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k-TUPLE DOMINATION IN KNESER GRAPHS[1]

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Abstract

Given a positive integer k, a k-tuple dominating set of a graph G is a subset of vertices $D \subseteq V(G)$ such that every vertex of G has at least k neighbors in D. The k-tuple domination number of G, denoted $\gamma_{\times k}(G)$, is the minimum cardinality of a k-tuple dominating set of G. In this paper we determine all the minimum k-tuple dominating sets for the Kneser graphs $K(n, r)$ with n large enough with respect to r. In addition, we relate k-tuple dominating sets and 2-packings in Kneser graphs, and we compute the 2-packing number of $K(3r-2, r)$ for $r \geq 3$. Finally, we obtain minimum sized k-tuple dominating sets of $K(n, 2)$ for $n \geq \Theta(\sqrt{k})$.

Keywords: Kneser graphs, multiple domination, k-tuple domination, 2-packings.

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1. Introduction

Given a simple graph G, let $N_G(v)$ denote the *open neighbourhood* of a vertex v in G and $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighbourhood* of v in G. When the graph G is clear from the context, we may omit the subscripts and simply write $N(v)$ and $N[v]$. Furthermore, let $\delta(G)$ be the *minimum degree* among all the vertices of graph G. A *dominating set* in G is a subset $D \subseteq V(G)$ such that every vertex $v \in V(G)$ verifies that $|N[v] \cap D| \geq 1$. The *domination number* of G, denoted

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$\gamma(G)$, is the minimum cardinality of a dominating set in G. Domination in graphs has been extensively studied in graph theory, and there is rich literature on this subject (see e.g. [15, 16, 17]).

Some of the most studied variations of domination introduced an integer k, such as k-tuple domination [6, 7, 11, 25]. Formally, given a graph G and a positive integer k, a set $D \subseteq V(G)$ is called a *k-tuple dominating set* of G if for every vertex $v \in V(G)$, we have $|N_G[v] \cap D| \geq k$. The *k-tuple domination number* of G is the minimum cardinality of a k-tuple dominating set of G, and is denoted by $\gamma_{\times k}(G)$. A *$\gamma_{\times k}$-set* is a k-tuple dominating set with cardinality $\gamma_{\times k}(G)$. The k-tuple domination number is only defined for graphs with $k \leq \delta(G) + 1$. An excellent brief survey on k-tuple domination appears in the book Topics in Domination in Graphs [15], as part of a chapter devoted to the study of multiple domination. Liao and Chang [21] studied the problem from an algorithmic point of view and proved that, fixed k, determining the k-tuple domination number is NP-complete (even for split graphs and for bipartite graphs). Considering these unfavorable outcomes, exploring how this parameter behaves within classes of graphs exhibiting a nice combinatorial structure, like Kneser graphs, represents a natural direction for further research.

For positive integers $r \leq n$, we denote by $[r..n]$ and $[n]$ the sets $\{r, \ldots, n\}$ and $\{1, \ldots, n\}$, respectively. If $n \geq 2r$, the *Kneser graph* $K(n, r)$ has as vertices the r-subsets of $[n]$ and two vertices are adjacent in $K(n, r)$ if and only if they are disjoint. This class of graphs gained prominence due to the Erdös-Ko-Rado theorem [10], which determined the independence number of the Kneser graph $K(n, r)$ to be $\binom{n-1}{r-1}$, as a result on extremal combinatorics. Lovász's proof of Kneser's conjecture [20, 22], later complemented by Matoušek's combinatorial proof [23], provided a determination of the chromatic number of Kneser graphs. Numerous other graph invariants have been investigated in Kneser graphs. One of them is the domination number [12, 14, 18, 24]. In particular, the value of the domination number of $K(n, r)$ is determined for n large enough in [24].

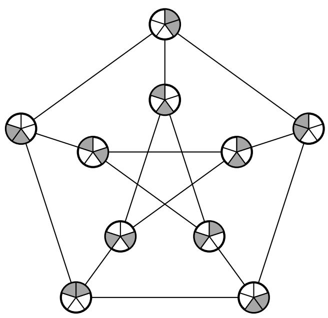

Figure 1. Kneser graph $K(5, 2)$.

Theorem 1 [24, Theorem 2.2]**.** *If $n \geq r^2 + r$, then $\gamma(K(n,r)) = r + 1$.*

It is also known that $\gamma(K(n,2)) = 3$ for $n \geq 4$ [18]. Furthermore, for the remaining cases of the domination number of Kneser graphs partial results are provided, but a complete solution for the domination number, similar to the case of chromatic and independence numbers, has not yet been achieved. After these results, different variations of domination have been explored in Kneser graphs. For instance, Grundy domination numbers and the related zero forcing numbers [4], the total dominator chromatic number [19] and Roman domination graph invariants [28]. In addition, another kind of multiple domination, called *k-domination*, has been studied and results on k-domination in Kneser graphs appear in [5] after a first version of the present paper [9] was released.

As we have mentioned, in this work we focus on the study of k-tuple domination in Kneser graphs. In Section 2 we extend the result in Theorem 1 (notice that $\gamma_{\times 1}(G) = \gamma(G)$) by proving that $\gamma_{\times k}(K(n,r)) = k + r$ if and only if $n \geq r(k+r)$ and we characterize the $\gamma_{\times k}$-sets for these cases. Besides, we prove that $\gamma_{\times k}(K(n,r))$ is not decreasing with respect to n, so we conclude that $\gamma_{\times k}(K(n,r))$ is at least $k + r + 1$ if $n < r(k+r)$. In addition, we calculate $\gamma_{\times k}(K(n,r))$ for k large enough. In Section 3 we analyze k-tuple dominating sets of Kneser graphs, from its relationship with 2-packings in graphs. We show that 2-packings in Kneser graphs are closely connected to intersecting set families. Using this link, we compute the 2-packing number of $K(n,r)$ for $n = 3r - 2$ and $r \geq 3$. Then we obtain $\gamma_{\times k}$-sets for the odd graphs $K(7,3)$ and $K(11,5)$ for every k, from the Steiner systems $S(2,3,7)$ (Fano plane) and $S(4,5,11)$, respectively. Furthermore, in the case of $K(7,3)$, we give all the $\gamma_{\times k}$-sets. Finally, in Section 4 we give a characterization of the k-tuple dominating sets for $K(n,2)$ that lead us to provide $\gamma_{\times k}(K(n,2))$ for $k \leq a\,\frac{n-3}{4}$, with $a = n - 4$ if n is even and $a = n - 6 + (n \bmod 4)$ if n is odd. We also design an ILP formulation to obtain some k-tuple dominating sets whose cardinality meet the lower bounds given previously. Table 1 shows the values of $\gamma_{\times k}(K(n,2))$ for $n \leq 26$ and $k \leq 60$.

2. Monotonicity and Results for n Large Enough

Similarly to other works on domination in Kneser graphs, we use $\gamma_{\times k}(n,r)$ to denote the k-tuple domination number of the Kneser graph $K(n,r)$. Since Kneser graphs are regular graphs and each vertex in $K(n,r)$ has degree $\binom{n-r}{r}$, then the Kneser graph $K(n,r)$ can only have a k-tuple dominating set if $k \leq \delta(K(n,r)) + 1 = \binom{n-r}{r} + 1$. Furthermore, if $k = \binom{n-r}{r} + 1$, the entire set of vertices in $K(n,r)$ is the only k-tuple dominating set. The Kneser graph $K(n,1)$ is isomorphic to the complete graph K_n. In this case, $\gamma_{\times k}(n,1) = k$ for $k \in [n]$. On the other hand, if $n = 2r$, then the Kneser graph $K(2r,r)$ is isomorphic to $\frac{1}{2}\binom{2r}{r}$ copies of K_2. It follows that $\gamma_{\times 1}(2r,r) = \frac{1}{2}\binom{2r}{r}$, $\gamma_{\times 2}(2r,r) = \binom{2r}{r}$, and no k-tuple dominating set exists for these graphs if $k \geq 3$. Thus, for the subsequent discussion, we consider $n \geq 2r+1$ and $r \geq 2$.

The known general bounds for k-tuple domination number of graphs [6, 7, 11, 25] are not efficient to determine the $\gamma_{\times k}(n,r)$ for $n \geq 2r+1$. Moreover, some of these upper bounds applied to $\gamma_{\times k}(n,r)$ are increasing with respect to n whereas, as we will prove in Theorem 2, the parameter $\gamma_{\times k}(n,r)$ is decreasing with respect to n.

Note that if D is a $(k+1)$-tuple dominating set of a graph G and $v \in D$, then $D \setminus \{v\}$ is a k-tuple dominating set ($k \geq 1$). Therefore, for any graph G, we have $\gamma_{\times k}(G) < \gamma_{\times (k+1)}(G)$ whenever $\delta(G) \geq k-1$. Then it turns out that if $n \geq 2r+1$ and $2 \leq k \leq \binom{n-r}{r}$, we have $\gamma_{\times k}(n,r) < \gamma_{\times (k+1)}(n,r)$, i.e., $\gamma_{\times k}(n,r)$ is strictly increasing with respect to k. In addition, in general, there is no monotonicity for the k-tuple domination number with respect to induced subgraphs. However, in the case of Kneser graphs, if $2r \leq m < n$, $K(m,r)$ is an induced subgraph of $K(n,r)$ and we will show that $\gamma_{\times k}(m,r) \geq \gamma_{\times k}(n,r)$ for every $k \geq 2$ if $k \leq \binom{m-r}{r}+1$. This is, for each fixed $k \geq 2$, $\gamma_{\times k}(n,r)$ is decreasing with respect to n.

Theorem 2. *If D is a k-tuple dominating set of $K(n,r)$ with $k \geq 2$, then D is a k-tuple dominating set of $K(n+1,r)$. In consequence, we have $\gamma_{\times k}(n+1,r) \leq \gamma_{\times k}(n,r)$ for every n.*

Proof. Let D be a k-tuple dominating set of $K(n,r)$. In order to see that D is a k-tuple dominating set of $K(n+1,r)$, it is enough to show that every vertex u of $K(n+1,r)$ containing the element $n+1$ satisfies

$$|N_{n+1}[u] \cap D| \geq k,$$

where $N_t[u]$ denotes the closed neighborhood of the vertex u in the graph $K(t,r)$.

Let $\tilde{u}$ be a $(r-1)$-subset of $[n]$ and $u = \tilde{u} \cup \{n+1\}$. Let us define, for $b \in [n] \setminus \tilde{u}$, $u_b = \tilde{u} \cup \{b\}$. Note that if $b \in [n] \setminus \tilde{u}$, then

$$|N_{n+1}[u] \cap D| = |\{w \in D : |w \cap \tilde{u}| = 0\}| \geq |N_n(u_b) \cap D| \geq \begin{cases} k-1, & \text{if } u_b \in D, \\ k, & \text{if } u_b \notin D. \end{cases}$$

Therefore, if there exists $b \in [n] \setminus \tilde{u}$ such that $u_b \notin D$, then $|N_{n+1}[u] \cap D| \geq k$.

Otherwise, consider u_b for some $b \in [n] \setminus \tilde{u}$. We have $u_b \in D$ and in consequence, $|N_n(u_b) \cap D| \geq k-1$. Consider $z \in N_n(u_b) \cap D$, and let x be an element from z. Now, consider the vertex u_x. Note that $w \cap \tilde{u} = \emptyset$ for every $w \in N_n(u_x) \cap D$. Besides, $\tilde{u} \cap z = \emptyset$ since $z \in N_n(u_b)$. It follows that

$$|N_{n+1}[u] \cap D| = |\{w \in D : |w \cap \tilde{u}| = 0\}| \geq |N_n(u_x) \cap D| + 1 \geq k.$$

Therefore D is a k-tuple dominating set of $K(n+1,r)$. ■

In Theorem 1 it is shown that if $n \geq r^2 + r$, then $\gamma_{\times 1}(n,r) = \gamma(K(n,r)) = r+1$. Moreover, for every k we state the following result.

Lemma 3. *Let n, k, and r be positive integers such that $n \geq r(k+r)$. Then $\gamma_{\times k}(n,r) = k+r$.*

Proof. Let $D = \{u_1, \ldots, u_{k+r}\}$ be a set of vertices of $K(n,r)$ such that $u_i \cap u_j = \emptyset$ for all $i \neq j$. This is possible since $n \geq r(k+r)$. Consider a vertex u of $K(n,r)$. If $u \in D$, then there exists $i \in \{1, \ldots, k+r\}$ such that $u = u_i$. Since $u \cap u_j = \emptyset$ for all $j \neq i$, we have $D \setminus \{u_i\} \subseteq N(u)$. Thus,

$$|D \cap N[u]| = |D| = k + r \geq k.$$

If $u \notin D$, we have $|\{i : u_i \cap u \neq \emptyset\}| \leq r$. So

$$|D \cap N[u]| \geq |D| - r = k.$$

Therefore D is a k-tuple dominating set of $K(n,r)$ with cardinality $k+r$, and $\gamma_{\times k}(n,r) \leq k+r$.

Now, assume that there exists a k-tuple dominating set D with cardinality $|D| = k+r-1$. Let us consider r distinct vertices $u_1, \ldots, u_r$ of D. Let $a_1 \in u_1$ and for each $2 \leq i \leq r$ we choose $a_i \in u_i$ such that $a_i \notin \{a_1, \ldots, a_{i-1}\}$. Let $w = \{a_1, \ldots, a_r\}$. If $w \neq u_i$ for every $i \in [r]$, then we have $w \cap u_i \neq \emptyset$, and so $u_i \notin N[w]$ for all $i \in [r]$. It turns out that

$$|D \cap N[w]| \leq |D| - r \leq k - 1,$$

contradicting the fact that D is a k-tuple dominating set.

If $w = u_j$ for some j, note that we can choose $b \in [n] \setminus (\bigcup_{i=1}^{r} u_i)$. This is possible since $|\bigcup_{i=1}^{r} u_i| \leq n-1$. In fact, $u_j \subseteq \left(\bigcup_{i \neq j} u_i\right) \cup \{a_j\}$ and then

$$\left|\bigcup_{i=1}^{r} u_i\right| \leq \left|\bigcup_{i \neq j} u_i\right| + 1 \leq r(r-1) + 1 < r(k+r) \leq n.$$

Let $w' = [w \setminus \{a_j\}] \cup \{b\}$. We have $w' \neq u_i$ for every $i \in [r]$. Besides, $a_i \in w' \cap u_i$ for $i \neq j$, and $w' \cap u_j = w \setminus \{a_j\}$. Therefore $w' \cap u_j \neq \emptyset$. Thus, $u_i \notin N[w']$ for all $i \in [r]$. Then

$$|D \cap N[w']| \leq |D| - r \leq k - 1,$$

which contradicts the fact that D is a k-tuple dominating set.

Therefore, it does not exist a k-tuple dominating set of $K(n,r)$ with cardinality less than $k+r$. We conclude that $\gamma_{\times k}(n,r) = k+r$. ■

Note that the condition $n \geq r(k+r)$ in Lemma 3 guarantees the existence of a set of $k+r$ pairwise disjoint vertices of $K(n,r)$. In the next result, we state that, unless $k=1$ and $r=2$, these set families are all the $\gamma_{\times k}$-sets in $K(n,r)$ for $n \geq r(k+r)$.

We introduce the following notation, which will be used throughout the paper. Given a set of vertices D in $K(n,r)$ and $x \in [n]$, the occurrences of the element x in D, denoted by $i_x(D)$, represent the number of vertices in D that contain the element x. In other words, $i_x(D)$ is the cardinality of the set $\{u \in D : x \in u\}$. For a positive integer a, we define $X_a(D)$ as the set of elements in $[n]$ such that their occurrences in D are equal to a, i.e., $X_a(D) = \{x \in [n] : i_x(D) = a\}$. Similarly, we define $X_{\overline{a}}^{\geq}(D) = \{x \in [n] : i_x(D) \geq a\}$, and $X_{\overline{a}}^{\leq}(D) = \{x \in [n] : i_x(D) \leq a\}$. When the set D is clear from the context, we shall omit it in the notation. It is important to note that the sum of the occurrences of all elements in D is equal to r times the cardinality of D, i.e., $\sum_{x \in [n]} i_x = r|D|$.

Lemma 4. *Let k and r be positive integers with $k \geq 2$ if $r=2$, and let $n \geq 2r+1$. If D is a k-tuple dominating set of $K(n,r)$ with cardinality $k+r$, then the vertices of D are pairwise disjoint.*

Proof. Let D be a k-tuple dominating set of $K(n,r)$ with cardinality $k+r$. Assume that the vertices in D are not pairwise disjoint. Let $a \in [n]$ such that $i_a = \max_{x \in [n]} i_x$. Under our assumption, $i_a \geq 2$.

If $i_a \geq 3$, let u_1, u_2, u_3 in D such that $a \in u_1 \cap u_2 \cap u_3$. Let us consider $u_4, \ldots, u_{r+1}$ vertices in $D \setminus \{u_1, u_2, u_3\}$. Let $b_1 = a$. For $j = 2, \ldots, r-1$, we choose $b_j \in u_{j+2} \setminus \{b_1, \ldots, b_{j-1}\}$. Let $b = \{b_1, \ldots, b_{r-1}\}$. Since $n \geq 2r+1$, it follows that $|[n] \setminus b| = n - (r-1) \geq r+2$. Thus, there exists $x \in [n]$ such that $w = b \cup \{x\} \neq u_j$ for $j = 1, \ldots, r+1$. As a result, for every j we have $w \cap u_j \neq \emptyset$ and $w \neq u_j$. Then,

$$|D \cap N[w]| \leq |D| - |\{u_1, \ldots, u_{r+1}\}| = |D| - (r+1) = k-1, \tag{1}$$

which contradicts the fact that D is a k-tuple dominating set. So, $i_a = 2$. Let u_1 and u_2 be the two vertices that contain the element a.

If $r > 2$, let $u_3, \ldots, u_{r+1}$ be vertices in $D \setminus \{u_1, u_2\}$. Let $b_1 = a$ and for $j = 2, \ldots, r$, we choose $b_j \in u_{j+1} \setminus \{b_1, \ldots, b_{j-1}\}$. Let $w = \{b_1, \ldots, b_r\}$. Note that b is a vertex that is not adjacent to any u_j for $j = 1, \ldots, r+1$. If $w \neq u_j$ for each j, then (1) holds, contradicting the fact that D is k-tuple dominating. Then $w = u_j$ for some j. Since the only vertices that contain the element $b_1 = a$ are u_1 and u_2, without loss of generality, assume $w = u_1$. Then we have $b_j \in u_1 \cap u_{j+1}$ for every $j = 1, \ldots, r$. Let $x \in u_2 \setminus w$ and let $w' = w \setminus \{b_1\} \cup \{x\} = \{x, b_2, \ldots, b_r\}$. Let us see that $w' \neq u_j$ and $w' \cap u_j \neq \emptyset$ for every j. In fact, as $b_j \in u_1 \cap u_{j+1}$ for $j = 2, \ldots, r$, then $w' \cap u_j \neq \emptyset$ for every $j \in 1, \ldots, r+1$. Moreover, if $w' = u_\ell$ for some ℓ, then since $a \notin w'$, $\ell \geq 3$ and for $j \neq \ell$, $j \geq 2$ we have that

$b_{j-1} \in u_1 \cap u_j \cap u_\ell$. Thus, $i_{b_{j-1}} \geq 3$ but $\max_{x \in [n]} i_x = 2$. Therefore, $w' \neq u_j$ for every j, and in consequence (1) holds for w'.

If $r = 2$, we have $u_1 = \{a, c\}$, $u_2 = \{a, d\}$ for some $c, d \in [n]$. Since $k \geq 2$, there exists a vertex $u_3 \in D \setminus \{u_1, u_2\}$ such that $u_3 \neq \{c, d\}$. Let $x \in u_3 \setminus \{c, d\}$. Since $i_a = 2$, $x \neq a$. Consider the vertex $w = \{a, x\}$. We have $w \neq u_j$ and $w \cap u_j \neq \emptyset$ for every j. Then (1) holds.

Either $r > 2$ or $r = 2$, we arrive at a contradiction since D is a k-tuple dominating set.

Therefore, we conclude that the vertices in D are pairwise disjoint. ■

This result does not hold when $k = 1$ and $r = 2$. In fact, in [18] it is shown that the dominating sets of $K(n, 2)$ for $n \geq 5$ are the sets of 3 vertices that are either pairwise disjoint or mutually intersecting.

As a by product of Lemmas 3 and 4 we have the following result for n large enough.

Theorem 5. *For $k \geq 2$, $\gamma_{\times k}(n, r) = k + r$ if and only if $n \geq r(k + r)$.*

In addition, from monotonicity on n we have the following.

Corollary 6. *For $k \geq 2$, $\gamma_{\times k}(n, r) \geq k + r$. Moreover, if $n < r(k + r)$, then $\gamma_{\times k}(n, r) \geq k + r + 1$.*

The remaining of this section is devoted to obtain $\gamma_{\times k}$-sets for $k = \binom{n-r}{r} - t$ when n is large enough with respect to both r and t. As we have mentioned, if $k = \binom{n-r}{r} + 1$, the only $\gamma_{\times k}$-set of $K(n, r)$ is the set of vertices itself. On the other hand, when $n \geq 3r - 1$, the diameter of the Kneser graph $K(n, r)$ is equal to 2 [27]. In these cases, any pair of vertices in $K(n, r)$ are adjacent or they have a common neighbor. Thus, for $k = \binom{n-r}{r}$ and $n \geq 3r - 1$, it follows that $\gamma_{\times k}(n, r) = \binom{n}{r} - 1$. In a similar way, we prove that for a positive integer t, when $n \geq (t + 3)r - \left\lceil \frac{t+2}{2} \right\rceil$, every set of $t + 2$ vertices in $K(n, r)$ is contained in the closed neighborhood of some vertex and in consequence, for $k = \binom{n-r}{r} - t$, it follows that $\gamma_{\times k}(n, r) = \binom{n}{r} - (t + 1)$.

Remark 7. Let $t \geq 2$, and let S be a set of t vertices of $K(n, r)$ such that each vertex $u \in S$ intersects at least one vertex in $S \setminus \{u\}$. It holds

$$\left| \bigcup_{v \in S} v \right| \leq tr - \left\lceil \frac{t}{2} \right\rceil . \tag{2}$$

Lemma 8. *Let $t \geq 2$. If $n \geq (t+1)r - \left\lceil \frac{t}{2} \right\rceil$ and S is a set of t vertices of $K(n, r)$, then there exists a vertex w of $K(n, r)$ such that $S \subseteq N[w]$.*

Proof. Let t, r, n be positive integers such that $t \geq 2$ and $n \geq (t+1)r - \lceil \frac{t}{2} \rceil$, and let S be a set of t vertices of $K(n,r)$. If there exists $w \in S$ such that $w \cap u = \emptyset$ for every $u \in S \setminus \{w\}$, the result holds. Now, assume that each vertex of S intersects at least another vertex in S. Thus, by Remark 7, it follows (2). Since $n \geq (t+1)r - \lceil \frac{t}{2} \rceil$, we have $\left|[n] \setminus \bigcup_{v \in S} v\right| = n - \left|\bigcup_{v \in S} v\right| \geq r$ and there exists at least one vertex $w \in [n] \setminus \bigcup_{v \in S} v$. Since $w \cap v = \emptyset$ for each $v \in S$, we have $S \subseteq N[w]$, and the statement holds. ■

Theorem 9. *For a nonnegative integer t, $n \geq (t+3)r - \lceil \frac{t+2}{2} \rceil$ and $k = \binom{n-r}{r} - t$, it holds $\gamma_{\times k}(n,r) = \binom{n}{r} - (t+1)$. Moreover, for every set $S \subseteq V(K(n,r))$ with cardinality $t+1$, $V(K(n,r)) \setminus S$ is a $\gamma_{\times k}$-set.*

Proof. Let S be a set of vertices of $K(n,r)$ with cardinality $t+2$. Let us show that the set $D = V(K(n,r)) \setminus S$ is not a k-tuple dominating set. In fact, by Lemma 8, we have that there exists a vertex $w \in V(K(n,r))$ such that $S \subseteq N[w]$. Thus, $|N[w] \cap D| = |N[w]| - |S| = \binom{n-r}{r} + 1 - (t+2) = k - 1$. In consequence, $V(K(n,r)) \setminus S$ is not a k-tuple dominating set of $K(n,r)$. Therefore, $\gamma_{\times k}(n,r) \geq \binom{n}{r} - (t+1)$.

On the other hand, let S be any set of vertices of $K(n,r)$ with cardinality $t+1$, and $D = V(K(n,r)) \setminus S$. Let $u \in V(K(n,r))$. We have

$$|N[u] \cap D| = |N[u]| - \underbrace{|N[u] \cap S|}_{\leq |S|} \geq \binom{n-r}{r} + 1 - (t+1) = k.$$

Thus, D is a k-tuple dominating set, and $\gamma_{\times k}(n,r) \leq |D| = \binom{n}{r} - (t+1)$.

Therefore, $\gamma_{\times k}(n,r) = \binom{n}{r} - (t+1)$. ■

To end this section, we remark that the lower bound for n in the previous theorem is tight. Note that if $n = (t+3)r - \lceil \frac{t+2}{2} \rceil - 1$ and $k = \binom{n-r}{r} - t$, then $\gamma_{\times k}(n,r) < \binom{n}{r} - (t+1)$. In fact, we may consider the following set S. For t even,

$$S = \left\{ [\xi+1..\xi+r], [\xi+r..\xi+2r-1], \, \xi = (x-1)(2r-1), \text{ with } x \in \left[\frac{t}{2}+1\right] \right\}$$

and for t odd,

$$S = \left\{ [\xi+1..\xi+r], [\xi+r..\xi+2r-1], \, \xi = (x-1)(2r-1), \text{with } x \in \left[\left\lceil\frac{t}{2}\right\rceil+1\right] \right\}$$
$$\cup \left\{ [\xi+1..\xi+r], [\xi+r..\xi+2r-1], [\xi+2r..\xi+3r-2], \, \xi = \left\lceil\frac{t}{2}\right\rceil (2r-1) \right\}.$$

In both cases, S is a set of $t+2$ vertices and $V(K(n,r)) \setminus S$ is a k-tuple dominating set. Thus, $\gamma_{\times k}(n,r) \leq \binom{n}{r} - (t+2)$.

3. 2-Packings and k-Tuple Dominating Sets in Odd Graphs

Packings and dominating sets have been extensively studied in graphs. Given a graph G, a subset $S \subseteq V(G)$ is called a 2-*packing* of G if for every pair of vertices $u, v \in S$, their closed neighborhoods satisfy $N[u] \cap N[v] = \emptyset$. In other words, no two vertices in the 2-packing have any common neighbor. The 2-*packing number* of a graph G, denoted by $\rho(G)$, is the maximum cardinality of a 2-packing in G, see e.g. [3, 8]. In [6], the author establishes a relationship between the 2-packing number and the k-tuple domination number.

Theorem 10 [6, Theorem 2.3]**.** *Let $k \geq 2$. For any graph G of order n and $\delta(G) \geq k$,*

$$k\rho(G) \leq \gamma_{\times k}(G) \leq n - \rho(G).$$

It is known that when $n \geq 3r-1$ with $r \geq 2$, the diameter of the Kneser graph $K(n,r)$ is equal to 2 [27]. Therefore, in these cases the 2-packing number of the Kneser graph $K(n,r)$ is equal to 1. Thus, for the remaining of this section, we consider $2r+1 \leq n \leq 3r-2$. We use $\rho(n,r)$ to denote the 2-packing number of the Kneser graph $K(n,r)$.

Let S be a 2-packing in $K(n,r)$. If u and v are two vertices in S, then $u \cap v \neq \emptyset$. Besides, notice that if $|u \cap v| > (3r-1) - n$, then

$$|[n] \setminus (u \cup v)| = |[n]| - |u| - |v| + |u \cap v| \geq r,$$

this implies that there exists a vertex $w \in N(u) \cap N(v)$, contradicting the fact that S is a 2-packing.

Remark 11. Let $2r+1 \leq n \leq 3r-2$. A set S of r-subsets of $[n]$ is a 2-packing of $K(n,r)$ if and only if for every pair $u, v \in S$, it holds that

$$1 \leq |u \cap v| \leq (3r-1) - n.$$

Some results on extremal combinatorics and intersecting families can be used to give upper bounds for the 2-packing number of Kneser graphs. In fact, from [26], we have

$$\rho(n,r) \leq \binom{n}{(3r-1)-n}$$

and

$$\rho(n,r) \leq \sum_{i=0}^{(3r-1)-n} \binom{n-1}{i}.$$

In the case $n = 3r-2$ both bounds remain

$$\rho(n,r) \leq n,$$

which is tight only for $r = 3$, as we will show in the next Theorem.

Theorem 12. *Let n and r be positive integers such that $r \geq 3$ and $n = 3r-2$. Then,*

$$\rho\,(n,r) = \begin{cases} 7, & \text{if } r = 3, \\ 5, & \text{if } r = 4, \\ 3, & \text{if } r \geq 5. \end{cases} \tag{3}$$

Proof. Let $r \geq 3$, $n = 3r-2$ and S a 2-packing of $K(n,r)$. From Remark 11, we have that for every pair of vertices $u, v \in S$, it holds $|u \cap v| = 1$.

Let us see that $i_x \leq 3$ for every $x \in [n]$. In fact, suppose $i_a \geq 4$ for some $a \in [n]$, and let u_1, u_2, u_3, u_4 be four vertices in S such that $a \in u_j$ for $j \in [4]$. We have $|u_j \setminus \{a\}| = r-1$ and $(u_j \setminus \{a\}) \cap (u_\ell \setminus \{a\}) = \emptyset$ for $j, \ell \in [4]$ with $j \neq \ell$. Then

$$n \geq \left|\bigcup_{j=1}^{4} u_j\right| = 1 + \sum_{j=1}^{4} |u_j \setminus \{a\}| = 4r-3 = n + \underbrace{r-1}_{>0},$$

which cannot be true. Thus, $i_x \leq 3$ for every $x \in [n]$. Then we have $|S| \leq \frac{3n}{r}$.

If $r = 3$, then $n = 7$ and the cardinality of a 2-packing S is at most 7. In fact, a set of 7 3-subsets of [7] which mutually intersect in exactly one element, is a Fano plane (see Figure 2). Thus, we have $\rho\,(7,3) = 7$.

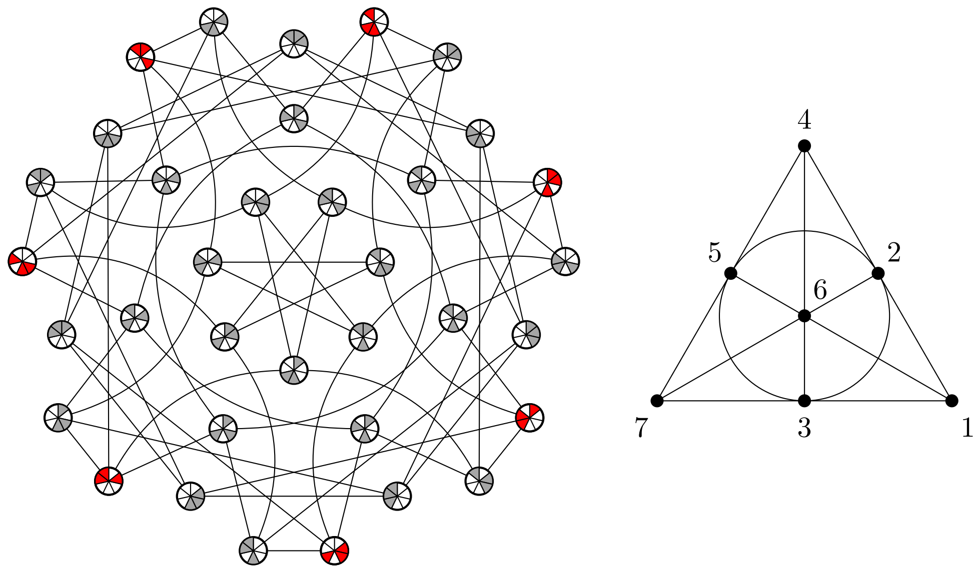

Figure 2. A maximum 2-packing in $K(7,3)$ and the corresponding Fano plane.

If $r \geq 4$, let us suppose that $i_a = 3$ for some $a \in [n]$ and let the vertices $u_1, u_2, u_3 \in S$ such that $a \in u_j$ for each j. Without loss of generality, let us consider $a = 1$ and

$$u_1 = \{1\} \cup [2..r], \qquad u_2 = \{1\} \cup [r+1..2r-1], \qquad u_3 = \{1\} \cup [2r..3r-2].$$

Suppose there exists another vertex w in S. $1 \notin w$ since i_1 is exactly 3. Then $w = \{a_1, \dots, a_r\}$ with $a_j \in [n] \setminus \{1\}$ for $j \in [r]$. Since $\{u_i \setminus \{1\}\}_{i=1}^{3}$ is a partition of $[n]\setminus\{1\}$, and $r \geq 4$, by the pigeonhole principle there exist at least two elements a_j and a_ℓ that belongs to u_k for some $k \in [3]$. This is, $|u_k \cap w| \geq 2$ but this leads to a contradiction with Remark 11.

Therefore, if there exists $a \in [n]$ such that $i_a = 3$, then $|S| \leq 3$. On the contrary, if $i_a \leq 2$ for every $a \in [n]$, we have $|S| \leq \frac{2n}{r}$.

If $r = 4$, then $n = 10$ and the cardinality of a 2-packing S is at most 5. In fact, we have $\rho\,(10, 4) = 5$ and if S is a maximum 2-packing of $K(10, 4)$, then, up to automorphism, we have

$$S = \big\{\{1,2,3,4\}, \{1,5,6,7\}, \{2,5,8,9\}, \{3,6,8,10\}, \{4,7,9,10\}\big\}.$$

If $r \geq 5$, suppose that $i_a \leq 2$ for every $a \in [n]$. Let $u_1, u_2, u_3 \in S$. Without loss of generality, we may assume that

$$u_1 = \{1,2\} \cup [4..r+1], \qquad u_2 = \{1,3\} \cup [r+2..2r-1], \qquad u_3 = \{2,3\} \cup [2r..3r-3].$$

The fourth vertex in S must contain exactly one element from each set u_1, u_2, u_3 and two elements from $[n] \setminus (u_1 \cup u_2 \cup u_3)$. However,

$$|[n] \setminus (u_1 \cup u_2 \cup u_3)| = (3r-2) - (3r-3) = 1.$$

We can conclude that $|S| \leq 3$, and $S = \{u_1, u_2, u_3\}$ is a 2-packing of maximum cardinality.

Therefore, if $r \geq 5$, $\rho\,(3r-2, r) = 3$. ■

As a by-product of the previous proof, we have the following result.

Corollary 13. *Let n and r be positive integers such that $r \geq 5$ and $n = 3r-2$. If S is a maximum 2-packing of $K(n, r)$, then, up to automorphism,*

$$S = \big\{\{1,2\} \cup [4..r+1], \{1,3\} \cup [r+2..2r-1], \{2,3\} \cup [2r..3r-3]\big\},$$

or

$$S = \big\{\{1\} \cup [2..r], \{1\} \cup [r+1..2r-1], \{1\} \cup [2r..3r-2]\big\}.$$

Let us notice that 2-packings in Kneser graphs are related to Steiner systems. A Steiner system $S(t, r, n)$ is a collection of r-subsets of $[n]$, called blocks, with the property that each t-subset of $[n]$ is contained in exactly one block. It is not hard to see that for $2r+1 \leq n \leq 3r-2$, if a Steiner system $S(3r-n, r, n)$ exists, then

$$\rho(n,r) \leq |S(3r-n,r,n)|.$$

This relationship is useful to study 2-packings in odd graphs. To this end, we consider perfect 1-codes in graphs [13]. A subset of vertices C of a graph G is a perfect 1-code of G if the family of closed neighbourhood $\{N[v]\}_{v\in C}$ is a partition of $V(G)$. Notice that if a perfect 1-code of a graph G exists, then it is also a maximum 2-packing of G. In particular, it is known that a set C is a Steiner system $S(r-1, r, 2r+1)$ if and only if C is a perfect 1-code in the odd graph $K(2r+1, r)$ [13]. Thus, the case $r = 3$ in Theorem 12 also follows from the fact that the Fano plane is a Steiner system $S(2, 3, 7)$. Similarly, since a Steiner system $S(4, 5, 11)$ exists [1], then the odd graph $K(11, 5)$ has a perfect 1-code and

$$\rho(11,5) = 66.$$

The well known conjecture due to Biggs [2] asserts that there is no perfect 1-code if $r \neq 3, 5$. Although it has been verified for some values of r, it has not yet been settled in general.

Notice that the existence of a perfect 1-code in the odd graph $K(2r+1, r)$ would give the following lower bound for the k-tuple domination number

$$\gamma_{\times k}(2r+1,r) \geq k\rho(2r+1,r) = \frac{\binom{2r+1}{r} k}{r+2},$$

which we will show is tight for $r = 3, 5$.

Lemma 14. *Let G be a vertex-transitive graph, and k a positive integer. If $\gamma_{\times k}(G) = k\rho(G)$ and D is a $\gamma_{\times k}$-set, then $|N[v] \cap D| = k$ for every $v \in V(G)$.*

Proof. Let S be a maximum 2-packing of G and D a $\gamma_{\times k}$-set of G. From definition $|N[u] \cap D| \geq k$ for every vertex $u \in V(G)$. On the other hand, the sets $\{N[u]\}_{u\in S}$ are pairwise disjoint. So, we have

$$\gamma_{\times k}(G) = |D| \geq \sum_{u\in S} \underbrace{|N[u] \cap D|}_{\geq k} \geq k|S| = k\rho(G) = \gamma_{\times k}(G). \tag{4}$$

In consequence, $|N[u] \cap D| = k$ for every $u \in S$.

Besides, let $v \in V(G)$. Since G is a vertex-transitive graph, it is possible to find a maximum 2-packing S' such that $v \in S'$. Thus, $|N[v] \cap D| = k$. ∎

As it is well-known that Kneser graphs are vertex-transitive graphs, we have that if $\gamma_{\times k}(n,r) = k\rho(n,r)$ and D is a $\gamma_{\times k}$-set of $K(n,r)$, then $|N[u] \cap D| = k$ for every $u \in V(K(n,r))$.

Remark 15. Let us notice that if D_k is a k-tuple dominating set of $K(n,r)$ with cardinality $|D_k| = k\rho(n,r)$, then the set $V(K(n,r)) \setminus D_k$ is a $\tilde{k}$-tuple dominating set of $K(n,r)$, with $\tilde{k} = \binom{n-r}{r} + 1 - k$. In fact, for every vertex $u \in V(K(n,r))$, by Lemma 14 we have

$$|N[u] \cap (V(K(n,r)) \setminus D_k)| = \underbrace{|N[u]|}_{\binom{n-r}{r}+1} - \underbrace{|N[u] \cap D_k|}_{k} = \binom{n-r}{r} + 1 - k.$$

It is known that there exist exactly two disjoint Fano planes. Thus the union of these two Fano planes is a $\gamma_{\times 2}$-set of $K(7,3)$. Moreover, we provide $\gamma_{\times k}$-sets of $K(7,3)$ for each $k \in [5]$.

Remark 16. For $r = 3$ and $n = 7$, from Theorem 12 it turns out that $\rho(7,3) = 7$. A bound for $\gamma_{\times k}(7,3)$ is given by Theorem 10 and we have $\gamma_{\times k}(7,3) \geq 7k$. In fact, let us see that this bound is tight, i.e., $\gamma_{\times k}(7,3) = 7k$ for $k \in [5]$.

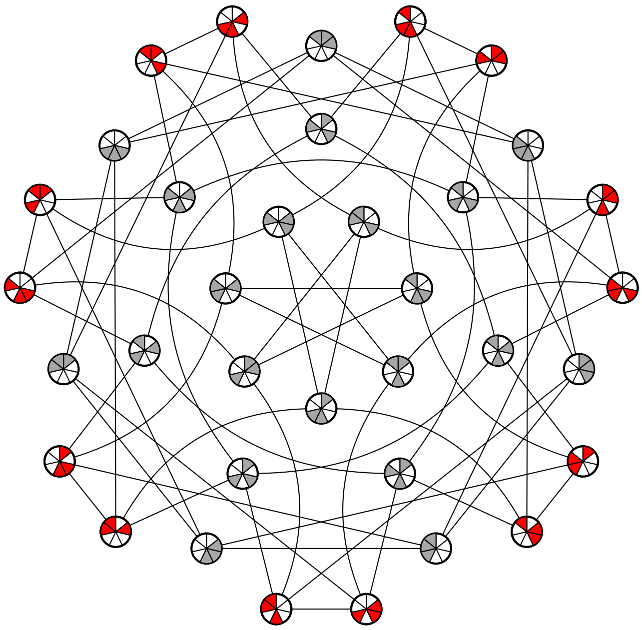

Figure 3. A $\gamma_{\times 2}$-set of $K(7,3)$.

To this end, it is enough to consider two disjoint Fano planes F_1, F_2 and the sets

$$D_1 = F_1, \qquad D_2 = F_1 \cup F_2, \qquad D_3 = V \setminus D_2, \qquad D_4 = V \setminus D_1, \qquad D_5 = V,$$

where $V = V(K(7,3))$. For each $k \in [5]$, D_k turns out to be a k-tuple dominating set with cardinality $|D_k| = 7k$ and in consequence is a $\gamma_{\times k}$-set.

In Remark 16 we provide $\gamma_{\times k}$-sets of $K(7,3)$. In fact, by exhaustive methods we can prove that these sets are unique up to automorphism. Regarding Steiner systems $S(4,5,11)$, it is also known that there are exactly two disjoint such Steiner systems. We know that the union of them is a $\gamma_{\times 2}$-set of $K(11,5)$. We also find, for $k \in \{3,4,5,6\}$, a $\gamma_{\times k}$-set of $K(11,5)$ with cardinality $66k = \rho\,(11,5)\,k$ (a construction of these sets can be found in [9]).

4. k-Tuple Domination in Kneser Graphs $K(n,2)$

The study of subsets of vertices satisfying certain restrictions in Kneser graphs $K(n,2)$ deserves particular attention, as this subclass of Kneser graphs has a remarkable structure. As we have mentioned, in [18] it is shown that $\gamma_{\times 1}\,(n,2) = 3$ for every $n \geq 4$. In this section we focus on the k-tuple domination number $\gamma_{\times k}\,(n,2)$ for $k \geq 2$. First, we provide a characterization of the k-tuple dominating sets of $K(n,2)$ in terms of the occurrences of the elements in $[n]$.

Lemma 17. *Let $n \geq 5$ and $D \subseteq V(K(n,2))$. Then, D is a k-tuple dominating set of $K(n,2)$ if and only if for every pair $a,b \in [n]$*

$$i_a + i_b \leq \begin{cases} |D| - k + 2, & \text{if } \{a,b\} \in D, \\ |D| - k, & \text{if } \{a,b\} \notin D. \end{cases}$$

Proof. Let D be a set of vertices of $K(n,2)$ and let $u = \{a,b\}$ be a vertex of $K(n,2)$. The number of vertices in D that contain either the element a or b is exactly $i_a + i_b - 1$ when $u \in D$ and $i_a + i_b$, otherwise.

On the one hand, if $u \in D$, then we have,

$$\begin{aligned} |D \cap N[u]| &= |D| - |\{v \in D : v \neq u \wedge v \cap u \neq \emptyset\}| \\ &= |D| - (|\{v \in D : a \in v \vee b \in v\}| - 1) \\ &= |D| - (i_a + i_b - 1) + 1 = |D| + 2 - (i_a + i_b). \end{aligned} \tag{5}$$

On the other hand, whether $u \notin D$ it holds

$$|D \cap N[u]| = |D| - |\{v \in D : a \in v \vee b \in v\}| = |D| - (i_a + i_b) = |D| - (i_a + i_b). \tag{6}$$

From (5) and (6) it turns out that D is a k-tuple dominating set if and only if for every pair of elements $a,b \in [n]$ it holds

$$\begin{cases} |D| + 2 - (i_a + i_b) \geq k, & \text{if } \{a,b\} \in D, \\ |D| - (i_a + i_b) \geq k, & \text{if } \{a,b\} \notin D, \end{cases}$$

i.e.,

$$i_a + i_b \leq \begin{cases} |D| - k + 2, & \text{if } \{a,b\} \in D, \\ |D| - k, & \text{if } \{a,b\} \notin D. \end{cases}$$

■

Remark 18. Note that if D is a k-tuple dominating set and $\{a,b\} \in D$, then from Lemma 17 together with the fact that $i_b \geq 1$, we have that for any $a \in [n]$, it holds $i_a \leq |D| - k + 1$.

From Theorem 5 with $r = 2$, we have that $\gamma_{\times k}(n,2) = k+2$ if and only if $n \geq 2(k+2)$. In addition, from monotonicity, it follows that for $k \geq 2$ and $n < 2(k+2)$, $\gamma_{\times k}(n,2) \geq k+3$. Moreover, in the following result we state that the only value of n for which this bound is tight is $n = 2k+3$. We introduce the following notation. Given a set A and a positive integer $r \leq |A|$, we denote by $\binom{A}{r}$ the set of all the r-subsets of A.

Theorem 19. *Let $k \geq 2$. Then $\gamma_{\times k}(n,2) = k+3$ if and only if $n = 2k+3$.*

Proof. Let $k \geq 2$ and $n \leq 2k+3$. Let us suppose that there exists a k-tuple dominating set $D = \{u_1, \ldots, u_{k+3}\}$ of $K(n,2)$ with cardinality $k+3$. We will prove that n must be equal to $2k+3$. Note that it is enough to prove that $n \geq 2k+3$. Let us consider a vertex u, and let $a \in u$. By Remark 18 we have $i_a \leq 4$. Moreover, we can prove the following.

Claim 20. *$i_a \leq 2$ for every $a \in [n]$.*

Proof. Suppose, on the contrary, that $i_a \geq 3$ for some $a \in [n]$.

Note that there exists $b \in [n] \setminus \{a\}$ such that $\{a,b\} \notin D$ and $i_b \geq 1$. In fact, if for every element $x \neq a$ with $i_x \geq 1$ we have $\{a,x\} \in D$, then there are at most $i_a + 1$ elements in $[n]$ that appear in vertices of D. Let x be an element different from a such that $i_x \geq 1$. It turns out that $\{a,x\} \in D$ and by Lemma 17 $i_a + i_x \leq 5$.

Then if $i_a = 4$, we have $i_x = 1$ and $k+3 = |D| = 4$ which contradicts that $k \geq 2$. On the other hand, if $i_a = 3$ then it turns out that $i_y \leq 2$ for every $y \neq a$ with $i_y \geq 1$, and we have

$$2(k+3) = 2|D| = \sum_{y=1}^{n} i_y \leq i_a + 2i_a = 9.$$

We also get $k+3 = 4$, which does not hold since $k \geq 2$.

Therefore, it is possible to choose $b \in [n]$ such that $i_b \geq 1$ and $\{a,b\} \notin D$. Thus $i_a \geq 3$, $i_b \geq 1$, and in consequence $i_a + i_b \geq 4$, which leads to a contradiction since by Lemma 17 we have $i_a + i_b \leq 3$.

We conclude $i_a \leq 2$ for all $a \in [n]$. □

Let $n_1 = |X_1|$ and $n_2 = |X_2|$. We have $n \geq n_2 + n_1$ and $2|D| = 2n_2 + n_1$. So

$$n \geq n_2 + n_1 = 2|D| - n_2 = 2(k+3) - n_2 = 2(k+2) + (2 - n_2).$$

Since $n < 2(k+2)$, it turns out that $n_2 \geq 3$. Let us see that n_2 is exactly 3.

Claim 21. $n_2 = 3$.

Proof. Suppose that a and b are two elements in $[n]$ such that $i_a = i_b = 2$, and let $v = \{a, b\}$. If $v \notin D$, by Lemma 17 $i_a + i_b \leq 3$ which leads to a contradiction. Thus, $v \in D$. If $n_2 \geq 4$, let us consider a_1, a_2, a_3, a_4 such that $i_{a_j} = 2$ for $1 \leq j \leq 4$. Then $\{a_1, a_j\} \in D$ for every $2 \leq j \leq 4$, and $i_{a_1} \geq 3$ which contradicts the fact that $i_{a_1} = 2$. Thus, $n_2 = 3$. □

Therefore we have $n \geq n_2 + n_1 = 2k + 3$. In consequence, it turns out that if $\gamma_{\times k}(n, 2) = k + 3$, then $n = 2k + 3$.

On the other hand, if $n = 2k+3$, then from Theorem 5 and monotonicity, we have $\gamma_{\times k}(n, 2) \geq k+3$. Now, let us see that there exists a k-tuple dominating set $\widehat{D}$ with cardinality $k + 3$. From the reasoning above, we have that $\widehat{D} = X_1 \cup X_2$, $|X_2| = 3$ and $\binom{X_2}{2} \subseteq \widehat{D}$. In fact, consider the set $\widehat{D}$ given by

$$\widehat{D} = \binom{[3]}{2} \cup \{\{2a, 2a+1\} : 2 \leq a \leq k+1\}.$$

$\widehat{D}$ is a k-tuple dominating set of $K(n, 2)$ with $|\widehat{D}| = k + 3$. Therefore, we have $\gamma_{\times k}(2k + 3, 2) = k + 3$.

We conclude that for $k \geq 2$, $\gamma_{\times k}(n, 2) = k + 3$ if and only if $n = 2k + 3$. ■

As a by-product of the proof of Theorem 19 we have that if $n = 2k + 3$ with $k \geq 2$, and D is a $\gamma_{\times k}$-set of $K(n, 2)$, then up to automorphism

$$D = \binom{[3]}{2} \cup \{\{2a, 2a+1\} : 2 \leq a \leq k+1\}.$$

Note that in Theorems 5 and 19, we provide $\gamma_{\times k}(n, 2)$ for $n \geq 2k + 3$. From monotonicity on n it follows that if $n \leq 2k+2$, then $\gamma_{\times k}(n, 2) \geq k+4$. Moreover, in the remaining of this section we will prove Theorem 22, where we obtain $\gamma_{\times k}(n, 2)$ if n is large enough with respect to k.

Theorem 22. *Let $n \geq 7$ and $\frac{n-3}{2} < k \leq a\,\frac{n-3}{4}$ where $a = n - 4$ if n is even and $a = n - 6 + (n \mod 4)$ if n is odd. Let $\alpha = \left\lceil \frac{2k}{n-3} \right\rceil$. It holds*

1. $\gamma_{\times k}(n, 2) = k + 2\alpha$, *if* $\frac{2}{\alpha}k + 4 \leq n < \frac{2}{\alpha-1}k + 3$;
2. $\gamma_{\times k}(n, 2) = k + 2\alpha + 1$, *if* $n = \left\lceil \frac{2}{\alpha}k \right\rceil + 3$.

Table 1 illustrates the results of Theorem 22 in different colors (item 1 in light blue and item 2 in blue).

In order to prove Theorem 22, we first give lower bounds for $\gamma_{\times k}(n, 2)$ in Proposition 23. We then demonstrate that these bounds are tight by constructing k-tuple dominating sets for $K(n, 2)$ with the desired cardinality. Since the construction of these sets is somewhat laborious, we present it in a separate section.

Proposition 23. *Let α, n be positive integers with $\alpha \geq 2$ and $n \geq 2\alpha + 3 + (\alpha \mod 2)$*

1. $\gamma_{\times k}(n,2) \geq k + 2\alpha$, *if* $\frac{2}{\alpha}k + 4 \leq n < \frac{2}{\alpha-1}k + 3$;
2. $\gamma_{\times k}(n,2) \geq k + 2\alpha + 1$, *if* $n = \lceil \frac{2}{\alpha}k \rceil + 3$.

Proof. Let α, n be positive integers with $\alpha \geq 2$ and $n \geq 2\alpha + 3 + (\alpha \mod 2)$. First, we prove the two following claims.

Claim 24. *If $K(n,2)$ admits a k-tuple dominating set with cardinality $k + 2\alpha$, then $\alpha n \geq 2k + 4\alpha$.*

Proof. Let D be a k-tuple dominating set of $K(n,2)$ with cardinality $|D| = k + 2\alpha$. Our goal is to prove that $\alpha n \geq 2(k + 2\alpha) = 2|D|$. Let us suppose, on the contrary, that $\alpha n < 2|D|$.

On the one side, by Lemma 17 for every pair of elements $a, b \in [n]$ we have

$$i_a + i_b \leq \begin{cases} |D| - k + 2 = 2(\alpha+1), & \text{if } \{a,b\} \in D, \\ |D| - k = 2\alpha, & \text{if } \{a,b\} \notin D. \end{cases} \tag{7}$$

Since $\alpha n < 2|D| = \sum_{x \in [n]} i_x$, we have that there is at least one element $a \in [n]$ for which $i_a \geq \alpha + 1$. For every other element $b \in X_{\alpha}^{\geq}$, it holds $i_a + i_b \geq 2\alpha + 1$, and by (7) $\{a, b\} \in D$.

If $|X_{\alpha}^{\geq}| > 1$, then we have that for every $x \in X_{\alpha}^{\geq}$, $i_x \leq \alpha + 2$. In fact, if for some element y it holds $i_y > \alpha + 2$, then by (7) for every other $x \in [n]$

$$i_x \leq 2(\alpha+1) - i_y < 2(\alpha+1) - (\alpha+2) = \alpha,$$

and y would be the only element in $X_{\alpha}^{\geq}$, but $|X_{\alpha}^{\geq}| > 1$.

Let us see that $|X_{\alpha}^{\geq}| > 1$. Otherwise, $X_{\alpha}^{\geq} = \{a\}$ with $i_a \geq \alpha + 1$ and in consequence

$$\begin{aligned} \alpha n < 2|D| = \sum_{x \in [n]} i_x = i_a + \sum_{x \in [n] \setminus \{a\}} \overbrace{i_x}^{\leq \alpha - 1} &\leq i_a + (n-1)(\alpha-1) \\ &= i_a + \alpha n - n - \alpha + 1. \end{aligned}$$

Then, $i_a > n + (\alpha - 1) \geq n$, which cannot be true since $i_x \leq n - 1$ for every element $x \in [n]$. Therefore, $|X_{\alpha}^{\geq}| > 1$. In consequence, $i_x \leq \alpha + 2$ for every $x \in X_{\alpha}^{\geq}$. What is more, if $y \in X_{\alpha+2}$, then for every $x \in X_{\alpha}^{\geq}$ we have $i_x \leq 2(\alpha+1) - i_y = \alpha$ and it turns out that $i_x = \alpha$. Then either $X_{\alpha}^{\geq} = X_{\alpha} \cup X_{\alpha+1}$ or $X_{\alpha}^{\geq} = X_{\alpha} \cup X_{\alpha+2}$ with $|X_{\alpha+2}| = 1$.

Case 1. $X_\alpha^{\geq} = X_\alpha \cup X_{\alpha+1}$. Since for $a \in X_{\alpha+1}$ and every $b \in X_\alpha^{\geq}$ we have $\{a,b\} \in D$, then $|X_\alpha^{\geq}| \leq i_a + 1 = \alpha + 2$, and we have

$$\begin{aligned}2|D| &= \sum_{x\in[n]} i_x = \sum_{x\in X_\alpha^{\geq}} i_x + \sum_{x\in X_{\alpha-1}^{\leq}} i_x \leq |X_\alpha^{\geq}|(\alpha+1) + \underbrace{|X_{\alpha-1}^{\leq}|}_{n-|X_\alpha^{\geq}|}(\alpha-1)\\ &= n(\alpha-1) + 2|X_\alpha^{\geq}| \leq n\alpha - n + 2(\alpha+2).\end{aligned}$$

Since $2|D| \geq \alpha n + 1$, it turns out that $n \leq 2(\alpha+2) - 1$. By hypothesis, we have

$$n \geq \begin{cases} 2(\alpha+2), & \text{if } \alpha \text{ is odd,}\\ 2(\alpha+2)-1, & \text{if } \alpha \text{ is even.}\end{cases}$$

Thus, α is even and $n = 2(\alpha+2) - 1$. Therefore

$$\alpha n + 1 \leq 2|D| \leq n\alpha - n + 2(\alpha+2) = \alpha n + 1.$$

It turns out that $2|D| = \alpha n + 1$. But $2|D|$ is even whereas $\alpha n + 1$ is odd.

Case 2. $X_\alpha^{\geq} = X_\alpha \cup X_{\alpha+2}$ with $X_{\alpha+2} = \{a\}$. Since for every $b \in X_\alpha$ we have $\{a,b\} \in D$, then $|X_\alpha^{\geq}| \leq \alpha + 3$, and we have

$$\begin{aligned}2|D| &= \sum_{x\in[n]} i_x = i_a + \sum_{x\in X_\alpha} i_x + \sum_{x\in X_{\alpha-1}^{\leq}} i_x \leq (\alpha+2) + \underbrace{|X_\alpha|}_{|X_\alpha^{\geq}|-1}\alpha + \underbrace{|X_{\alpha-1}^{\leq}|}_{n-|X_\alpha^{\geq}|}(\alpha-1)\\ &= n(\alpha-1) + |X_\alpha^{\geq}| + 2 \leq n\alpha - n + \alpha + 5.\end{aligned}$$

Since $2|D| > \alpha n$, it turns out that $n < \alpha + 5$. By hypothesis, $n \geq 2\alpha + 3$, then $2\alpha + 3 < \alpha + 5$, that implies $\alpha < 2$, which cannot be true since $\alpha \geq 2$.

In both cases, we arrive at a contradiction. Therefore, we can conclude that $\alpha n \geq 2k + 4\alpha$ as claimed. □

Claim 25. *If $K(n,2)$ admits a k-tuple dominating set with cardinality $k+2\alpha-1$, then $(\alpha-1)n \geq 2k + 3(\alpha-1)$.*

Proof. Let D be a k-tuple dominating set of $K(n,2)$ with cardinality $|D| = k+2\alpha-1$. We are intended to prove that $(\alpha-1)n \geq 2k+3(\alpha-1)$ or equivalently $(\alpha-1)n+\alpha+1 \geq 2|D|$. Let us suppose, on the contrary, that $(\alpha-1)n+\alpha+1 < 2|D|$.

By Lemma 17 for every pair of elements $a, b \in [n]$ we have

$$i_a + i_b \leq \begin{cases} |D| - k + 2 = 2\alpha + 1, & \text{if } \{a,b\} \in D,\\ |D| - k = 2\alpha - 1, & \text{if } \{a,b\} \notin D.\end{cases} \tag{8}$$

Since $(\alpha-1)n < 2|D| = \sum_{x\in[n]} i_x$, we have that there is at least one element $a\in[n]$ for which $i_a \ge \alpha$. For every other element $b\in X_\alpha^{\ge}$, it holds $i_a+i_b\ge 2\alpha$, and by (8) $\{a,b\}\in D$.

Let us see that $|X_\alpha^{\ge}|>1$. On the contrary, $X_\alpha^{\ge}=\{a\}$ and

$$(\alpha-1)n+\alpha+1 < 2|D| = \sum_{x\in[n]} i_x = i_a + \sum_{x\neq a} \underbrace{i_x}_{\le \alpha-1} \le i_a + (\alpha-1)n-(\alpha-1).$$

Thus, $i_a > 2\alpha$, which cannot be true since $i_x\le 2\alpha$ for every element $x\in[n]$ by Remark 18. Therefore, $|X_\alpha^{\ge}|\ge 2$.

Then, we have that for every $x\in X_\alpha^{\ge}$, $i_x\le\alpha+1$. In fact, if for some element y it holds that $i_y>\alpha+1$, then by (8) for every other $x\in[n]$

$$i_x \le 2\alpha+1-i_y < 2\alpha+1-(\alpha+1)=\alpha,$$

and y would be the only element in $X_\alpha^{\ge}$, but $|X_\alpha^{\ge}|>1$. Therefore, $i_x\le\alpha+1$ for every $x\in X_\alpha^{\ge}$. What is more, there is at most one element in $X_{\alpha+1}$ since if $x,y\in X_{\alpha+1}$, then $i_x+i_y=2\alpha+2$ and this cannot be true by (8).

Therefore, either $X_\alpha^{\ge}=X_\alpha$ or $X_\alpha^{\ge}=X_\alpha\cup X_{\alpha+1}$ with $|X_{\alpha+1}|=1$.

Case 1. $X_\alpha^{\ge}=X_\alpha$. For every pair $a,b\in X_\alpha$ we have $\{a,b\}\in D$ since $i_a+i_b=2\alpha$ and it holds (8). Thus $\binom{X_\alpha}{2}\subseteq D$ and we have $|X_\alpha|\le\alpha+1$. Then

$$\begin{aligned}2|D| &= \sum_{x\in[n]} i_x = \sum_{x\in X_\alpha} i_x + \sum_{x\in X_{\alpha-1}^{\le}} i_x \le |X_\alpha|\alpha + \underbrace{|X_{\alpha-1}^{\le}|}_{n-|X_\alpha|}(\alpha-1)\\ &= n(\alpha-1)+|X_\alpha| \le n(\alpha-1)+\alpha+1.\end{aligned}$$

But we have assumed that $2|D|>n(\alpha-1)+\alpha+1$.

Case 2. $X_\alpha^{\ge}=X_\alpha\cup X_{\alpha+1}$ with $X_{\alpha+1}=\{a\}$. For every other $b\in X_{\alpha-1}^{\ge}$ we have $i_a+i_b\ge 2\alpha$ thus $\{a,b\}\in D$ and in consequence $|X_{\alpha-1}^{\ge}|\le\alpha+2$. If $X=X_{\alpha-1}\cup X_\alpha$, we have

$$\begin{aligned}2|D| &= \sum_{x\in[n]} i_x = i_a + \sum_{x\in X} i_x + \sum_{x\in X_{\alpha-2}^{\le}} i_x \le (\alpha+1) + \underbrace{|X|}_{|X_{\alpha-1}^{\ge}|-1}\alpha + \underbrace{|X_{\alpha-2}^{\le}|}_{n-|X_{\alpha-1}^{\ge}|}(\alpha-2)\\ &= n(\alpha-2)+2|X_{\alpha-1}^{\ge}|+1 \le n(\alpha-1)-n+2\alpha+5.\end{aligned}$$

Since $2|D|>(\alpha-1)n+\alpha+1$, it turns out that $n<\alpha+4$, but by hypothesis $n\ge 2\alpha+3\ge\alpha+5$, since $\alpha\ge 2$.

In both cases, we arrive at a contradiction. Therefore, we can conclude that $(\alpha-1)n\ge 2k+3(\alpha-1)$ as claimed. □

Assume that $n = \left\lceil \frac{2}{\alpha}k \right\rceil + 3$. Suppose that $K(n,2)$ admits a k-tuple dominating set with cardinality $k+2\alpha$. From the first claim, it follows that $\alpha n \geq 2k+4\alpha$, or, equivalently,

$$n \geq \frac{2}{\alpha}k + 4,$$

which does not hold. Thus, there does not exist any k-tuple dominating set with $k+2\alpha$ vertices and

$$\gamma_{\times k}(n,2) \geq k + 2\alpha + 1.$$

Similarly, if $\frac{2}{\alpha}k+4 \leq n < \frac{2}{\alpha-1}k+3$ and $K(n,2)$ admits a k-tuple dominating set with cardinality $k+2\alpha-1$, from the second claim it follows that $(\alpha-1)n \geq 2k+3(\alpha-1)$, i.e.,

$$n \geq \frac{2}{\alpha-1}k + 3,$$

which is not true. Then there does not exist any k-tuple dominating set with $k+2\alpha-1$ vertices and

$$\gamma_{\times k}(n,2) \geq k + 2\alpha.$$

■

Construction of k-tuple dominating sets for $K(n,2)$

In order to provide k-tuple dominating sets of $K(n,2)$ whose cardinalities achieve the lower bounds for $\gamma_{\times k}(n,2)$ given in Proposition 23, let us introduce the following definition.

Definition. Let $m \in \mathbb{N}$. For $i \in \mathbb{N}$, $i < \frac{m}{2}$, we define $D_i^{[m]}$ as the set

$$D_i^{[m]} = \big\{\{\xi, \xi + i\} : \xi \in [m]\big\},$$

where the sums are taken modulo m. And for $\alpha \in \mathbb{N}$, $\alpha \geq 2$, such that $m > \alpha$, let $a = \left\lfloor \frac{\alpha}{2} \right\rfloor$. We define $D^{m,\alpha}$ as the set given by

$$D^{m,\alpha} = \left(\bigcup_{i=1}^{a} D_i^{[m]} \right) \cup \widehat{D},$$

with

$$\widehat{D} = \begin{cases} \big\{\{\xi, \xi + \left\lfloor \frac{m}{2} \right\rfloor\}, \xi \in \left[\left\lfloor \frac{m}{2} \right\rfloor\right]\big\}, & \text{if } \alpha \text{ is odd,} \\ \emptyset, & \text{if } \alpha \text{ is even.} \end{cases}$$

Example 26. • Let $\alpha = 3$, $m = 6 > \alpha$ and $a = \left\lfloor \frac{\alpha}{2} \right\rfloor = 1$. The set $D^{m,\alpha} = D^{6,3}$ is given by

$$D^{6,3} = D_1^{[6]} \cup \widehat{D},$$

where

$$\begin{aligned} D_1^{[6]} &= \big\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,1\}\big\}, \\ \widehat{D} &= \big\{\{1,4\},\{2,5\},\{3,6\}\big\}. \end{aligned}$$

$D^{6,3}$ is a set of vertices of $K(n,2)$ for every $n \geq 6$ and the occurrences of the elements in $[n]$ for the set $D^{6,3}$ are

$$i_x = \begin{cases} 3 = \alpha, & \text{if } x \in [6], \\ 0, & \text{otherwise.} \end{cases}$$

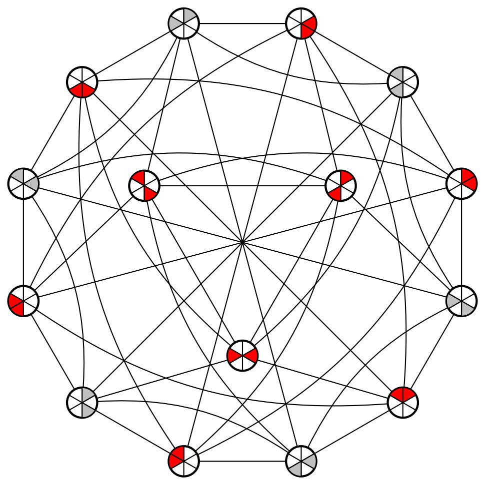

Figure 4. Set $D^{6,3}$ in the Kneser graph $K(6,2)$.

Let $\alpha = 4$, $m = 14 > \alpha$ and $a = \left\lfloor \frac{\alpha}{2} \right\rfloor = 2$. The set $D^{m,\alpha} = D^{14,4}$ is given by

$$D^{14,4} = D_1^{[14]} \cup D_2^{[14]},$$

where

$$\begin{aligned} D_1^{[14]} = \big\{&\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{8,9\}, \\ &\{9,10\},\{10,11\},\{11,12\},\{12,13\},\{13,14\},\{14,1\}\big\}, \\ D_2^{[14]} = \big\{&\{1,3\},\{2,4\},\{3,5\},\{4,6\},\{5,7\},\{6,8\},\{7,9\},\{8,10\}, \\ &\{9,11\},\{10,12\},\{11,13\},\{12,14\},\{13,1\},\{14,2\}\big\}. \end{aligned}$$

$D^{14,4}$ is a set of vertices of $K(n,2)$ for every $n \geq 14$ and the occurrences of the elements in $[n]$ for the set $D^{14,4}$ are

$$i_x = \begin{cases} 4 = \alpha, & \text{if } x \in [14], \\ 0, & \text{otherwise.} \end{cases}$$

• Let $\alpha = 3$, $m = 11 > \alpha$, and $a = \lfloor \frac{\alpha}{2} \rfloor = 1$. The set $D^{m,\alpha} = D^{11,3}$ is given by

$$D^{11,3} = D_1^{[11]} \cup \widehat{D},$$

where

$$\begin{aligned} D_1^{[11]} &= \big\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\\ &\quad \{7,8\},\{8,9\},\{9,10\},\{10,11\},\{11,1\}\big\},\\ \widehat{D} &= \big\{\{1,6\},\{2,7\},\{3,8\},\{4,9\},\{5,10\}\big\}. \end{aligned}$$

$D^{11,3}$ is a set of vertices of $K(n,2)$ for every $n \geq 11$ and the occurrences of the elements in $[n]$ for the set $D^{11,3}$ are

$$i_x = \begin{cases} 3 = \alpha, & \text{if } x \in [10], \\ 2 = \alpha - 1, & \text{if } x = 11, \\ 0, & \text{otherwise.} \end{cases}$$

Let us observe that $D^{m,\alpha} \subseteq \binom{[m]}{2}$. In consequence, $D^{m,\alpha}$ is a set of vertices of $K(n,2)$ for every $n \geq m$, and any element in $[m]$ has at most α occurrences in $D^{m,\alpha}$. In fact, $m \geq \alpha + 1 > 2a$, i.e., $a < \frac{m}{2}$. So, each element $x \in [m]$ appears in exactly two vertices of $D_i^{[m]}$ for each $1 \leq i \leq a$. Moreover, for $i \neq j$, $D_i^{[m]} \cap D_j^{[m]} = \emptyset$ since $i + j \leq 2a < m$. On the other hand, for α odd, a vertex in $\widehat{D}$ is not in $D_i^{[m]}$ for any i. Furthermore, every element $x \in [m]$ appears in exactly one vertex of $\widehat{D}$ if m is even, and every element $x \in [m-1]$ appears in exactly one vertex of $\widehat{D}$ if m is odd.

Thus, for both α and m odd we have

$$i_x\left(D^{m,\alpha}\right) = \begin{cases} \alpha, & \text{if } x \in [m-1], \\ \alpha - 1 = 2a, & \text{if } x = m, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

and if either α or m are even, then

$$i_x\left(D^{m,\alpha}\right) = \begin{cases} \alpha, & \text{if } x \in [m], \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

In any case, we have

$$|D^{m,\alpha}| = \frac{1}{2} \sum_{x \in [n]} i_x = \left\lfloor \frac{\alpha m}{2} \right\rfloor. \tag{11}$$

Lemma 27. *Let $\alpha, k \in \mathbb{N}$ with $\alpha \geq 2$ and $n \geq \alpha + 2$. If $n \geq \frac{2}{\alpha}k + 4$, then there exists a k-tuple dominating set of $K(n, 2)$ with cardinality $k + 2\alpha$.*

Proof. Due to Lemma 17 a set D is a k-tuple dominating set if and only if it verifies, for every pair of elements x and y in $[n]$

$$i_x + i_y \leq \begin{cases} |D| - k + 2 = 2\alpha + 2, & \text{if } \{x, y\} \in D, \\ |D| - k = 2\alpha, & \text{if } \{x, y\} \notin D. \end{cases}$$

Therefore, it is enough to find a set D of vertices with cardinality $k + 2\alpha$ such that $i_x \leq \alpha$ for each $x \in [n]$. Let $a = \left\lfloor \frac{\alpha}{2} \right\rfloor$. Since $n \geq \alpha + 2 > 2a + 1$, we have $a < \left\lfloor \frac{n}{2} \right\rfloor$. Thus, let us consider the set $D^{n,\alpha}$ as in Definition 4. Note that $|D^{n,\alpha}| = \left\lfloor \frac{\alpha n}{2} \right\rfloor$ by (11). Moreover, $|D^{n,\alpha}| \geq k + 2\alpha$. In fact, as $\frac{2}{\alpha}k + 4 \leq n$, we have $\alpha n \geq 2(k + 2\alpha)$. If either α or n are even, then

$$2|D^{n,\alpha}| = n\alpha \geq 2(k + 2\alpha).$$

On the other hand, if both α and n are odd, then

$$2|D^{n,\alpha}| = n\alpha - 1 \geq 2(k + 2\alpha) - 1.$$

As $2|D^{n,\alpha}|$ is an even integer and $2(k + 2\alpha) - 1$ is an odd integer, it turns out that $2|D^{n,\alpha}| > 2(k + 2\alpha) - 1$ and in consequence $2|D^{n,\alpha}| \geq 2(k + 2\alpha)$.

In any case, $|D^{n,\alpha}| \geq k + 2\alpha$ as claimed.

Note that $i_x(D^{n,\alpha}) \leq \alpha$ for every $x \in [n]$, by (9) and (10). Thus, eliminating any $|D^{n,\alpha}| - (k + 2\alpha)$ vertices from $D^{n,\alpha}$ gives as a result a set of vertices D with cardinality $k + 2\alpha$ such that $i_x(D) \leq i_x(D^{n,\alpha}) \leq \alpha$ for every $x \in [n]$. This is, a k-tuple dominating set with cardinality $k + 2\alpha$.

Therefore, $K(n, 2)$ admits a k-tuple dominating set with cardinality $k+2\alpha$. ■

Lemma 28. *Let $\alpha, k \in \mathbb{N}$ with $\alpha \geq 2$ and $n \geq 2\alpha + 3$. If $n = \left\lceil \frac{2}{\alpha}k \right\rceil + 3$, then there exists a k-tuple dominating set of $K(n, 2)$ with cardinality $k + 2\alpha + 1$.*

Proof. Due to Lemma 17 a set D is a k-tuple dominating set if and only if for every pair of elements x and y in $[n]$ it holds

$$i_x + i_y \leq \begin{cases} |D| - k + 2 = 2\alpha + 3, & \text{if } \{x, y\} \in D, \\ |D| - k = 2\alpha + 1, & \text{if } \{x, y\} \notin D. \end{cases}$$

Therefore it is enough to find a set D of vertices with cardinality $k + 2\alpha + 1$ such that $\alpha \leq i_x \leq \alpha + 1$ for each $x \in [n]$, and $\{x, y\} \in D$ for every pair $x, y \in X_{\alpha+1}$.

Let $\lambda \in \mathbb{N}$ such that $k = \alpha\lambda + b$ with $0 \leq b \leq \alpha - 1$. And let $a = \left\lfloor \frac{\alpha}{2} \right\rfloor$. We will give such a set D in terms of b considering the cases $b = 0$, $1 \leq b \leq a$ and $a + 1 \leq b \leq \alpha - 1$.

In order to do so, let us consider for $h \in \mathbb{N}$, $1 \leq h < n - \alpha$ the set

$$D(h) = D^{n-h,\alpha} \cup \binom{[n-h+1..n]}{2}. \tag{12}$$

Let us note that if $(n-h)\alpha$ is even, then using (10) we have that each element $x \in [n-h]$ has exactly α occurrences in $D^{n-h,\alpha}$ and none in $\binom{[n-h+1..n]}{2}$, and each $x \in [n-h+1..n]$ has exactly $h-1$ occurrences in $\binom{[n-h+1..n]}{2}$ and none in $D^{n-h,\alpha}$. In consequence

$$i_x(D(h)) = \begin{cases} \alpha, & \text{if } x \in [n-h], \\ h-1, & \text{if } x \in [n-h+1..n]. \end{cases} \tag{13}$$

Furthermore, by (11) we have

$$|D(h)| = \left|D^{n-h,\alpha}\right| + \left|\binom{[n-h+1..n]}{2}\right| = \frac{(n-h)\alpha}{2} + \binom{h}{2}. \tag{14}$$

In the first case, $b = 0$, we will prove that for $h = \alpha + 2$ the set $D(h)$ is itself a k-tuple dominating set with cardinality $k + 2\alpha$. In the remaining cases, we will consider the set $D(h)$ for $h = 2b + 2$ and $h = 2b - \alpha + 2$ respectively and modify them with the aim of giving k-tuple dominating sets of the desired cardinality.

Case 1. $b = 0$. Let $h = \alpha + 2$. Since $n \geq 2\alpha + 3$, we have $n - h \geq \alpha + 1 > \alpha$. So, we consider the set $D = D(h)$ as in (12). Note that $n = \left\lceil \frac{2}{\alpha}k \right\rceil + 3 = 2\lambda + 3$. Thus, for α odd, we have both n and h are odd and in consequence $n - h$ is even. Thus, $(n-h)\alpha$ is even. From (13) it turns out that $\alpha \leq i_x \leq \alpha + 1$ for each $x \in [n]$, and $\{x, y\} \in D$ for every pair $x, y \in X_{\alpha+1}$. Thus, D is a k-tuple dominating set. What is more,

$$\begin{aligned} 2|D| &= \sum_{x \in [n]} i_x = (n-h)\alpha + h(\alpha+1) = \alpha n + h \\ &= \alpha\left(\frac{2}{\alpha}k + 3\right) + \alpha + 2 = 2(k + 2\alpha + 1). \end{aligned}$$

Therefore, $|D|$ is a k-tuple dominating set with cardinality $k + 2\alpha + 1$.

Case 2. $1 \leq b \leq a$. Let $h = 2b + 2$. h is an even integer between 4 and $2a + 2 \leq \alpha + 2$. Since $n \geq 2\alpha + 3$, we have $n - h \geq (2\alpha + 3) - (\alpha + 2) = \alpha + 1 > \alpha$. So, we consider the set $D(h)$ as in (12).

Note that $n = \left\lceil \frac{2}{\alpha}k \right\rceil + 3 = 2\lambda + 4$. Thus, both n and h are even and in consequence also $n - h$. From (14),

$$\begin{aligned} 2|D(h)| &= (n-h)\alpha + h(h-1) = (n\alpha + h) - h\underbrace{(\alpha - h + 2)}_{\alpha - 2b} \\ &= 2(k + 2\alpha + 1) - \underbrace{h(\alpha - 2b)}_{\geq 0}, \end{aligned} \tag{15}$$

since $2(k+2\alpha+1) = 2k+4\alpha+2 = 2(\lambda\alpha+b)+4\alpha+2 = (2\lambda+4)\alpha+(2b+2) = n\alpha+h$.

If $b < a$, let us consider the following set D.

$$D = \left[D(h) \setminus \underbrace{\left(\bigcup_{\substack{1\le j\le b+1 \\ 1\le \xi \le \alpha-2b}} \{\{\xi, \xi+j\}\} \right)}_{D_1} \right] \cup \underbrace{\left(\bigcup_{\substack{1\le j\le b+1 \\ 1\le \xi \le \alpha-2b}} \{\{\xi, n-h+2j-1\}, \{\xi+j, n-h+2j\}\} \right)}_{D_2},$$

where the sums in D_1 are taken modulo $n-h$.

Let us see that $D_1 \subseteq D(h)$. In fact, we have that $b+1 \le a$, so for each j it holds $\{\xi, \xi+j\} \in D_j^{[n-h]} \subseteq D(h)$. Moreover, $\alpha - 2b < \alpha < n-h$, so for fixed j, the vertices $\{\xi, \xi+j\}$ are different. In consequence, $|D_1| = (b+1)(\alpha-2b)$.

On the other hand, $D_2 \cap D(h) = \emptyset$ since no vertex in $D(h)$ has an element in $[n-h]$ and other in $[n-h+1..n]$. Besides the vertices in D_2 are different. Thus, $|D_2| = 2(b+1)(\alpha-2b)$.

Note that for each $x \in [n]$, we have

$$i_x(D) = i_x(D(h)) - i_x(D_1) + i_x(D_2).$$

For $x \in [n-h]$, we have $i_x(D_1) = i_x(D_2)$, and in consequence

$$i_x(D) = i_x(D(h)) = \alpha,$$

and for $x \in [n-h+1..n]$, we have $i_x(D_1) = 0$ and $i_x(D_2) = \alpha - 2b$, and in consequence

$$i_x(D) = \underbrace{i_x(D(h))}_{h-1} + \alpha - 2b = \alpha + 1.$$

We have $\alpha \le i_x(D) \le \alpha+1$ for each $x \in [n]$, and $\{x, y\} \in D$ for every pair $x, y \in X_{\alpha+1}$. Thus, D is a k-tuple dominating set. Furthermore,

$$|D| = |D(h)| - |D_1| + |D_2| = k + 2\alpha + 1.$$

Therefore, $|D|$ is a k-tuple dominating set with cardinality $k+2\alpha+1$.

If $b = a$ and α is even then the set $D(h)$ is itself a k-tuple dominating set with cardinality $k+2\alpha+1$ since (15) holds. If α is odd, then let us consider the following set D.

$$D = \left[D(h) \setminus \underbrace{\left(\bigcup_{1 \le \xi \le b+1} \{\{\xi, \xi+1\}\} \right)}_{D_1} \right]$$
$$\cup \underbrace{\left(\bigcup_{1 \le \xi \le b+1} \{\{\xi, n-h+2\xi-1\}, \{\xi+1, n-h+2\xi\}\} \right)}_{D_2}.$$

Note that $D_1 \subseteq D_1^{[n-h]} \subseteq D(h)$. Moreover, since α is odd, we have $\alpha = 2a+1 = 2b+1$. Thus, $n-h \ge (2\alpha+3) - (2b+2) = \alpha+2 > b+1$. So, the vertices in D_1 are different and $|D_1| = b+1$. As in the case $b < a$, $D_2 \cap D(h) = \emptyset$ and the vertices in D_2 are different. Thus, $|D_2| = 2b+2$. We have that for each $x \in [n]$ it holds

$$i_x(D) = i_x(D(h)) - i_x(D_1) + i_x(D_2).$$

We have $i_x(D_1) = i_x(D_2)$ if $x \in [n-h]$ and $i_x(D_1) = 0$, $i_x(D_2) = 1$ if $x \in [n-h+1..n]$. In consequence

$$i_x(D) = \begin{cases} \alpha, & \text{if } x \in [n-h], \\ h = \alpha+1, & \text{if } x \in [n-h+1..n]. \end{cases}$$

This is, $\alpha \le i_x(D) \le \alpha+1$ for each $x \in [n]$, and $\{x, y\} \in D$ for every pair $x, y \in X_{\alpha+1}$. D is a k-tuple dominating set with cardinality

$$|D| = |D(h)| - |D_1| + |D_2| = k + 2\alpha + 1.$$

Case 3. $a+1 \le b \le \alpha-1$. Let $h = 2b-\alpha+2$. We have $3 \le h \le \alpha$, and h has the same parity as α, and $n-h \ge (2\alpha+3)-\alpha = \alpha+3 > \alpha$. So, we consider the set $D(h)$ as in (12).

Note that $n = \left\lceil \frac{2}{\alpha} k \right\rceil + 3 = 2\lambda+5$. Besides, if α is odd, then both n and h are odd and in consequence $n-h$ is even. So, in any case, $(n-h)\alpha$ is even. From (14),

$$\begin{aligned} 2|D(h)| &= (n-h)\alpha + h(h-1) = (n\alpha+h) - h\underbrace{(\alpha-h+2)}_{2\alpha-2b} \\ &= 2(k+2\alpha+1) - \underbrace{2h(\alpha-b)}_{>0}, \end{aligned}$$

since $2(k+2\alpha+1) = 2k+4\alpha+2 = 2(\lambda\alpha+b)+4\alpha+2 = (2\lambda+5)\alpha+(2b-\alpha+2) = n\alpha+h$.

The procedure is similar to Case 2. This is, we eliminate $h(\alpha-b)$ vertices from $D(h)$ and add another $2h(\alpha-b)$ vertices in order to get a k-tuple dominating set with cardinality $k+2\alpha+1$.

If α is even, let us consider the following set D.

$$D=\left[D(h)\setminus\underbrace{\left(\bigcup_{\substack{1\le j\le b-a+1\\1\le\xi\le 2\alpha-2b}}\{\{\xi,\xi+j\}\}\right)}_{D_1}\right]$$
$$\cup\underbrace{\left(\bigcup_{\substack{1\le j\le b-a+1\\1\le\xi\le 2\alpha-2b}}\{\{\xi,n-h+2j-1\},\{\xi+j,n-h+2j\}\}\right)}_{D_2},$$

where the sums in D_1 are taken modulo $n-h$.

Let us see that $D_1\subseteq D(h)$. In fact, we have that $b-a+1\le\alpha-a=a$, so for each j it holds $\{\xi,\xi+j\}\in D_j^{[n-h]}\subseteq D(h)$. And, since $n\ge 2\alpha+3$ and $h=2b-\alpha+2$, we have $n-h\ge 3\alpha-2b+1>2\alpha-2b$. So, $2\alpha-2b<\alpha<n-h$, thus for fixed j, the vertices $\{\xi,\xi+j\}$ are different. In consequence, $|D_1|=(b-a+1)(2\alpha-2b)=h(\alpha-b)$.

On the other hand, $D_2\cap D(h)=\emptyset$ since no vertex in $D(h)$ has an element in $[n-h]$ and other in $[n-h+1..n]$. Thus, the vertices in D_2 are different and it turns out that $|D_2|=2(b-a+1)(2\alpha-2b)=2h(\alpha-b)$.

If α is odd, let us consider the set $D=[D(h)\setminus D_1]\cup D_2$, where

$$D_1=\left(\bigcup_{\substack{1\le j\le b-a\\1\le\xi\le 2\alpha-2b}}\{\{\xi,\xi+j\}\}\right)\cup\left(\bigcup_{1\le\xi\le\alpha-b}\left\{\left\{\xi,\xi+\frac{n-h}{2}\right\}\right\}\right),$$
$$D_2=\left(\bigcup_{\substack{1\le j\le b-a\\1\le\xi\le 2\alpha-2b}}\{\{\xi,n-h+2j-1\},\{\xi+j,n-h+2j\}\}\right)$$
$$\cup\left(\bigcup_{1\le\xi\le\alpha-b}\left\{\{\xi,n\},\left\{\xi+\frac{n-h}{2},n\right\}\right\}\right),$$

and the sums in D_1 are taken modulo $n-h$.

Let us see that $D_1\subseteq D(h)$. In fact, we have that $b-a\le\alpha-1-a=a$, so for each j it holds $\{\xi,\xi+j\}\in D_j^{[n-h]}\subseteq D(h)$. And, since $n\ge 2\alpha+3$ and $h=2b-\alpha+2$, we have $n-h\ge 3\alpha-2b+1>2\alpha-2b$. So, $2\alpha-2b<\alpha<n-h$, thus for fixed j, the vertices $\{\xi,\xi+j\}$ are different. On the other hand, for

$1 \leq \xi \leq \alpha - b < \frac{n-h}{2}$, we have $\left\{\xi, \xi + \frac{n-h}{2}\right\} \in \widehat{D} \subseteq D(h)$. In consequence,

$$|D_1| = (b-a)(2\alpha - 2b) + (\alpha - b) = \underbrace{(2b - 2a + 1)}_{h}(\alpha - b) = h(\alpha - b).$$

Besides, $D_2 \cap D(h) = \emptyset$ since no vertex in $D(h)$ has an element in $[n-h]$ and other in $[n-h+1..n]$. Thus, the vertices in D_2 are different and it turns out that $|D_2| = 2(b-a)(2\alpha - 2b) + 2(\alpha - b) = 2h(\alpha - b)$.

Either if α is even or odd, the set D is a k-tuple dominating set with cardinality $k + 2\alpha + 1$. In fact, note that for each $x \in [n]$ we have

$$i_x(D) = i_x(D(h)) - i_x(D_1) + i_x(D_2).$$

For $x \in [n-h]$, we have $i_x(D_1) = i_x(D_2)$, and in consequence

$$i_x(D) = i_x(D(h)) = \alpha.$$

And for $x \in [n-h+1..n]$, we have $i_x(D_1) = 0$ and $i_x(D_2) = 2\alpha - 2b$, and in consequence

$$i_x(D) = \underbrace{i_x(D(h))}_{h-1} + 2\alpha - 2b = \alpha + 1.$$

We have $\alpha \leq i_x(D) \leq \alpha + 1$ for each $x \in [n]$, and $\{x, y\} \in D$ for every pair $x, y \in X_{\alpha+1}$. Thus, D is a k-tuple dominating set. Furthermore,

$$|D| = |D(h)| - |D_1| + |D_2| = k + 2\alpha + 1.$$

Therefore, either if α is even or odd, we obtain a k-tuple dominating set with cardinality $k + 2\alpha + 1$. ■

Combining Proposition 23 with Lemmas 27 and 28 yields the following result.

Theorem 29. *Let $\alpha, n \in \mathbb{N}$ with $\alpha \geq 2$ and $n \geq 2\alpha + 3 + (\alpha \mod 2)$. We have,*

1. $\gamma_{\times k}(n, 2) = k + 2\alpha$, *if* $\frac{2}{\alpha}k + 4 \leq n < \frac{2}{\alpha-1}k + 3$,
2. $\gamma_{\times k}(n, 2) = k + 2\alpha + 1$, *if* $n = \left\lceil \frac{2}{\alpha}k \right\rceil + 3$.

Let us see two remarks about the statement of Theorem 29.

Note that condition $\frac{2}{\alpha}k + 3 \leq n < \frac{2}{\alpha-1}k + 3$ is equivalent to $\alpha = \left\lceil \frac{2k}{n-3} \right\rceil$. Further, for $\alpha = \left\lceil \frac{2k}{n-3} \right\rceil \geq 2$, the inequality $n \geq 2\alpha + 3 + (\alpha \mod 2)$ is equivalent to $\frac{n-3}{2} < k \leq a\,\frac{n-3}{4}$ where $a = n-4$ if n is even and $a = n - 6 + (n \mod 4)$ if n is odd, with $n \geq 7$. Thus, Theorem 29 can be restated as Theorem 22.

Finally, it is worth noting that whereas in Theorem 22 the values of $\gamma_{\times k}(n, 2)$ are computed for $\frac{n-3}{2} < k \leq a\,\frac{n-3}{4}$, with $a = n-4$ if n is even and $a = n - 6 + (n \mod 4)$ if n is odd, Theorems 5 and 19 provide the values of $\gamma_{\times k}(n, 2)$ for $k \leq \frac{n-3}{2}$. In consequence, we have determined $\gamma_{\times k}(n, 2)$ for all $2 \leq k \leq a\,\frac{n-3}{4}$.

5. Further Remarks

Regarding $\gamma_{\times k}$-sets of $K(n,2)$ for large values of k, recall that for $k = \binom{n-2}{2} + 1$, we have $\gamma_{\times k}(n,2) = \binom{n}{2}$, whereas for $k = \binom{n-2}{2}$, $\gamma_{\times k}(n,2) = \binom{n}{2} - 1$. Together with Theorem 9 for $r = 2$, we have the following.

Corollary 30. *For $t \in \mathbb{N}$, if $k = \binom{n-2}{2} - t$ and $n \geq 2(t+3) - \left\lceil \frac{t+2}{2} \right\rceil$, then $\gamma_{\times k}(n,2) = k + 2n - 4$.*

Besides, in order to complete the study of $\gamma_{\times k}$-sets of $K(n,2)$ for large values of k, we have also studied properties of the $\gamma_{\times k}$-sets of $K(n,2)$ for $k \in \left\{\binom{n-4}{2} + 2, \binom{n-3}{2} + 1\right\}$. In fact, we obtain the following result. We omit the proof here for the sake of readability (it can be found in [9]).

Proposition 31. • *If $k = \binom{n-4}{2} + 2$ and $6 \leq n \leq 10$, then $\gamma_{\times k}(n,2) = \binom{n-2}{2} + 1$. Moreover, $D = \binom{[n-2]}{2} \cup \big\{\{n-1, n\}\big\}$ is a $\gamma_{\times k}$-set.*

• *If $k = \binom{n-3}{2} + 1$ and $n \geq 5$, then $\gamma_{\times k}(n,2) = \binom{n-1}{2}$. Moreover, $D = \binom{[n-1]}{2}$ is a $\gamma_{\times k}$-set.*

Notice that although the set $\binom{[n-2]}{2} \cup \big\{\{n-1, n\}\big\}$ given in the first item of Proposition 31 is a k-tuple dominating set of $K(n,2)$ for every $n \geq 6$, it is not true that it is a $\gamma_{\times k}$-set for every n. For instance, for $n = 11$ and $k = 23$ the set

$$D = \binom{[6]}{2} \cup \binom{[7..11]}{2} \cup \tilde{D},$$

where $\tilde{D} = \big\{\{1,7\}, \{2,7\}, \{2,8\}, \{3,8\}, \{3,9\}, \{4,9\}, \{4,10\}, \{5,10\}, \{5,11\}\big\}$, is a k-tuple dominating set with cardinality $\binom{n-2}{2} = \binom{9}{2} = 36$.

With the aim of summarizing the results in this paper for $K(n,2)$, Table 1 contains several values for $\gamma_{\times k}(n,2)$. Some of these values are obtained taking into account Lemma 17, Theorem 2, and upper bounds obtained by Integer Linear Programming. In this regard, we consider the following ILP formulation for the k-tuple dominating problem. The set of variables is given by $\{x_u, u \in V\}$, where

$$x_u = \begin{cases} 1, & \text{if } u \in D, \\ 0, & \text{otherwise,} \end{cases}$$

and the ILP model is formulated as

$$\begin{aligned} & \min \sum_{u \in V} x_u \\ & \text{s/t} \sum_{u \in N[v]} x_u \geq k, \quad \forall\, v \in V, \\ & x_v \in \{0,1\}, \qquad \forall\, v \in V. \end{aligned} \tag{16}$$

$k\backslash n$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a	3^a
2	6^d	6^e	6^g	5^c	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b	4^b
3		9^d	7^e	7^b	7^b	6^c	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b	5^b
4		10^d	10^e	9^c	8^b	8^b	8^b	7^c	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b	6^b
5			13^d	11^e	10^c	9^b	9^b	9^b	9^b	8^c	7^b	7^b	7^b	7^b	7^b	7^b	7^b	7^b	7^b	7^b	7^b	7^b	7^b
6			14^d	14^f	12^g	11^c	10^b	10^b	10^b	10^b	10^b	9^c	8^b	8^b	8^b	8^b	8^b	8^b	8^b	8^b	8^b	8^b	8^b
7			15^d	15^e	13^g	13^g	12^c	11^b	11^b	11^b	11^b	11^b	11^b	10^c	9^b	9^b	9^b	9^b	9^b	9^b	9^b	9^b	9^b
8				17^f	16^e	14^g	14^b	13^c	12^b	12^b	12^b	12^b	12^b	12^b	12^b	11^c	10^b	10^b	10^b	10^b	10^b	10^b	10^b
9				19^d	17^g	16^f	15^b	15^b	14^c	13^b	13^b	13^b	13^b	13^b	13^b	13^b	13^b	12^c	11^b	11^b	11^b	11^b	11^b
10				20^d	20^f	18^f	17^c	16^b	16^b	15^c	14^b	14^b	14^b	14^b	14^b	14^b	14^b	14^b	14^b	13^c	12^b	12^b	12^b
11				21^d	21^e	20^f	19^g	18^c	17^b	17^b	16^c	15^b	15^b	15^b	15^b	15^b	15^b	15^b	15^b	15^b	15^b	14^c	13^b
12					22^g	22^e	20^g	19^c	18^b	18^b	18^b	17^c	16^b	16^b	16^b	16^b	16^b	16^b	16^b	16^b	16^b	16^b	16^b
13					25^d	23^g	21^g	21^b	20^c	19^b	19^b	19^b	18^c	17^b	17^b	17^b	17^b	17^b	17^b	17^b	17^b	17^b	17^b
14					26^d	24^g	24^f	22^b	22^b	21^c	20^b	20^b	20^b	19^c	18^b	18^b	18^b	18^b	18^b	18^b	18^b	18^b	18^b
15					27^d	27^f	25^g	24^c	23^b	22^c	21^b	21^b	21^b	21^b	20^c	19^b	19^b	19^b	19^b	19^b	19^b	19^b	19^b
16					28^d	28^e	26^g	25^c	24^b	24^b	23^c	22^b	22^b	22^b	22^b	21^c	20^b	20^b	20^b	20^b	20^b	20^b	20^b
17						29^g	29^e	27^g	26^c	25^b	25^b	24^c	23^b	23^b	23^b	23^b	22^c	21^b	21^b	21^b	21^b	21^b	21^b
18						32^d	30^g	28^g	27^c	26^b	26^b	25^c	24^b	24^b	24^b	24^b	24^b	23^c	22^b	22^b	22^b	22^b	22^b
19						33^d	31^g	29-30	29^g	28^c	27^b	27^b	26^c	25^b	25^b	25^b	25^b	25^b	24^c	23^b	23^b	23^b	23^b
20						34^d	32-33	30-32	30^g	29^c	28^b	28^b	28^b	27^c	26^b	26^b	26^b	26^b	26^b	25^c	24^b	24^b	24^b
21						35^d	33-35	31-33	31^g	31^g	30^c	29^b	29^b	28^c	27^b	27^b	27^b	27^b	27^b	27^b	26^c	25^b	25^b
22						36^d	36^e	32-34	32-33	32^g	31^c	30^b	30^b	30^b	29^c	28^b	28^b	28^b	28^b	28^b	28^b	27^c	26^b
23							37^g	33-36	33-35	33^g	33^b	32^c	31^b	31^b	31^b	30^c	29^b	29^b	29^b	29^b	29^b	29^b	28^c
24							38-39	34-38	34-36	34-35	34^b	33^c	32^b	32^b	32^b	31^c	30^b	30^b	30^b	30^b	30^b	30^b	30^b
25							41^d	35-39	35-37	35-36	35^b	35^b	34^c	33^b	33^b	33^b	32^c	31^b	31^b	31^b	31^b	31^b	31^b
26							42^d	37-40	37-39	37-38	37^c	36^b	35^c	34^b	34^b	34^b	34^b	33^c	32^b	32^b	32^b	32^b	32^b
27							43^d	38-42	38-41	38-39	38^c	37^b	37^b	36^c	35^b	35^b	35^b	34^c	33^b	33^b	33^b	33^b	33^b
28							44^d	40-44	40-42	40-41	40^g	39^c	38^b	37^c	36^b	36^b	36^b	36^b	35^c	34^b	34^b	34^b	34^b
29							45^d	45^e	41-43	41-42	41^g	40^c	39^b	39^b	38^c	37^b	37^b	37^b	37^b	36^c	35^b	35^b	35^b
30								46^g	42-45		42^g	41^c	40^b	40^b	39^c	38^b	38^b	38^b	38^b	37^c	36^b	36^b	36^b
31								47-48	43-47	43-45	43^g	43^b	42^c	41^b	41^b	40^c	39^b	39^b	39^b	39^b	38^c	37^b	37^b
32								50^d	44-48	44-46	44-45	44^b	43^c	42^b	42^b	41^c	40^b	40^b	40^b	40^b	40^b	39^c	38^b
33								51^d	45-49	45-48	45-47	45^b	45^b	44^c	43^b	43^b	42^c	41^b	41^b	41^b	41^b	40^c	39^b
34								52^d	47-50	47-49	47-48	47^c	46^b	45^c	44^b	44^b	43^c	42^b	42^b	42^b	42^b	42^b	41^c
35								53^d		48-51	48-49	48^c	47^b	46^c	45^b	45^b	45^b	44^c	43^b	43^b	43^b	43^b	43^b
36								54^d	49-54	49-52	49-50	49^c	48^b	48^b	47^c	46^b	46^b	45^c	44^b	44^b	44^b	44^b	44^b
37								55^d	55^e	51-53	51-52	51^g	50^c	49^b	48^c	47^b	47^b	47^b	46^c	45^b	45^b	45^b	45^b
38									56^g	52-55	52-54		51^c	50^b	50^b	49^c	48^b	48^b	47^c	46^b	46^b	46^b	46^b
39									57^g	53-57	53-55		52^c	51^b	51^b	50^c	49^b	49^b	49^b	48^c	47^b	47^b	47^b
40									60^d	54-58	54-56			53^c	52^b	51^c	50^b	50^b	50^b	49^c	48^b	48^b	48^b
41									61^d	55-59	55-57			54^c	53^b	53^b	52^c	51^b	51^b	51^b	50^c	49^b	49^b
42									62^d	56-60	56-59			55^c	54^b	54^b	53^c	52^b	52^b	52^b	51^c	50^b	50^b
43									63^d	57-62					56^c	55^b	55^b	54^c	53^b	53^b	53^b	52^c	51^b
44									64^d						57^c	56^b	56^b	55^c	54^b	54^b	54^b	53^c	52^b
45									65^d	59-65					58^c	57^b	57^b	56^c	55^b	55^b	55^b	55^b	54^c
46									66^d	66^e					60^b	59^c	58^b	58^b	57^c	56^b	56^b	56^b	55^c
47										67^g					61^b	60^c	59^b	59^b	58^c	57^b	57^b	57^b	57^b
48										68^g					62^b	61^c	60^b	60^b	60^b	59^c	58^b	58^b	58^b
49										69-70					63^b	63^b	62^c	61^b	61^b	60^c	59^b	59^b	59^b
50										72^d					65^c	64^b	63^c	62^b	62^b	61^c	60^b	60^b	60^b
51										73^d					66^c	65^b	64^c	63^b	63^b	63^b	62^c	61^b	61^b
52										74^d					67^c	66^b	66^b	65^c	64^b	64^b	63^c	62^b	62^b
53										75^d						68^c	67^b	66^c	65^b	65^b	65^b	64^c	63^b
54										76^d						69^c	68^b	67^c	66^b	66^b	66^b	65^c	64^b
55										77^d						70^c	69^b	69^b	68^c	67^b	67^b	66^c	65^b
56										78^d	78^e					71^c	70^b	70^b	69^c	68^b	68^b	68^b	67^c
57																73^b	72^c	71^b	70^c	69^b	69^b	69^b	68^c
58																74^b	73^c	72^b	72^b	71^c	70^b	70^b	70^b
59																75^b	74^c	73^b	73^b	72^c	71^b	71^b	71^b
60											84^d					76^b	76^b	75^c	74^b	73^c	72^b	72^b	72^b

Table 1. $\gamma_{\times k}\,(n,2)$ for some values of n and k.

We implemented it in CPLEX solver [29] to obtain k-tuple dominating sets that allowed us to determine an upper bound of $\gamma_{\times k}(n, 2)$ for certain values of n and k, which together with the lower bounds obtained using Lemma 17 and Theorem 2, were tight. Notice that we have determined $\gamma_{\times k}(n, 2)$ for every n and $k \leq 18$. For the general case, fixed k, the amount of values of n for which $\gamma_{\times k}(n, 2)$ remains unknown is $\Theta(\sqrt{k})$. However, for some of them we have upper bounds arising from solving the ILP.

Table 1 shows the values of $\gamma_{\times k}(n, 2)$ stated by the results in this paper. Rows correspond to values of k while columns correspond to values of n. The superscript in each entry of the table indicates from which result it follows, and the bounds arise from k-tuple dominating sets obtained using CPLEX solver for (16) and monotonicity. (a) Domination number [18]. (b) Theorem 22.1. (c) Theorem 22.2. (d) Corollary 30. (e) Proposition 31. (f) Lower bound from Lemma 17 and upper bound from $\gamma_{\times k}$-sets found by ILP (16). (g) Lower bound from Monotonicity (Theorem 2) and upper bound from $\gamma_{\times k}$-sets found by ILP (16).

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