a contradiction, that $Q[S]$ has at most one face that is not homeomorphic to a 2-cell. Since S is a cut of G, $Q[S]$ has a 2-cell face f_1 that contains a connected component of $G - S$ by Lemma 8. Furthermore, $\deg(f_1)$ is even by Lemma 7. Then, we have $\deg(f_1) \geq 8$ by Lemma 13. If $\deg(f_1) = 8$, then f_1 contains the unique vertex v of G by Lemma 13. Since $k \in \{6, 7\}$, ∂f_1 is not a cycle. This contradicts that G is simple since $\deg_G(v) = 8$. Therefore, we have $\deg(f_1) \geq 10$.

Next, we discuss the other face f_2 that contains a connected component of G; f_2 might not be a 2-cell face. By the minimality of S, $V(\partial f_2) = S$. (Observe that f_2 contains at least one vertex of G since no edge of $Q(G)$ other than those on the boundary is inside f_2; recall that $Q[S]$ is the induced subgraph of $Q(G)$.) Hence f_2 is a k'-face where k' is an even number is at least k by Lemma 7.

Then the inequality $2|E(Q[S])| \geq 4(|F(Q[S])| - 2) + k' + 10 = 4|F(Q[S])| + k' + 2$ holds. On the other hand, we have $k - |E(Q[S])| + |F(Q[S])| \geq 0$, and hence $k + |F(Q[S])| \geq |E(Q[S])|$ holds. By combining the former inequality, we obtain $2k - (k' + 2) \geq 2|F(Q[S])|$. Since $|F(Q[S])| \geq 2$, the equality in the inequality above must hold, and it implies that $Q[S]$ is a 2-cell embedding. Thus, $Q[S]$ has exactly two faces f_1 and f_2 with $\deg(f_1) = 10$ and $\deg(f_2) \in \{6, 8\}$. However, $\deg(f_2)$ must be at least 10 by the argument above, a contradiction. ■

Lemma 15. *Let G be an 8-regular* O1TG, *and let S be a 6-cut of G. Then $Q[S]$ is the union of two essential 3-cycles that are homotopic to each other.*

Proof. By Lemma 14, $Q[S]$ has at least two faces f_1 and f_2 that are not homeomorphic to 2-cells. Assume that $\partial f_1 = W_1 \cup \cdots \cup W_l$, where each W_i is a boundary component that is a closed walk in $Q[S]$. First, we suppose $l = 1$. In this case, ∂f_1 consists of the unique boundary component, and f_1 contains a handle. Under the condition, f_2 that is not a 2-cell face cannot exist. Thus, we conclude that $l \geq 2$.

Now we take a simple closed curve γ_i along W_i inside f_1 for each $i \in \{1, \ldots, l\}$. Suppose that γ_i is trivial on T^2 for some $i \in \{1, \ldots, l\}$, say γ_1 without loss of generality. In this case, note that $V(W_1) \cap V(W_i) = \emptyset$ for each $i \neq 1$. If the interior of γ_1 contains a vertex other than those of W_1, then, W_1 would become a smaller cut set, contrary to S being minimal. On the other hand, if the interior of γ_1 does not contain any vertex other than those of W_1, then it also concludes a contradiction since $V(W_2) \cup \cdots \cup V(W_l)$ is a smaller cut set of G. As a result, γ_i is not trivial on T^2 for each $i \in \{1, \dots, l\}$.

Since γ_1 and γ_2 do not have any intersection, they are homotopic to each other on T^2. That is, γ_1 can be shifted through f_1 and aligned to γ_2, and the trajectory itself forms an annular region. Furthermore, this implies that no other boundary component can exist in the annular region, and hence we have $l = 2$; observe that if there exists W_3 in the annular region, then γ_3 must be trivial, a contradiction. That is, the annular region is actually f_1.

Since G is simple, we have $|V(W_1)| \geq 3$ and $|V(W_2)| \geq 3$. Furthermore, if $V(W_1) \cap V(W_2) \neq \emptyset$, then the other non-2-cell face f_2 does not exist on T^2; observe that the same argument also holds for f_2, and we can take an essential simple closed curve in f_2. Thus, we have $|V(W_1)| = |V(W_2)| = 3$. Furthermore, each of W_1 and W_2 is a 3-cycle under the situation, and hence we have got our desired conclusion. Observe that f_2 also has the boundary components W_1 and W_2, which are 3-cycles. ■

Lemma 16. *Every* 8*-regular* O1TG *has no minimal* 7*-cut.*

Proof. Suppose, for a contradiction, that an 8-regular O1TG G has a minimal 7-cut S. Then, by Lemma 14, $Q[S]$ has at least two faces f_1 and f_2 that are not homeomorphic to 2-cells. Actually, the same argument as that in the previous proof holds, and hence, we have $\partial f_1 = W_1 \cup W_2$ with $|V(W_1)| \geq 3$ and $|V(W_2)| \geq 3$, and $V(W_1) \cap V(W_2) = \emptyset$. Without loss of generality, we may assume that $|V(W_1)| = 3$, that is, W_1 is a cycle of length 3. Even if W_2 is not a cycle, then W_2 contains an essential cycle C. (In the case where W_2 is not a cycle, we can divide W_2 into two shorter closed walks, one of which is essential. Otherwise, W_2 would become trivial. By repeating this division, we can eventually obtain an essential cycle. For example, see an example in Figure 2, where $W_2 = w_4w_5w_6w_7w_6w_4$ and $C = w_4w_5w_6w_4$.) Since $V(W_1) \cap V(W_2) = \emptyset$, W_1 and C are homotopic, and hence $|C| = 3$ by Proposition 6. Then $V(W_1) \cup V(C)$ would become a smaller cut, contrary to S being minimal. ■

As follows, 4-regular quadrangulations of T^2 are completely classified in [3], where $Q(p, r, q)$ is a commonly used notation that represents those graphs. (Every 4-regular quadrangulations of T^2 can be cut along edges into a $p \times q$ rectangle, and the parameter r represents the deviation when reconstructing the torus from the annulus with two boundary cycles of length q. See [3, 9, 10, 11, 12] for details.) In particular, q represents the length of the meridian cycle of $Q(p, r, q)$. Note that to keep $Q(p, r, q)$ simple, we need the condition $q \geq 3$ at least. (See the center of Figure 1. The figure depicts $Q(5, 0, 4)$.)

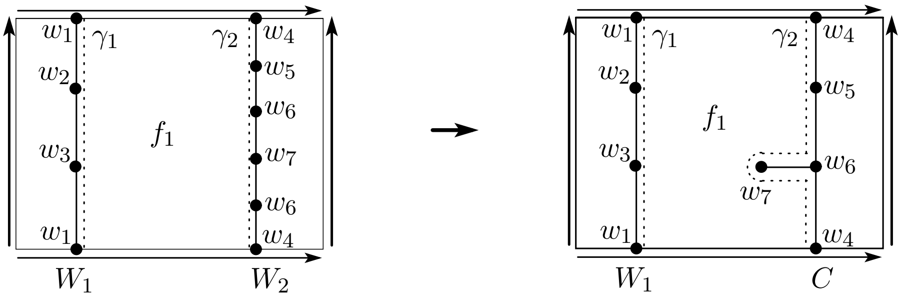

Figure 2. Homotopic closed walks W_1 and W_2 containing C.

Theorem 17 [3]**.** *Every 4-regular quadrangulation of T^2 is isomorphic to $Q(p, r, q)$ for some integers $p \geq 1$, $q \geq 3$ and $r \geq 0$.*

Now we prove our first main result in the paper.

***Proof of Theorem* 1.** By Lemmas 12 and 16, we have $\kappa(G) \in \{6, 8\}$. Then, we show the latter part of the theorem below. The necessity is trivial, and hence we discuss the sufficiency of the statement. Let G be an 8-regular O1TG that is not 8-connected. As above, we discuss the case of $\kappa(G) = 6$. Then there exists a 6-cut S of G. By Lemma 15, $Q[S]$ is the union of two essential 3-cycles that are homotopic to each other, and hence G is isomorphic to $Q(p, r, 3)$ ($p \geq 4$ and $r \geq 0$); observe that each annular region of $T^2 - Q[S]$ contains vertices of G, otherwise S would not become a cut of G. ■

4. Connectivity of Not 8-Regular O1TGs

In this section, we discuss the connectivity of not 8-regular O1TGs. At the beginning of the section, we prove the following theorem. (For the cases of the sphere and the projective plane, that is, for O1PGs and O1PPGs, the proofs are provided separately in [4] and [7], respectively.)

Theorem 18. *Every optimal 1-embedded graph on a closed surface is 4-connected.*

Proof. Let G be an optimal 1-embedded graph on a closed surface F^2. Suppose, for a contradiction, that G has a 3-cut S. Then, we have $|F(Q[S])| \geq 2$ by Lemma 8. Further, we have $|E(Q[S])| \geq 2|F(Q[S])|$ by Lemma 7. This is a contradiction since we clearly have $|E(Q[S])| \leq 3$. ■

The following is also a generalization of the results in [4, 7].

Lemma 19. *Let G be an optimal 1-embedded graph on F^2, and let S be a 4-cut of G. Then, one of the following holds.*

(I) *$Q[S]$ has exactly three 4-gonal faces each of which contains a vertex of G. In this case, $Q[S]$ has exactly six edges, and F^2 is nonorientable.*

(II) *$Q[S]$ has exactly two faces each of which contains a vertex of G. Furthermore, one of them is a 4-gonal face.*

Proof. We put $S = \{s_1, s_2, s_3, s_4\}$. By Lemma 8, we have $|F(Q[S])| \geq 2$. Furthermore, $|E(Q[S])| \geq 2|F(Q[S])|$ by Lemma 7, and hence we obtain $|F(Q[S])| \leq 3$; recall that $|V(Q[S])| = |S| = 4$, and then $|E(Q[S])| \leq 6$. First, we consider the case of $|F(Q[S])| = 3$. In this case, we immediately have $|E(Q[S])| = 6$ by the inequality above, and this implies that $Q[S]$ has exactly three 4-faces; that is, three 4-gonal faces. Let f_1, f_2 and f_3 denote such three 4-faces, and we may assume that $f_1 = s_1s_2s_3s_4$ without loss of generality. We may further assume that s_1s_2 is shared by two faces f_1 and f_2. If $f_2 = s_1s_2s_3s_4$, then $Q[S]$ must be a 4-cycle on the sphere by the simplicity of G, contradicting the existence of f_3. Thus, we have $f_2 = s_1s_2s_4s_3$, and F^2 contains a Möbius band that crosses edges s_1s_2 and s_3s_4. This is actually (I) in the statement.

Next, we consider the case of $F(Q[S]) = 2$. Let f_1 and f_2 be two faces of $Q[S]$. Suppose, for a contradiction, that $\deg(f_1) = \deg(f_2) = 6$ with $|E(Q[S])| = 6$; the other cases clearly admit a 4-face by a similar argument as above. Now, we consider the following two cases: Case (i) ∂f_1 has at least two boundary components, and Case (ii) ∂f_1 has exactly one boundary component.

We first discuss Case (i). In this case, we may assume that $\partial f_1 = W_1 \cup W_2$ with $V(W_1) = \{s_1, s_2, s_3\}$ and $V(W_2) = \{s_1, s_2, s_4\}$. Under the situation, a 4-cycle $s_1s_3s_2s_4s_1$ cuts off the region obtained from f_1 by identifying s_1s_2 from the other part, which is nothing but f_2, contradicting that f_2 has size 6.

Next, we assume Case (ii). By Lemma 9 and the simplicity of G, each s_i appears on ∂f_1 at most twice. Then, we may assume that $\partial f_1 = s_1s_2s_3s_1s_2s_4s_1$ without loss of generality; observe that the element of S that appears on ∂f_1 twice must be on the diagonal position. Similar to the previous case, the cycle $s_2s_3s_1s_4s_2$ is a surface separating 4-cycle; that is, $f_2 = s_2s_3s_1s_4$ is a 4-gonal face of $Q[S]$, a contradiction. ■

Next, we discuss minimal 5-cuts in O1TGs.

Lemma 20. *Let G be an* O1TG, *and let S be a minimal 5-cut of G. Then $Q[S]$ has a 6-face f such that $V(\partial f) = S$ and f contains a connected component of $G - S$.*

Proof. By Lemma 7, every face of $Q[S]$ has even size. Note that since S is minimal, $Q[S]$ has no 4-face that contains a connected component of $G - S$. By Lemma 8, $|F(Q[S])| \geq 2$ holds, and hence $Q[S]$ has at least two faces with size at least 6. Then the inequality $|F(Q[S])| \leq 3$ follows from (ii) of Lemma 10 with

$|S| = 5$, $\chi(F^2) = 0$, $p \geq 2$ and $q \geq 3$. By combining $|F(Q[S])| \leq 3$ and Euler's formula $5 - |E(Q[S])| + |F(Q[S])| \geq 0$, we obtain $|E(Q[S])| \leq 8$.

First, we consider the case of $|E(Q[S])| = 8$. Then all the equalities of inequalities above hold, and we have $|F(Q[S])| = 3$. Thus $Q[S]$ has the unique 4-face and two 6-faces under our assumption, and hence there exists a 6-face of $Q[S]$ that contains a connected component of $G - S$. If $|E(Q[S])| \leq 7$, then we have $|F(Q[S])| = 2$ since $Q[S]$ has at least two faces with size at least 6. In the same way, we can easily find our desired 6-face; note that $|E(Q[S])| \geq 6$ by our assumption. ■

Now we prove our second main result in the paper.

***Proof of Theorem* 2.** First, we show (i) in the statement. By Theorem 18, the sufficiency is trivial, and hence we discuss the necessity below. Let S be a 4-cut of an O1TG G. By Lemma 19, $Q[S]$ contains a 4-gonal 2-cell face containing a vertex since the surface is T^2 now; observe that T^2 is orientable, and that any surface separating cycle is trivial on T^2. Therefore, (i) of the statement holds.

Next, we show (ii). First, we discuss the sufficiency. We put $C_1 = xy_1z_1x$ and $C_2 = xy_2z_2x$, which are two homotopic cycles in the statement. By Proposition 6, both C_1 and C_2 are essential. Firstly, consider the 2-cell region R bounded by $C_1 \cup C_2$. We may assume that the boundary closed walk of R is $xy_1z_1xz_2y_2x$. If R contains no vertex of G, then R has the unique diagonal edge of $Q(G)$, say y_1z_2 without loss of generality; that is, R contains exactly two faces of $Q(G)$. However in this case, a crossing edge xz_2 in R and a non-crossing edge xz_2 on the boundary of R would form multiple edges, a contradiction. Similarly, the other region, which is outside of R, must contain a vertex of G. Therefore, $\{x, y_1, y_2, z_1, z_2\}$ is a 5-cut of G.

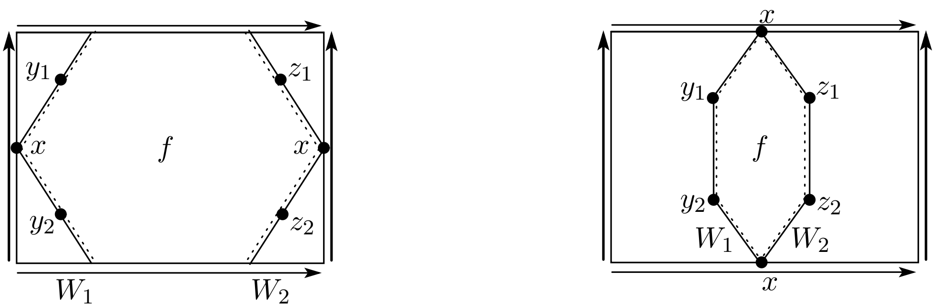

Figure 3. Homotopic 3-cycles W_1 and W_2.

Secondly, we discuss the necessity, by putting a minimal 5-cut $S = \{x, y_1, y_2, z_1, z_2\}$. By Lemma 20, $Q[S]$ has a 6-face f such that $V(\partial f) = S$ and f contains a connected component of $G - S$. Similar to the proof of Lemma 19, we consider the following two cases. Case (i) ∂f has at least two boundary components, and Case (ii) ∂f has exactly one boundary component. First, we discuss Case (i). In this case, we may assume that $\partial f = W_1 \cup W_2$ with $V(W_1) = \{x, y_1, y_2\}$ and $V(W_2) = \{x, z_1, z_2\}$; note that ∂f has exactly two connected components, otherwise f would not become a 6-face. Then, we clearly have our desired two 3-cycles; by Proposition 6 and by our former arguments. Under the situation, observe that the region bounded by $xy_1y_2xz_2z_1x$ other than f is homeomorphic to a 2-cell, where $W_1 = xy_1y_2x$ and $W_2 = xz_1z_2x$ are homotopic to each other in this direction (see the left-hand side of Figure 3).

Next, we consider Case (ii). We may assume that only x appears exactly twice on ∂f; since $|V(\partial f)| = 5$ now. By Lemma 9 and the simplicity of G, x must appear on the diagonal position of ∂f. Then, we may assume that $\partial f = xy_1y_2xz_2z_1x$, and hence we have our desired two 3-cycles also in this case; indeed using Proposition 6 (see the right-hand side of Figure 3). ■

5. Matching Extendability of O1TGs

Let G be an optimal 1-embedded graph on F^2, and let W be a closed walk consisting of only non-crossing edges that bounds a 2-cell region D. If D contains an odd number of vertices, then we call D an *odd weighted region*. In particular, if W is a cycle, then W is a *barrier cycle*. A barrier cycle of length k is called a *barrier k-cycle*.

The following lemma is a generalization of Lemma 2.3 in [14] that is often used in matching theory; this can be easily derived from Tutte's 1-factor theorem.

Lemma 21 [4]**.** *Let G be a graph of even order and $\{e_1, \dots, e_{m+1}\}$ be a matching of G which is not extendable. Then there exists $S \subseteq V(G)$ such that*

(i) $S \supset \bigcup_{i=1}^{m+1} V(e_i)$, *and*
(ii) $|S| \leq C_o(G - S) + 2m$.

In fact, by assuming m-extendability of a graph G, the equality of (ii) in Lemma 21 holds (see e.g., [4]). In the remaining part of the section, we prove the following three main results concerning matching extendability, using tools proved in the former sections.

***Proof of Theorem* 3.** First, we show the necessity. Suppose that an O1TG G contains a subgraph H in the statement. Since H is a quadrangulation of T^2, $|V(H)| = |F(H)|$ holds, and hence we have $|V(H)| = C_o(G - H)$ by the assumption. Let e be an edge in H. Then $G' = G - V(e)$ has a cut $V(H) \setminus V(e)$ such that $G' - (V(H) \setminus V(e))$ has exactly $|V(H)|$ odd components. Thus G'

does not have a perfect matching by Tutte's 1-factor theorem. That is, G is not 1-extendable.

Next, we discuss the sufficiency. Let G be an O1TG that is not 1-extendable, and assume that e is an edge of G that is not extendable. Then there exists $S \subset V(G)$ such that (i) $V(e) \subset S$ and (ii) $|S| \leq C_o(G-S)$ by Lemma 21 for $m = 0$. Now we consider $Q[S]$ on T^2. By Lemma 11, we have $2|F(Q[S])| \geq |E(Q[S])|$ with $m = 0$ and $\chi(F^2) = 0$. On the other hand, $|E(Q[S])| \geq 2|F(Q[S])|$ holds by Lemma 7, and hence we obtain $2|F(Q[S])| = |E(Q[S])|$. Actually, Lemma 11 is obtained by using Euler's formula, and this equality implies that $Q[S]$ is a 2-cell embedding, and it further indicates that $Q[S]$ is a quadrangulation of T^2. It is well-known that the number of vertices equals the number of faces of a quadrangulation of T^2, and hence we have $|S| = |F(Q[S])|$. Then, $|F(Q[S])| \leq C_o(G-S)$ by (ii) above, and thus $|F(Q[S])| = C_o(G-S)$ by Lemma 8. Hence each face of $Q[S]$ contains the unique odd component of $G - S$, that is, each face of $Q[S]$ is an odd weighted region. Therefore, $Q[S]$ is a subgraph H in the statement. ■

***Proof of Theorem* 4.** Any two independent edges on a barrier 4-cycle are not extendable, and hence the necessity clearly holds. Thus, we discuss the sufficiency of the statement below. Let G be an O1TG that is not 2-extendable, and assume that e_1 and e_2 are independent edges of G that are not extendable. Then there exists $S \subset V(G)$ such that (i) $V(e_1) \cup V(e_2) \subset S$ and (ii) $|S| \leq C_o(G-S)+2$ by Lemma 21 for $m = 1$.

Now we consider $Q[S]$ on T^2. By Lemma 11 with $m = 1$, we have $2|F(Q[S])| + 2 \geq |E(Q[S])|$. This inequality with Lemma 7 implies that there exist at most two faces of $Q[S]$ with size at least 6; that is, the others are all 4-faces. If $|S| \geq 5$, that is, $C_o(G-S) \geq 3$, then there exists a 4-gonal face f of $Q[S]$ that contains an odd component of $G - S$. Since any surface separating closed curve on T^2 is trivial, and hence, ∂f is our desired barrier 4-cycle. On the other hand, if $|S| = 4$, then there is also a barrier 4-cycle by (II) of Lemma 19 since T^2 is an orientable closed surface. Thus, we are done. ■

We use the following lemma in the proof of Theorem 5.

Lemma 22. *Let G be a $2q$-connected optimal 1-embedded graph on F^2 with $q \geq 3$ and let $\{e_1, \ldots, e_{m+1}\}$ be a matching of G which is not extendable. Then we have the following*

$$C_o(G-S) \leq \frac{2m - \chi(F^2)}{q-2}.$$

Proof. Then there exists $S \subset V(G)$ such that (i) $S \supset \bigcup_{i=1}^{m+1} V(e_i)$ and (ii) $|S| \leq C_o(G-S) + 2m$ of Lemma 21. Now we consider $Q[S]$ on F^2. Since G is $2q$-connected, $Q[S]$ has at least $C_o(G-S)$ faces with size at least $2q$ by

Lemma 8. By (ii) of Lemma 10, $|S| - \chi(F^2) + (2-q)C_o(G-S) \geq |F(Q[S])|$ holds, and thus $C_o(G-S) + 2m - \chi(F^2) + (2-q)C_o(G-S) = 2m - \chi(F^2) + (3-q)C_o(G-S) \geq |F(Q[S])|$. Since $|F(Q[S])| \geq C_o(G-S)$ by Lemma 8, we have $2m - \chi(F^2) \geq (q-2)C_o(G-S)$, and hence the inequality in the statement holds. ■

***Proof of Theorem* 5.** The necessity is trivial and hence we prove the sufficiency of the statement; recall the observation in the introduction. Let G be an 8-regular O1TG. Suppose, for a contradiction, that G is not 3-extendable, and assume that $M = \{e_1, e_2, e_3\}$ is a matching that is not extendable. Then there exists $S \subset V(G)$ such that (i) $V(M) \subset S$ and (ii) $|S| \leq C_o(G-S) + 4$ by Lemma 21 for $m = 2$. Since G is 8-regular, $\kappa(G) \in \{6, 8\}$ by Theorem 1. First, suppose that $\kappa(G) = 8$. Then the inequality $C_o(G-S) \leq 2$ holds by Lemma 22 with $q = 4$, $m = 2$ and $\chi(T^2) = 0$, and hence we have $|S| \leq 6$ by (ii) above. This contradicts $\kappa(G) = 8$.

Hence we assume that $\kappa(G) = 6$ and hence $Q(G)$ is isomorphic to $Q(p, r, 3)$ ($p \geq 4$ and $r \geq 0$) by Theorem 1. Then the inequality $C_o(G-S) \leq 4$ holds by Lemma 22, with $q = 3$, $m = 2$ and $\chi(T^2) = 0$, and hence $|S| \leq 8$ holds by (ii) above. First, we consider the case of $|S| = 6$. By Lemma 15, $Q[S]$ is the union of two essential 3-cycles C_1 and C_2 that are homotopic to each other. Since $S = V(M)$, one edge in M joins vertices of C_1 and C_2. This implies that one of the two annular faces of $Q[S]$ does not contain a vertex of G, contradicting Lemma 8.

Next, we discuss the case of $|S| = 7$. By Lemma 16, S contains a 6-cut S', and we put $S \setminus S' = \{v\}$. By Lemma 15 again, $Q[S']$ is the union of two essential 3-cycles. Let H_1 and H_2 be connected components of $G - S'$. Under the situation, $H_i' = Q[V(H_i)]$ is a Cartesian product of a 3-cycle and a k-path ($k \geq 0$) for each $i \in \{1, 2\}$. Then each of H_1' and H_2' is either a 3-cycle (if $k = 0$) or 3-connected (if $k \geq 1$). Thus, $H_i - v$ is connected for i such that $v \in V(H_i)$. Therefore $C_o(G-S) \leq 2$ holds, contradicting $C_o(G-S) \geq 3$ obtained from (ii) above. Finally, we consider the case of $|S| = 8$. Similarly, by (ii) above, we have $C_o(G-S) \geq 4$, that is, $C_o(G-S) = 4$. By the argument in the proof of Lemma 22, all the equalities hold, and hence $Q[S]$ is a 2-cell embedding, and has exactly four faces, each of which is 6-gonal, and contains an odd component of $G - S$. This clearly contradicts Lemma 13. ■

6. Remarks

In this paper, we have discussed connectivity and matching extendability of O1TGs. We aim to investigate similar problems on optimal 1-embedded graphs on the Klein bottle, and further extend our discussion to such graphs on general closed surfaces. Actually, we proved some results that work for optimal 1-embedded graphs on general closed surfaces, for example, Theorem 18, Lemmas 19 and 22. Furthermore, there are facts that do not hold for general closed surfaces but do apply to optimal 1-embedded graphs on the torus and the Klein bottle, that is, closed surfaces with Euler characteristic 0. For example, Lemmas 12, 14 and 20 are among them, and further, we can confirm that the assertion of Theorem 3 also holds for optimal 1-embedded graphs on the Klein bottle. These results are expected to be very helpful for our future research. However, on the other hand, there are many claims that require us to re-examine the discussion using properties specific to the Klein bottle. Specifically, it is known that the Klein bottle has a non-trivial surface separating simple closed curve, which is known as an *equator*, and understanding its impact on our problem seems to be the first step of our future work.

We believe that, at least, every optimal 1-embedded graph on the Klein bottle has no minimal 7-cut. In other words, we propose the following conjecture, which is similar to the proposition for O1TGs.

Conjecture 23. *Every* 8*-regular optimal* 1*-embedded graph G on the Klein bottle has connectivity either* 6 *or* 8*.*

Similar to the case of the torus, 4-regular quadrangulations on the Klein bottle are well classified in [10], and it would be even better if we could describe our claims using those classifications. Furthermore, we would like to investigate the matching extendability of optimal 1-embedded graphs on the Klein bottle as a direction for future work. In particular, we know that there exists an 8-regular optimal 1-embedded graph on this surface that is not 3-extendable.

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