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ON INFINITE SEQUENCES OF MINIMAL GRAPHS CONTAINING MONOCHROMATIC TRIANGLES FOR ANY EDGE 2-COLORING

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Abstract

We present an infinite sequence of "minimal" graphs over every *odd* (at least nine) vertices, each of which contains monochromatic triangles for any edge 2-coloring. This result complements generalized Graham graphs, which constitutes an infinite sequence of minimal such graphs over every *even* (at least eight) vertices. These two results give an infinite sequence of minimal such graphs over *every* natural number (at least eight) of vertices.

Keywords: Ramsey theory, edge 2-coloring, monochromatic triangle, Graham graph.

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1. Introduction

Ramsey theory, initiated by Ramsey [13], is one of the most important areas of combinatorics. Ramsey theory studies how many elements of some structure there need to be to guarantee that a particular property on the structure holds. (See [8] as a classical textbook.) The simplest problem in graph theoretic Ramsey theory is to ask for the minimum number of vertices of complete graphs, say, K_n over n vertices, such that there is (at least) one monochromatic triangle (i.e., K_3) for any edge 2-coloring. The answer to this question is six, that is, K_6. Thus, any graph containing K_6 satisfies such a property. Here, we say that a graph $G = (V, E)$ is (p, q)-*Ramsey* if for every 2-coloring to E, there exists a K_p of the first color or a K_q of the second color. We focus on the case of $p = q = 3$, i.e., $(3,3)$-*Ramsey* graphs.

From this fact, it is natural to ask for a structure of graphs with such a property that *does not* contain K_6, which was posed by Erdős and Hajnal [5]. In what follows, we consider graphs that do not contain K_6, called K_6-*free* graphs. Answering the question, Graham [7] presented a K_6-free $(3,3)$-Ramsey graph, called here the *Graham graph*, which is on eight vertices. In fact, it is unique of all K_6-free $(3,3)$-Ramsey graphs on at most eight vertices.

More generally, a graph is *minimal* (with respect to the $(3,3)$-Ramsey) if the graph does not properly contain any $(3,3)$-Ramsey graph. Thus, K_6 as well as the Graham graph are both minimal. Following the Graham graph, Nenov [10] presented a minimal graph on nine vertices (as a graph next to the Graham graph), called here the *Nenov graph*, and it is unique of all minimal graphs on nine vertices. All minimal graphs on at most thirteen vertices are listed in [1].

Along the research of this stream, it furthermore is natural to ask if for *each* natural number $n \geq 8$, there exists a minimal $(3,3)$-Ramsey graph over n vertices. Indeed, a partial answer to this question has been given by Burr, Erdös, and Lovász [2]. For each pair of positive integers $p, q \geq 3$, there exist infinitely many minimal (p,q)-Ramsey graphs[1]. Moreover, Nenov and Khadzhiivanov [12] proposed a simple construction which gives an infinite sequence of minimal $(3,3)$-Ramsey graphs over every *even* (at least six) vertices. (We also find it in [6, 14]. See the related results below.) The graphs are $K_3 \otimes C_{2r+1}$ for all $r \geq 1$, where C_n is the cycle graph over n vertices, and the operation $\otimes$ of $G_1 \otimes G_2$ for two graphs G_1, G_2 is some kind of graph products of adding edges to all the pairs of vertices between G_1 and G_2. In fact, the graphs of $r = 1, 2$ are identical to K_6 and the Graham graph, respectively. For each *odd* number n (at least nine), Nenov [11] showed the existance of a minimal $(3,3)$-Ramsey graph with n vertices. However, his proof is not constructive. In this paper, we present a (rather) simple construction which is a new constructive proof of the above result, as follows, where the graph $G(r)$ is defined in the next subsection.

Theorem 1. *For every positive integer $r \geq 2$, the graph $G(r)$ is a minimal $(3,3)$-Ramsey graph with $2r + 5$ vertices.*

Our idea

Before presenting our idea, we note that the graph $K_3 \otimes G$ cannot be minimal $(3,3)$-Ramsey for any graph G over *even* vertices. This comes from the fact that G is non-bipartite if and only if $K_3 \otimes G$ is $(3,3)$-Ramsey. The direction of the only if part above is the lemma itself in [14]. For the sake of complementing the lemma as well as showing the impossibility, we prove the direction of the if part

[1] An asymptotic result is given in [3].

in the next section. Suppose that $K_3 \otimes G$ (over odd vertices) for some graph G (over even vertices) is $(3,3)$-Ramsey. Then G must be non-bipartite from the if part of the fact, and hence G contains an odd cycle. Thus, the graph $K_3 \otimes G$ contains $K_3 \otimes C_{2r+1}$ for some $r \geq 1$, and hence it is not minimal. Therefore, we cannot attain our goal simply by extending the construction by [6, 12, 14].

Our construction is inspired by a structure of the Nenov graph. Let K_4^- be the graph obtaining from K_4 by deleting an edge of K_4. Then the Nenov graph is the graph $K_4^- \otimes' C_5$, where the operation $\otimes'$ is some kind of graph products as follows. Let $\{v_1, \ldots, v_4\}$ (respectively, $\{u_1, \ldots, u_5\}$) be the vertices of K_4^- (respectively, C_5). See Figure 1, where indices i and j of vertices v_i and u_j are displayed. Then, about edges between K_4^- and C_5,

- for $i = 2, 3$, we add an edge between v_i and u_j for all $j = \{1, \ldots, 5\}$,
- for $i = 1, 4$, we add an edge between v_1 (respectively, v_4) and $u_1, \ldots, u_4$ (respectively, u_1, u_4, u_5.).

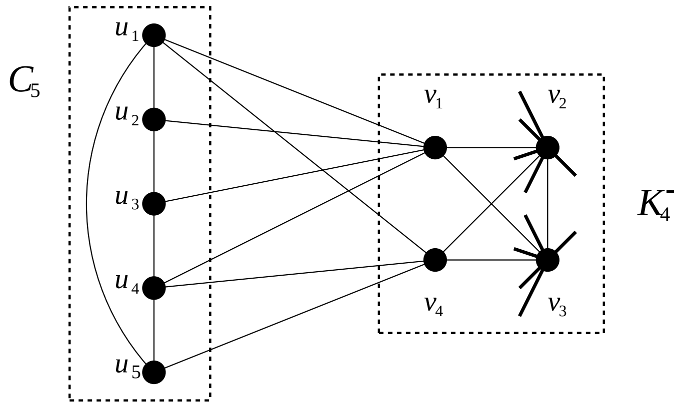

Figure 1. Nenov graph.

See the following figure, where edges in the first item above (i.e., on v_2 and v_3) are omitted. Once we figure out this structure of the Nenov graph, it is not hard to attain our goal by extending this graph, replacing C_5 with C_{2r+1} for $r \geq 2$. Note here that K_4^- is the graph obtained from two triangles by identifying two vertices and the edge between them. The graphs $K_3 \otimes C_{2r+1}$ make use of the triangle, and our construction also makes use of two triangles, but it is not so simple as the graphs $K_3 \otimes C_{2r+1}$ to show that those are minimal $(3,3)$-Ramsey. Indeed, the edges between K_4^- and C_{2r+1} are (somewhat) asymmetric, and used in an elaborate way. The graphs $G(r)$ generated by our construction are $K_4^- \otimes' C_{2r+1}$ for all $r \geq 2$, where the operation $\otimes'$ is a generalization of the case of $r = 2$, i.e., the Nenov graph, in particular, v_1 is connected to vertices u_j for $j \in \{1, \ldots, 2r-1, 2r\}$, and v_4 is connected to vertices u_j for $j \in \{1, 2r, 2r+1\}$.

The vertices v_2 and v_3 are connected to all the vertices of C_{2r+1}. (See Figure 2 in the main section.)

Related results

As is mentioned above, following the Graham graph, Nenov [10] presented a minimal graph on nine vertices. They focused on the "minimality" with respect to $(3,3)$-Ramsey, that is, a deletion of any edge gives a non-$(3,3)$-Ramsey graph. This implies the definition of minimal with respect to $(3,3)$-Ramsey, as is mentioned above. On the other hand, [6, 14] studied on the "maximality" with respect to $(3,3)$-Ramsey, which is called *co-critical* there. That is, an addition of any edge gives a $(3,3)$-Ramsey graph. In this vein of the notion, a co-critical graph is *minimal* if a deletion of any *vertex* is not co-critical. Note that the minimality for co-critical is different from that for $(3,3)$-Ramsey. Thus, a minimal co-critical graph may contain a co-critical graph as a proper subgraph. If a minimal co-critical graph does not properly contain any co-critical graph, the graph is called *strongly* minimal. It is not known [4] whether there exists an infinite sequence of strongly minimal co-critical graphs while there does exist for minimal co-critical graphs [6]. In constructing co-critical graphs (not necessarily minimal), the maximum degree as small as possible has been focused. Szabó [14] proposed a construction in which the maximum degree is $O(n^{3/4})$, slightly improving $O(n^{3/4}\log n)$ [6], while the lower bound is $\Omega(n^{1/2})$.

2. Preliminaries

In this paper, we mostly follow the standard notation and concepts of graph theory. For example, P_n, C_n, and K_n are a path graph, a cycle graph, and a complete graph on n vertices, respectively. For a graph $G = (V,E)$, a path $v_1,\ldots,v_k \in V$ (respectively, a cycle $u_1,\ldots,u_\ell,u_1 \in V$) in G is denoted by $P_k = v_1v_2\cdots v_k$ (respectively, $C_\ell = u_1u_2\cdots u_\ell$). Thus, an edge $e \in E$ is denoted by $e = uv$ for the end vertices u and v. We call K_3 a *triangle*. We denote by K_4^- the graph obtaining from K_4 by deleting an edge of K_4. For a graph $G = (V,E)$, the set of vertices of G is denoted by $V(G)$, and the set of edges of G by $E(G)$. (That is, $V = V(G)$ and $E = E(G)$.) We say that a graph $G = (V,E)$ *contains* a graph $G' = (V',E')$ if $V' \subseteq V$ and $E' \subseteq E$. Furthermore, for a subset $E' \subseteq E$, we (crudely) denote the graph $G' = (V, E \setminus E')$ by $G' = G \setminus E'$.

In this paper, we focus on the simplest case of graph theoretic Ramsey theory, that is, edge 2-colorings and monochromatic triangles. In what follows, we use red and blue as two colors.

Definition 1. For natural numbers p,q, a graph $G = (V,E)$ is *(p,q)-Ramsey* if for every edge 2-coloring, there exists a K_p colored with red or a K_q colored with blue.

Definition 2. For two graphs G_1, G_2, we denote by $G_1 \otimes G_2$ the graph obtained by completely joining a copy of G_1 to a copy of G_2, i.e., adding edges between $V(G_1)$ and $V(G_2)$.

Lemma 2 [12, 14]. *For any graph G, if G is non-bipartite, $K_3 \otimes G$ is $(3,3)$-Ramsey.*

Lemma 3. *For any graph G, if G is bipartite, $K_3 \otimes G$ is non-$(3,3)$-Ramsey.*

Proof. Let $G = (X, Y, E)$ be a bipartite graph, where $[X, Y]$ is the bipartition of G. Let $V(K_3) = \{v_x, v_y, v_z\}$. We color $E(K_3 \otimes G)$ in the following way. First, color (v_x, v_z), (v_y, v_z) with red and (v_x, v_y) with blue. Then color every edge of G with red. Finally,

1. color (v_x, v) with blue for every $v \in X$ and (v_x, v) with red for every $v \in Y$,
2. color (v_y, v) with red for every $v \in X$ and (v_y, v) with blue for every $v \in Y$,
3. color (v_z, v) with blue for every $v \in X \cup Y$.

Then it is easy to check there is no monochromatic triangle in $K_3 \otimes G$ under this coloring. Therefore, $K_3 \otimes G$ is non-$(3,3)$-Ramsey. ∎

Corollary 1. *Any graph G is non-bipartite if and only if $K_3 \otimes G$ is $(3,3)$-Ramsey.*

Fact 1. *Any edge 2-coloring of K_5 with no monochromatic triangle must consist of two monochromatic Hamilton cycles. This implies that for each vertex v of K_5, exactly two edges incident to v are colored with red, and the others with blue.*

3. Proof of Theorem 1

The graphs generated by our construction are $G(r) = K_4^- \otimes' C_{2r+1}$ for all $r \geq 2$, where the operation $\otimes'$ is the one as defined in Introduction. (See Figure 2, where indices i and j of vertices v_i and u_j are displayed.) That is, v_1 is connected to vertices u_j for $j \in \{1, \ldots, 2r-1, 2r\}$, and v_4 is connected to vertices u_j for $j \in \{1, 2r, 2r+1\}$. The vertices v_2 and v_3 are connected to all the vertices of C_{2r+1}. The subscripts of vertices of K_4^- and C_{2r+1} are taken modulo 4 and $2r+1$, respectively. We show that the graph $G(r)$ is $(3,3)$-Ramsey and minimal in this order.

3.1. $(3,3)$-Ramsey

In this subsection, we show that there is at least one monochromatic triangle for any edge 2-coloring of $G(r)$. Fix the color of v_2v_3, say (without loss of generality) red. Suppose that $G(r)$ is already edge 2-colored by red and blue. Assume to the contrary that $G(r)$ has no monochromatic triangle under the coloring. In what

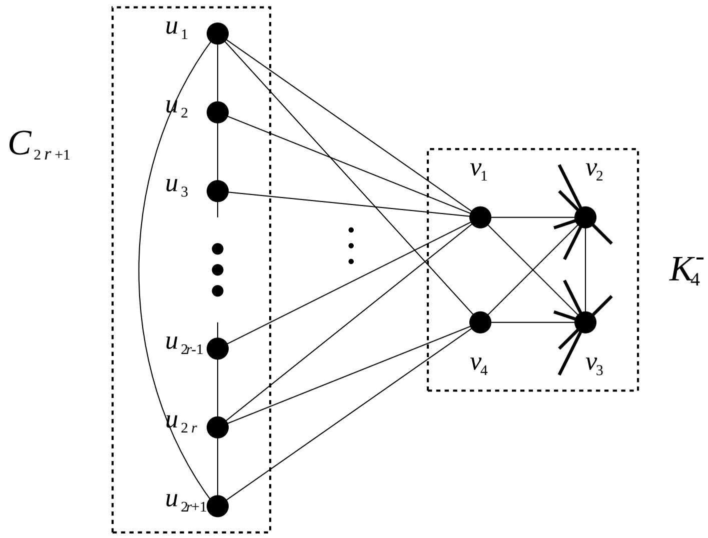

Figure 2. The graph $K_4^- \otimes' C_{2r+1}$ for all $r \geq 2$.

follows, we simply say that an edge is red (or blue) if the edge is colored with red (or blue).

Let $X = \{v_1v_2, v_1v_3, v_2v_4, v_3v_4\}$, that is, $E(K_4^-) \setminus \{v_2v_3\}$. By symmetry of v_2 and v_3, it suffices to consider the following edge 2-colorings of K_4^- so that $G(r)$ has no monochromatic triangle.

1. All edges in X are blue.
2. v_1v_2 is red and any other in X is blue.
3. v_3v_4 is red and any other in X is blue.
4. v_1v_2, v_2v_4 are red and any other in X is blue.
5. v_1v_2, v_3v_4 are red and any other in X is blue.

Before deriving a contradiction for each case, we present several observations of the edge 2-coloring of $G(r)$, which are necessary conditions for $G(r)$ to have no monochromatic triangle.

Observation 1. *For any $i \in \{1, 2, \ldots, 2r+1\}$, if v_2u_i (respectively, v_3u_i) is red, then v_3u_i (respectively, v_2u_i) must be blue.* (*Otherwise, $v_2v_3u_i$ would be monochromatic.*)

Observation 2. *If u_iu_{i+1} is red (respectively, blue) for some $i \in \{1, 2, \ldots, 2r+1\}$, then at least one of v_ju_i and v_ju_{i+1} (if any) is blue (respectively, red) for each $j \in \{2, 3\}$.* (*Otherwise, $v_ju_iu_{i+1}$ would be monochromatic.*)

Observation 3. *If v_1v_2, v_1v_3 are blue, then v_1u_j is red for any $j \in \{1, 2, \ldots, 2r\}$. (This is because of Fact 1 on K_5 over $v_1, v_2, v_3, u_j, u_{j+1}$ except for $j = 2r$.) Similarly, if v_2v_4, v_3v_4 are blue, then v_4u_j is red for any $j \in \{1, 2r, 2r+1\}$.*

Observation 4. *If the colors of v_1v_2 and v_1v_3 are different, then the colors of v_1u_j and v_1u_{j+1} are different for any $j \in \{1, 2, \ldots, 2r-1\}$. (This is because of Fact 1 on K_5 over $v_1, v_2, v_3, u_j, u_{j+1}$.) Similarly, if the colors of v_2v_4 and v_3v_4 are different, then the colors of v_4u_j and v_4u_{j+1} are different for any $j \in \{2r, 2r+1\}$.*

Case (1). For any $i \in \{1, 2, \ldots, 2r+1\}$, v_1u_i and v_4u_i (if any) are red, that comes from Observation 3, and hence all edges of C_{2r+1} are blue so that $v_1u_iu_{i+1}$ for $i \in \{1, \ldots, 2r-1\}$, $v_4u_{2r}u_{2r+1}$, and $v_4u_{2r+1}u_1$ are all not monochromatic. Then, by Observation 2, at least $r+1$ edges between v_j and C_{2r+1} are colored with red for each $j \in \{2, 3\}$. This implies that there is a monochromatic triangle $v_2v_3u_i$ for some $i \in \{1, 2, \ldots, 2r+1\}$, a contradiction.

Case (2). Firstly, we note v_2u_i must be blue for all $i \in \{1, 2, \ldots, 2r\}$, which comes from Fact 1 on K_5 over $v_1, v_2, v_3, u_i, u_{i+1}$ for each $i \in \{1, \ldots, 2r-1\}$. Then u_iu_{i+1} is red for any $i \in \{1, 2, \ldots, 2r-1\}$. On the other hand, by Observation 3, v_4u_i is red for all $i \in \{1, 2r, 2r+1\}$, and hence $u_{2r}u_{2r+1}$ and $u_{2r+1}u_1$ are blue. Here, there are two possibilities by Observation 4; (a) for any odd $i \in \{1, 3, \ldots, 2r-1\}$, v_1u_i and v_1u_{i+1} are red and blue, respectively, or (b) vice-versa. In case of (a), v_3u_i is red if i is even so that $v_1v_3u_i$ is not monochromatic for even $i \in \{1, \ldots, 2r\}$. On the other hand, v_3u_i is blue if i is odd so that $v_3u_iu_{i+1}$ is not monochromatic for odd $i \in \{1, \ldots, 2r\}$. However, since $u_{2r+1}u_1$, v_2u_1, v_3u_1 are all blue, v_2u_{2r+1} and v_3u_{2r+1} must be both red so that $v_2u_1u_{2r+1}$ and $v_3u_1u_{2r+1}$ are not chromatic. In this case, $v_2v_3u_{2r+1}$ is monochromatic, a contradiction. In case of (b), the proof is almost same as that for the case (a). The difference is the reason that v_3u_{2r+1} is red. Since v_3u_i must be blue if i is even for $i \in \{1, \ldots, 2r\}$, v_3u_{2r+1} must be red so that $v_3u_{2r}u_{2r+1}$ is not monochromatic. Therefore, we have the same contradiction as the case (a).

Case (3). Similar to Case (1), for any $i \in \{1, 2, \ldots, 2r\}$, v_1u_i is red, and hence u_ju_{j+1} for any $j \in \{1, 2, \ldots, 2r-1\}$ is blue. On the other hand, we note v_3u_i must be blue for each $i \in \{1, 2r, 2r+1\}$, which comes from Fact 1 on K_5 over $v_2, v_3, v_4, u_i, u_{i+1}$ for each $i \in \{2r, 2r+1\}$. In particular, v_3u_1 and v_3u_{2r} are fixed to blue. Since u_ju_{j+1} for all $j \in \{1, 2, \ldots, 2r-1\}$ is blue, by applying Observation 2 and Observation 1 (in this order alternately), the colors of v_3u_i and v_3u_{i+1} are alternate (v_2u_i and v_2u_{i+1} also), and hence the colors of v_3u_1 and v_3u_{2r} must be different, a contradiction.

Case (4). Similar to Case (2), v_2u_i must be blue for all $i \in \{1, 2, \ldots, 2r+1\}$, and hence all edges of C_{2r+1} are red. (The inclusion of u_{2r+1} is differ from Case (2).) Moreover, (by Observation 4) there are two possibilities; (a) for any odd

$i \in \{1, \ldots, 2r\}$, v_1u_i and v_1u_{i+1} are red and blue, respectively, or (b) vice-versa. In case of (a), v_3u_i is red (respectively, blue) if i is even (respectively, odd) for any $i \in \{1, 2, \ldots, 2r+1\}$. In particular, v_3u_{2r+1} is blue. Then, since v_3u_1 is blue, v_4u_1 must be red, which implies v_4u_1, v_4u_{2r} are red and v_4u_{2r+1} is blue (by Observation 4), which implies $v_3v_4u_{2r+1}$ is monochromatic, a contradiction. In case of (b), the proof is almost same as that for the case (a). Note that v_3u_{2r+1} must be blue so that $v_3u_1u_{2r+1}$ is not chromatic. The difference is the reason that v_4u_{2r} is red. Since v_3u_i must be blue if i is even for $i \in \{1, \ldots, 2r\}$, v_4u_{2r} must be red so that $v_3v_4u_{2r}$ is not monochromatic. Therefore, we have the same contradiction as the case (a).

Case (5). Similar to Case (2), v_2u_i must be blue for all $i \in \{1, 2, \ldots, 2r\}$, and v_3u_i must be blue for each $i \in \{1, 2r, 2r+1\}$, and hence u_iu_{i+1} is red for any $i \in \{1, 2, \ldots, 2r+1\}$. Then v_1u_1 and v_1u_{2r} must be red so that $v_1v_3u_1$ and $v_1v_3u_{2r}$ are not monochromatic. Then v_1u_i and v_1u_{i+1} are red and blue for any odd $i \in \{1, \ldots, 2r-1\}$, respectively, but however, this contradicts that v_1u_{2r} is red.

Therefore, since we have a contradiction for each case, that is, there is at least one monochromatic triangle, we conclude that $G(r)$ is $(3,3)$-Ramsey.

3.2. Minimality

In this subsection, we show that for any edge $e \in E(G(r))$, $G(r) \setminus \{e\}$ is not $(3,3)$-Ramsey, that is, there is an edge 2-coloring of $G(r) \setminus \{e\}$ without monochromatic triangles. An edge 2-coloring of a graph is *good* if the coloring does not contain a monochromatic triangle. The following theorem is useful for our proof. For a graph G, a function $c : V(G) \to \{1, 2, \ldots, k\}$ is a *vertex k-coloring* of G if $c(u) \neq c(v)$ for any $uv \in E(G)$.

Theorem 4 [9]**.** *If a graph has a vertex* 5*-coloring, then it has a good edge* 2*-coloring.*

In what follows, e denotes the edge deleted from $G(r)$. For subgraphs of $G(r)$ (or subsets of $V(G(r))$), H_1 and H_2, we denote by $E(H_1, H_2)$ the set of edges between vertices of H_1 and H_2. We divide the proof into the following two cases.

Case (i). $e \notin \{v_1u_1, v_1u_{2r}\}$. In this case, we show that $G(r) \setminus \{e\}$ has a vertex 5-coloring c of $G(r) \setminus \{e\}$ depending on which edge is e, as follows.

(i-1) $e = v_2v_3$. Since $K_4^- \setminus \{e\}$ and C_{2r+1} have a vertex 2-coloring and a vertex 3-coloring, respectively, $G(r) \setminus \{e\}$ clearly has a vertex 5-coloring.

(i-2) $e = v_1v_3$. We assign colors to vertices as follows; $c(v_2) = 1$, $c(v_4) = c(u_{2r-1}) = 2$, $c(v_1) = c(v_3) = 3$, $c(u_{2r}) = c(u_i) = 4$ and $c(u_{2r+1}) = c(u_{i+1}) = 5$ for any odd $i \in \{1, 2, \ldots, 2r-3\}$ except for u_{2r-1}.

(i-3) $e = v_2v_4$. We assign colors to vertices as follows; $c(v_1) = c(u_{2r+1}) = 1$, $c(v_2) = c(v_4) = 2$, $c(v_3) = 3$, $c(u_i) = 4$ and $c(u_{i+1}) = 5$ for any odd $i \in \{1, 2, \dots, 2r-1\}$.

(i-4) $e \in E(C_{2r+1})$. We may assume $e = u_1u_2$ since any other case can be proved similarly. Then we assign colors to vertices as follows; $c(v_1) = c(v_4) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(u_1) = c(u_i) = 4$ and $c(u_{i+1}) = 5$ for any even $i \in \{2, 4, \dots, 2r\}$.

(i-5) $e \in E(\{v_2, v_3\}, C_{2r+1})$. We may assume $e = v_2u_1$ since any other case can be proved similarly. Then we assign colors to vertices as follows; $c(v_1) = c(v_4) = 1$, $c(v_2) = c(u_1) = 2$, $c(v_3) = 3$, $c(u_i) = 4$ and $c(u_{i+1}) = 5$ for any even $i \in \{2, 4, \dots, 2r\}$.

(i-6) $e \in E(\{v_1, v_4\}, C_{2r+1}) \setminus \{v_1u_1, v_1u_{2r}\}$. We first assume $e = v_1u_i$ for $i \in \{2, 3, \dots, 2r-1\}$. Then $c(v_1) = c(v_4) = c(u_i) = 1$, $c(v_2) = 2$, $c(v_3) = 3$ and we assign colors 4 and 5 to vertices in $V(C_{2r+1}) \setminus \{u_i\}$ alternately as previous cases. Next we assume $e = v_4u_i$ for $i \in \{1, 2r, 2r+1\}$. If $i = 2r+1$, then $G(r) \setminus \{e\}$ has a vertex 5-coloring similarly to the previous case. If $i \neq 2r+1$, then $c(v_1) = c(u_{2r+1}) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(v_4) = c(u_i) = 4$, and we assign colors 4 and 5 to vertices in $V(C_{2r+1}) \setminus \{u_i, u_{2r+1}\}$ alternately so that the neighbor of u_i (not u_{2r+1}) has color 5.

As above, since $G(r) \setminus \{e\}$ has a vertex 5-coloring for each case, $G(r) \setminus \{e\}$ has a good edge 2-coloring by Theorem 4.

Case (ii). $e \in \{v_1u_1, v_1u_{2r}\}$. In this case, since $G(r) \setminus \{e\}$ does not have a vertex 5-coloring, we directly construct a good edge 2-coloring. We color several edges as follows; all edges in $\{v_1v_2, v_2v_3, v_2v_4, v_3u_{2r+1}, v_4u_1, v_4u_{2r}\} \cup E(C_{2r+1})$ are red and ones in $\{v_1v_3, v_3v_4, v_3u_1, v_3u_{2r}, v_4u_{2r+1}\} \cup E(\{v_2\}, C_{2r+1})$ are all blue. Then we complete a good edge 2-coloring depending on which of v_1u_1 and v_1u_{2r} is e, as follows.

- If $e = v_1u_1$, then v_1u_i is red (respectively, blue) if i is even (respectively, odd) for any $i \in \{2, \dots, 2r\}$ and v_3u_i is blue (respectively, red) if i is even (respectively, odd) for any $i \in \{2, \dots, 2r-1\}$.
- If $e = v_1u_{2r}$, then v_1u_i is red (respectively, blue) if i is odd (respectively, even) for any $i \in \{1, 2, \dots, 2r-1\}$ and v_3u_i is red (respectively, blue) if i is even (respectively, odd) for any $i \in \{2, \dots, 2r-1\}$.

Therefore, since $G(r) \setminus \{e\}$ has a good edge 2-coloring for each case, we conclude that $G(r)$ is minimal.

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