

ON THE WORD-REPRESENTABILITY OF K_m - K_n GRAPHS

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Abstract

Word-representable graphs are a class of graphs that can be represented by words, where edges and non-edges are determined by the alternation of letters in those words. Several papers in the literature have explored the word-representability of split graphs, in which the vertices can be partitioned into a clique and an independent set. In this paper, we initiate the study of the word-representability of graphs in which the vertices can be partitioned into two cliques. We provide a complete characterization of such word-representable graphs in terms of forbidden subgraphs when one of the cliques has a size of at most four. In particular, if one of the cliques is of size four, we prove that there are seven minimal non-word-representable graphs.

Keywords: word-representable graph, semi-transitive graph, semi-transitive orientation.

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1. INTRODUCTION

Two letters x and y alternate in a word w if after deleting from w all letters but the copies of x and y we either obtain a word $xyxy\cdots$ or a word $yxyx\cdots$ (of even or odd length). A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y , $x \neq y$, alternate in w if and only if $xy \in E$; each letter in V must appear in w . The unique minimal (by the number of vertices) non-word-representable graph on 6 vertices is the wheel graph W_5 (which is the cycle graph B_5 with an all adjacent vertex added), while there are 25 non-word-representable graphs on 7 vertices and they can be found in [18].

The introduction of word-representable graphs was influenced by alternation word digraphs used in [17], which served as a tool to study the celebrated Perkins semigroup, B_2^1 , a central concept in semigroup theory since 1960, particularly as a source of examples and counterexamples.

An orientation of a graph is *semi-transitive* if it is acyclic (there are no directed cycles), and for any directed path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k$, either there is no edge between v_0 and v_k , or $v_i \rightarrow v_j$ is an edge for all $0 \leq i < j \leq k$. An undirected graph is *semi-transitive* if it admits a semi-transitive orientation. A shortcut in an acyclically oriented graph G is an induced *non-transitive* subgraph G' on vertices $\{v_0, v_1, \dots, v_k\}$, $k \geq 3$, containing $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k$ and $v_0 \rightarrow v_k$ (that is, $v_i \rightarrow v_j$ is not present in G' for at least one pair of i and j with $0 \leq i < j \leq k$). The edge $v_0 \rightarrow v_k$ is called *shortcutting edge*.

A fundamental result in the area of word-representable graphs is the following theorem.

Theorem 1 [10]. *A graph is word-representable if and only if it admits a semi-transitive orientation.*

Corollary 2 [10]. *Any 3-colourable graph is word-representable.*

There is an extensive line of research in the literature on word-representable graphs [14]. Most relevant to our work is a series of papers [5, 11, 12, 13, 16] that explore the word-representability of split graphs. A split graph is a graph whose vertices can be partitioned into a clique and an independent set [8]. Split graphs appear in the literature in various contexts (see, for example, [7] and references therein).

In this paper, we extend the study of word-representability of split graphs to what we call K_m - K_n graphs by characterizing, in terms of forbidden subgraphs, word-representable K_m - K_n graphs for any $1 \leq m \leq 4$ and $n \geq 1$. A K_m - K_n graph is one that can be split into cliques K_m and K_n , where K_m is maximal; that is, there is no vertex in K_n that is connected to every vertex in K_m . Let

$V(K_m) = \{1, \dots, m\}$ and $V(K_n) = \{m+1, \dots, m+n\}$, where for a graph G , $V(G)$ is the set of vertices in G .

Both split graphs and K_m - K_n graphs are relevant to the *speed of hereditary classes of graphs* and their asymptotic structure, which have been extensively studied in the literature. These concepts were used in [6] to find asymptotics for the number of word-representable graphs. To elaborate, it is known that for every hereditary class X , that is different from the class of all finite graphs,

$$\lim_{n \rightarrow \infty} \frac{\log_2 X_n}{\binom{n}{2}} = 1 - \frac{1}{k(X)},$$

where $k(X)$ is a natural number known as the *index* of X . To define this notion let us denote by $\mathcal{E}_{i,j}$ the class of graphs whose vertices can be partitioned into at most i independent sets and j cliques. In particular, $\mathcal{E}_{p,0}$ is the class of p -colourable graphs. Then $k(X)$ is the largest k such that X contains $\mathcal{E}_{i,j}$ with $i+j=k$ for some i and j . This result was obtained independently by Alekseev [1] and Bollobás and Thomason [3, 4] and is known nowadays as the Alekseev-Bollobás-Thomason Theorem (see e.g. [2]).

1.1. Relevant tools to study word-representability of graphs

The following theorem is a special case of Theorem 5.4.7 in [14], and it allows the reduction of questions about the word-representability of arbitrarily large graphs within certain graph families to smaller, more manageable graphs.

Theorem 3 [14]. *Suppose u and v are two vertices in a graph G and neighbourhoods of u and v are the same, except u and v are allowed to be connected. Moreover, let G' be obtained from G by removing u . Then, G is word-representable if and only if G' is word-representable.*

The following two lemmas are instrumental for us in proving non-word-representability of graphs. A vertex in a graph G is a *source* (respectively, *sink*) if all edges incident with v are oriented outwards (respectively, towards) it.

Lemma 4 [15]. *Suppose that an undirected graph G has a cycle $C = x_1x_2 \dots x_mx_1$, where $m \geq 4$ and the vertices in $\{x_1, x_2, \dots, x_m\}$ do not induce a clique in G . If G is oriented semi-transitively, and $m-2$ edges of C are oriented in the same direction (i.e., from x_i to x_{i+1} or vice versa, where the index $m+1 := 1$), then the remaining two edges of C are oriented in the opposite direction.*

Lemma 5 [18]. *Suppose that a graph G is word-representable, and v is a vertex in G . Then, there exists a semi-transitive orientation of G where v is a source (or a sink).*

1.2. Our methodology and organization of the paper

Proving that a given graph is not word-representable, or equivalently, not semi-transitive, often involves examining all possible extensions of partial orientations and demonstrating that none results in a semi-transitive orientation. Lemmas 4 and 5 are crucial here because they dramatically reduce the number of orientations to consider. We refer to [18] for more details on this approach.

By a “line” of a proof we mean a sequence of instructions that directs us in orienting a partially oriented graph and necessarily ends with detecting a shortcut showing that this particular orientation branch will not produce a semi-transitive orientation. The idea is that if no branch produces a semi-transitive orientation, then the graph is non-semi-transitive.

Each proof begins with assumptions on orientations of certain edges, and there are four types of instructions.

- “ $Ba \rightarrow b$ (Copy x)” means “Branch on edge ab , orient the edge as $a \rightarrow b$, create a copy of the current version of the graph except orient the edge ab there as $b \rightarrow a$, and call the new copy x ; leave Copy x aside and continue to follow the instructions”. The instruction B occurs when no application of Lemma 4 is possible in the partially oriented graph.
- “ MCx ” means “Move to Copy x ”, where Copy x of the graph in question is a partially oriented version of the graph that was created at some point in the branching process. This instruction is always followed by an oriented edge $a \rightarrow b$ to remind the reader of the directed edge obtained after the application of the branching process.
- “ $Oa \rightarrow b$ ($Cabc$)” means orient the edge ab as $a \rightarrow b$ in the cycle abc to avoid a directed cycle. If instead of a triangle we see a longer cycle, then we deal with an application of Lemma 4 to get a cycle where all but two edges are oriented in one direction, and one of the remaining two edges is oriented in the opposite direction.
- “ $Oa \rightarrow b$ $Oc \rightarrow d$ ($Cxyz \dots$)” means that Lemma 4 is applied to cycle $xyz \dots$ to create new directed edges, $a \rightarrow b$ and $c \rightarrow d$.

Each line ends with “S: $xy \dots z$ ” indicating a shortcut with the shortcutting edge $x \rightarrow z$ is obtained.

We note that for smaller graphs, particularly those with relatively few edges, proofs of non-word representability can be obtained in an automated manner using user-friendly software [19]. Any proof of “reasonable size” can be easily verified [18].

For a class of graphs in question (with a fixed K_m), we consider the graph on $m + 2^m - 1$ vertices obtained from K_m by adding vertices that are connected to all possible subsets of vertices in K_m of size at most $m - 1$, ensuring that no two newly added vertices share the same neighbourhood. Then we create

the clique K_n , where $n = 2^m - 1$, from the newly added vertices and denote the resulting graph (obtained by adding the new edges and vertices to K_m) as H_m . By Theorem 3, finding all minimal non-word-representable subgraphs for this class of graphs is equivalent to finding all minimal non-word-representable subgraphs for H_m , which can be achieved by sequentially removing the vertices of H_m one by one in all possible ways. If H_m is word-representable then all graphs in the class are word-representable.

To shorten our proofs of Theorems 8 and 10, we first establish in Lemmas 7 and 9 the non-word-representability of all minimal non-word-representable graphs under consideration (as identified through computer experiments). We then use this result to argue that no other minimal non-word-representable graphs exist within the class. The latter is achieved through a branching process in which vertices are removed one by one, as described earlier. A branch terminates when a word-representable graph is obtained. At each step, the non-word-representability of a graph is demonstrated by explicitly identifying a minimal non-word-representable subgraph. Word-representability, on the other hand, can be established either by providing a semi-transitive orientation or by verifying that the graph is not among the 26 known non-word-representable graphs with at most 7 vertices. More commonly, however, this is done using publicly available, user-friendly, and independently developed software [9, 19], which significantly reduces the space required for this paper.

The paper is organized as follows. In Section 2, we characterize word-representable K_m - K_n graphs for $m \in \{1, 2, 3\}$. In Section 3, we characterize word-representable K_4 - K_n graphs. Two of the cases in the proof of Theorem 10 in Section 3, which together require more than five pages, have been moved to the Appendix. Finally, in Section 4, we provide concluding remarks.

2. K_m - K_n GRAPHS FOR $m \in \{1, 2, 3\}$

In what follows, if u is a vertex in a graph G and G' is a subgraph of G , then $N_{G'}(u)$ is the set of neighbours of u (the vertices connected to u by an edge) in G that belong to G' .

Theorem 6. *Any graph K_m - K_n , where $m \in \{1, 2\}$ and $n \geq 1$, is word-representable.*

Proof. Let $m = 1$. By definition, there is no edge between K_1 and K_n (otherwise, K_n would not be maximal), and the word $1123 \cdots (n+1)$ represents the graph K_1 - K_n .

Let $m = 2$. H_2 , the largest possible graph K_2 - K_n we need to consider, is on 5 vertices, $V(K_n) = \{3, 4, 5\}$, $N_{K_2}(3) = \emptyset$, $N_{K_2}(4) = \{1\}$, and $N_{K_2}(5) = \{2\}$,

where K_2 is the subgraph of H_3 given by the vertices in $\{1, 2\}$. Since H_3 is 3-colourable, by Corollary 2, it is word-representable. ■

Lemma 7. *The graph A_3 in Figure 1 is a minimal non-word-representable graph.*

Proof. The graph A_3 is Graph 17 in Figure 3 in [18] containing minimal non-word-representable graphs on 7 vertices. In particular, the minimality follows from the fact that removing any vertex we do not obtain W_5 , the only non-word-representable graph on 6 vertices. An alternative, short proof of the non-word-representability of A_3 goes as follows. Suppose, by Lemma 5, that vertex 10 is a source. Then, using Lemma 4 and the proof format from Section 1.2, we obtain the following, where “(10)” refers to the vertex labeled 10.

1. $B8 \rightarrow 9$ (Copy 2) $O8 \rightarrow 2$ (C2(10)98) $O3 \rightarrow 2$ (C28(10)3) $O3 \rightarrow 9$ (C2893) $O8 \rightarrow 4$ (C2(10)48) $O4 \rightarrow 9$ (C3(10)49) $O1 \rightarrow 9$ $O8 \rightarrow 1$ (C1948) S:819(10).
2. $MC2$ $9 \rightarrow 8$ $O2 \rightarrow 8$ (C2(10)98) $O2 \rightarrow 3$ (C28(10)3) $O9 \rightarrow 3$ (C2893) $O4 \rightarrow 8$ (C2(10)48) $O9 \rightarrow 4$ (C3(10)49) $O9 \rightarrow 1$ $O1 \rightarrow 8$ (C1948) S:918(10). ■

Theorem 8. *A K_3 - K_n graph is word-representable if and only if it does not contain A_3 in Figure 1 as an induced subgraph.*

Proof. The graph H_3 on 10 vertices is shown in Figure 1, where the vertices marked by squares represent the clique K_n , with its edges omitted for a clearer visual representation of the graph. The graph A_3 , also shown in Figure 1, is the only minimal forbidden subgraph of H_3 . Indeed, using symmetries, we only need to consider four cases, where the first vertex removed corresponds to the smallest label removed.

- Vertex 1 is removed. By Theorem 3, we can remove vertices 5, 8, 9 as they have the same neighbourhoods as, respectively, vertices 4, 6, 7. The obtained graph, A_1 , is on 6 vertices, and $A_1 \neq W_5$, so A_1 is word-representable. In Figure 1 we provide a semi-transitive orientation of A_1 , hence giving an alternative justification for word-representability of A_1 by Theorem 1. Therefore, this case does not give us any minimal non-word-representable subgraphs.
- Vertex 4 is removed. In Figure 1, we provide a semi-transitive orientation of the obtained graph, A_2 . The fact that the orientation is semi-transitive can be checked directly via considering directed paths of length 3 or more beginning at vertices 1, 3, 5, 8 and 9 and applying the definition of a semi-transitive orientation. However, a simpler way to verify this claim is to use any of the available user-friendly software [9, 19].
- Vertex 5 is removed. Note that the obtained graph contains the graph A_3 in Figure 1 as an induced subgraph, which is minimal non-word-representable by Lemma 7. To find other potentially minimal non-word-representable subgraphs, we need to ensure that A_3 is not a subgraph in the graph. This can be achieved

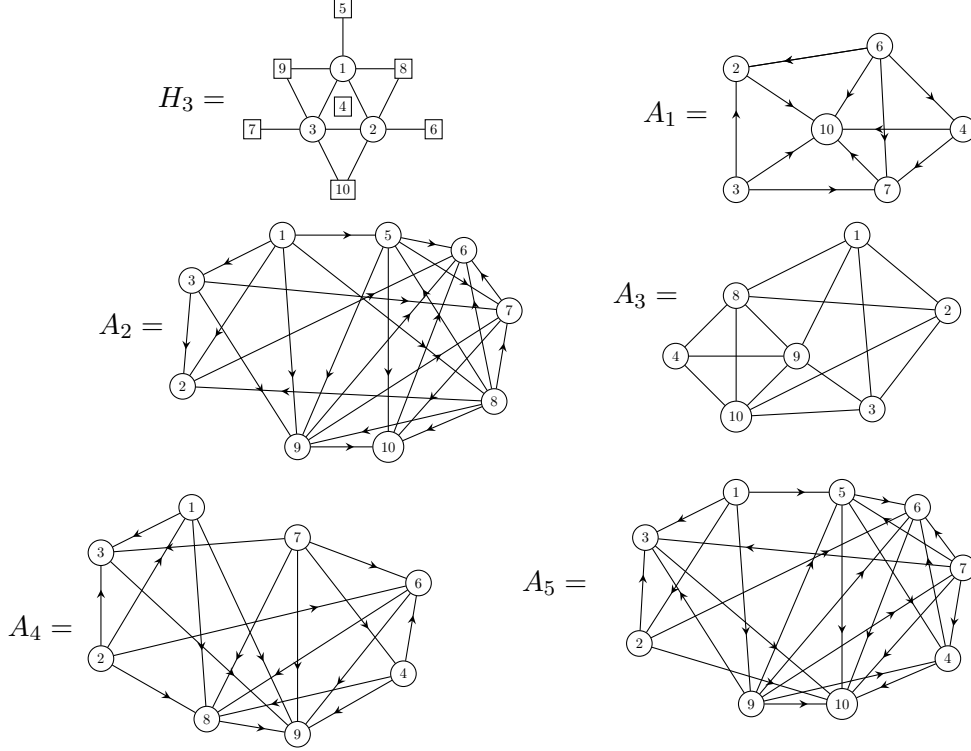


Figure 1. The graphs H_3 and A_1 – A_5 in the proof of Theorem 8. The vertices marked by squares in H_3 form a clique, with its edges omitted for a clearer visual representation of the graph.

by removing one of the vertices 8, 9, or 10. Since vertices 8 and 9 have the same neighborhoods, we only need to consider two cases.

- Vertex 8 is removed. The resulting graph is word-representable, as it is a subgraph of the word-representable graph A_5 , which is considered below and presented in Figure 1.

- Vertex 10 is removed. The obtained graph, A_4 , is word-representable, and its semi-transitive orientation is presented in Figure 1. Checking that the orientation is semi-transitive can be done directly, or via software [9, 19].

- Vertex 8 is removed. In Figure 1, we provide a semi-transitive orientation of the obtained graph, A_5 . Again, semi-transitivity of the orientation can be checked directly, but it is more easily done using software [9, 19]. ■

3. THE WORD-REPRESENTABILITY OF K_4 - K_n GRAPHS

Lemma 9. *The subgraphs of K_4 - K_n in Figure 2 are minimal non-word-representable.*

Proof. The minimality of B_1 and B_2 follows from the fact that removing any vertex in it we have a graph on 6 vertices different from W_5 , and hence word-representable. The minimality of B_3 – B_7 follows from the fact that removing any vertex in it we have a graph on 7 vertices different from any of the 25 non-word-representable graphs in Figure 3 in [18]. Alternatively, one can use the user-friendly software [9] to check that the obtained graphs by removing a single vertex are word-representable.

Next, we prove non-word-representability of B_1 – B_7 . In each case we use Lemma 5 to assume a particular vertex being a source, and the format of the proof presented in Section 1.2 (that relies on Lemma 4).

Graph B_1 . Let vertex 1 be a source.

1. B5→6 (Copy 2) O5→2 (C1652) O3→2 (C1523) O3→6 (C1632) O4→2 (C1524) O3→4 (C1634) O7→2 O3→7 (C2734) S:1372.
2. MC2 6→5 O2→5 (C1652) O2→3 (C1523) O6→3 (C1632) O2→4 (C1524) O4→3 (C1634) O2→7 O7→3 (C2734) S:1273.

Graph B_2 . Let vertex 1 be a source.

1. B6→7 (Copy 2) O6→3 (C1763) O2→3 (C1632) O2→5 (C1523) O6→5 (C1652) O7→5 (C1752) O2→4 (C1524) O7→4 (C1742) S:1674.
2. MC2 7→6 O3→6 (C1763) O3→2 (C1632) O5→2 (C1523) O5→6 (C1652) O5→7 (C1752) O4→2 (C1524) O4→7 (C1742) S:1476.

Graph B_3 . Let vertex 1 be a source.

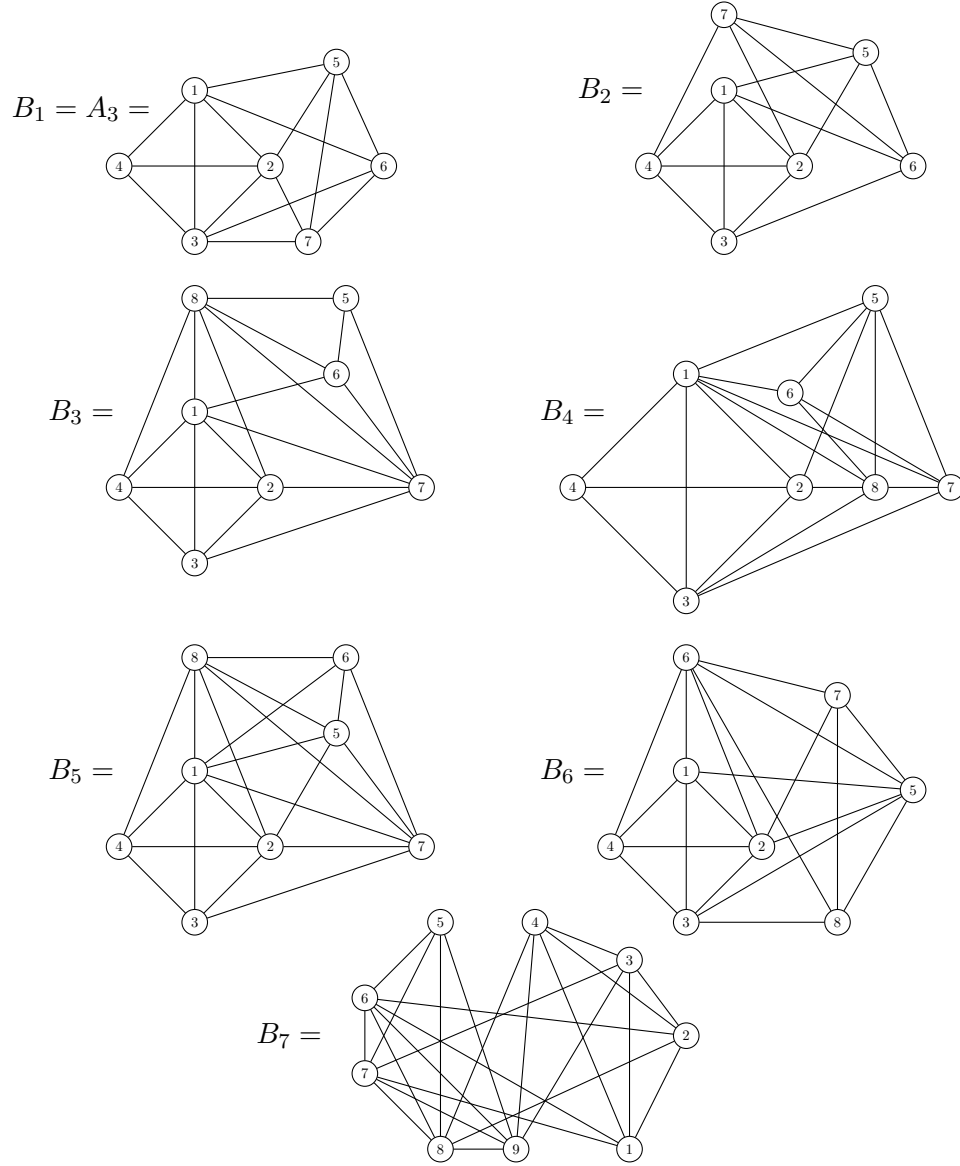
1. B6→8 (Copy 2) O3→8 (C1683) O3→2 (C1832) O4→8 (C1684) O4→2 (C1842) O7→2 (C1724) O7→6 (C1672) S:1768.
2. MC2 8→6 O8→3 (C1683) O2→3 (C1832) O8→4 (C1684) O2→4 (C1842) O2→7 (C1724) O6→7 (C1672) S:1867.

Graph B_4 . Let vertex 1 be a source.

1. B7→8 (Copy 2) O2→8 (C1782) O5→8 (C1582) O3→8 (C1583) O6→8 (C1683) O2→4 (C1824) O2→6 (C1624) O5→6 (C1562) O7→6 (C1762) O2→3 (C1623) O7→3 (C1732) O7→5 (C1573) O4→3 (C1734) S:1438.
2. MC2 8→7 O8→2 (C1782) O8→5 (C1582) O8→3 (C1583) O8→6 (C1683) O4→2 (C1824) O6→2 (C1624) O6→5 (C1562) O6→7 (C1762) O3→2 (C1623) O3→7 (C1732) O5→7 (C1573) O3→4 (C1734) S:1834.

Graph B_5 . Let vertex 1 be a source.

1. B7→8 (Copy 2) O7→3 (C1873) O7→5 (C1573) O7→2 (C1572) O7→6 (C1673) O4→2 (C1724) O6→2 (C1624) O6→5 (C1562) O3→2 (C1623) O8→2 (C1823) O8→5 (C1582) O4→3 (C1734) O4→8 (C1843) S:1485.

Figure 2. Minimal forbidden subgraphs in K_4 - K_n .

2. MC2 $8 \rightarrow 7$ $O3 \rightarrow 7$ (C1873) $O5 \rightarrow 7$ (C1573) $O2 \rightarrow 7$ (C1572) $O6 \rightarrow 7$ (C1673) $O2 \rightarrow 4$ (C1724) $O2 \rightarrow 6$ (C1624) $O5 \rightarrow 6$ (C1562) $O2 \rightarrow 3$ (C1623) $O2 \rightarrow 8$ (C1823) $O5 \rightarrow 8$ (C1582) $O3 \rightarrow 4$ (C1734) $O8 \rightarrow 4$ (C1843) S:1584.

Graph B_6 . Let vertex 1 be a source.

1. $B7 \rightarrow 8$ (Copy 2) $O7 \rightarrow 2$ (C1872) $O7 \rightarrow 5$ (C1572) $O7 \rightarrow 3$ (C1573) $O4 \rightarrow 2$ (C1724) $O4 \rightarrow 8$ (C1842) $O4 \rightarrow 3$ (C1734) $O5 \rightarrow 8$ (C1584) $O3 \rightarrow 8$ (C1583) $O3 \rightarrow 2$ (C1832) $O6 \rightarrow 2$ $O7 \rightarrow 6$ (C2673) S:1762.

2. MC2 $8 \rightarrow 7$ $O2 \rightarrow 7$ (C1872) $O5 \rightarrow 7$ (C1572) $O3 \rightarrow 7$ (C1573) $O2 \rightarrow 4$ (C1724) $O8 \rightarrow 4$ (C1842) $O3 \rightarrow 4$ (C1734) $O8 \rightarrow 5$ (C1584) $O8 \rightarrow 3$ (C1583) $O2 \rightarrow 3$ (C1832) $O2 \rightarrow 6$ $O6 \rightarrow 7$ (C2673) S:1267.

Graph B_7 . Let vertex 7 be a source.

1. $B7 \rightarrow 8$ (Copy 2) $O7 \rightarrow 3$ (C3987) $O4 \rightarrow 3$ (C3794) $O4 \rightarrow 8$ (C3784) $O7 \rightarrow 5$ (C3957) $O7 \rightarrow 6$ (C3967) $O5 \rightarrow 8$ (C4958) $O6 \rightarrow 8$ (C4968) $B5 \rightarrow 6$ (Copy 3) $O7 \rightarrow 1$ $O1 \rightarrow 6$ (C1756) S:7168.

2. MC3 $6 \rightarrow 5$ $O2 \rightarrow 8$ $O6 \rightarrow 2$ (C2856) S:7628.

3. MC2 $8 \rightarrow 7$ $O3 \rightarrow 7$ (C3987) $O3 \rightarrow 4$ (C3794) $O8 \rightarrow 4$ (C3784) $O5 \rightarrow 7$ (C3957) $O6 \rightarrow 7$ (C3967) $O8 \rightarrow 5$ (C4958) $O8 \rightarrow 6$ (C4968) $B5 \rightarrow 6$ (Copy 4) $O8 \rightarrow 2$ $O2 \rightarrow 6$ (C2856) S:8267.

4. MC4 $6 \rightarrow 5$ $O1 \rightarrow 7$ $O6 \rightarrow 1$ (C1756) S:8617. ■

The proof of the following theorem follows the same approach as the proof of Theorem 8. However, it involves a significantly larger number of cases. To keep the proof concise, we introduce a new proof format and move the two longest out of five cases to the Appendix.

Theorem 10. *A K_4 - K_n graph is word-representable if and only if it avoids the graphs B_1 – B_7 in Figure 2 as induced subgraphs.*

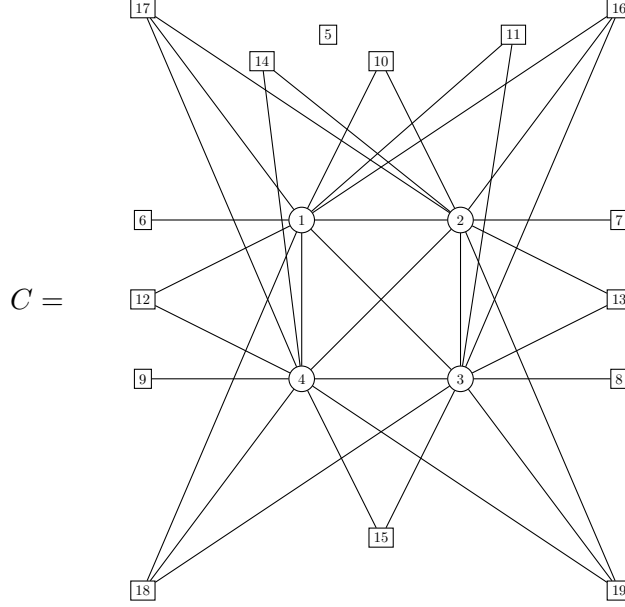
Proof. By Theorem 3, we can assume that the most general case of a graph K_4 - K_n is the graph C in Figure 3, where again we use square boxes to indicate vertices connected to each other in the larger clique.

Because of the symmetries, we need to consider four cases of removing a lexicographically smallest vertex to find all minimal non-word-representable subgraphs.

Case 1. Vertex 1 is removed (which is equivalent to removing vertex 2, 3, or 4). After removing vertices having the same neighbourhoods and renaming the vertices, we obtain a graph isomorphic to the graph H_3 in Figure 1, which is the most general K_3 - K_n graph considered in Theorem 8. Therefore, in this case, we obtain a single minimal non-word-representable subgraph, which is $A_3 = B_1$.

Case 2. Vertex 5 is removed; vertices 1–4 are retained. In this case, we obtain the minimal non-word-representable graphs B_1 – B_6 . See the Appendix for details.

Case 3. Vertex 6 is removed (which is equivalent to removing vertex 7, 8, or 9); vertices 1–5 are retained. In this case, we obtain the minimal non-word-representable graphs B_1 – B_7 . See the Appendix for details.

Figure 3. The most general case of a graph K_4-K_n .

Case 4. Vertex 10 is removed (which is equivalent to removing a vertex in $\{11, 12, 13, 14, 15\}$); vertices 1–9 are retained. Note that since there is an occurrence of B_7 formed by the vertices 1, 2, 3, 4, 5, 11, 12, 13, and 14, we do not need to consider the cases involving the removal of vertices labeled higher than 14 (and retaining the vertices 1–9 and 11–14). Moreover, by symmetry, removing vertex 11 is equivalent to removing a vertex in $\{12, 13, 14\}$, so without loss of generality, we assume that vertices 10 and 11 are removed. Using the fact that removing vertices 13 and 14 gives a graph isomorphic to one obtained by removing the vertices 13 and 15, there are six subcases to consider, and in all of them, we obtain graphs containing B_2 , formed by 2, 3, 4, 7, 8, 9, and 19: 10–17, 10–14.16–18, 10–13.16–18, 10.11.13.14.16–18, 10.11.14–18, and 10.11.14.16–18; see the Appendix for an explanation of the shorthand notation used here. Hence in Case 4, no other minimal non-word-representable graphs exist other than those in Figure 2.

Case 5. Vertex 16 is removed (which is equivalent to removing vertex 17, 18, or 19); vertices 1–15 are retained. This case does not give any new minimal non-word-representable subgraphs, as all graphs in this case contain, for example, a copy of the non-word-representable subgraph B_1 , formed by vertices 1, 2, 3, 4, 10, 11, and 13 as an induced subgraph.

Hence the graphs in B_1 – B_7 in Figure 2 are the only minimal forbidden non-word-representable subgraphs in K_4-K_n . ■

4. CONCLUDING REMARKS

In this paper, we extended the study of word-representability of split graphs to K_m - K_n graphs by characterizing word-representable K_m - K_n graphs, in terms of forbidden subgraphs, for any $1 \leq m \leq 4$ and $n \geq 1$. Extending our studies to the cases of $m \geq 5$ presents a challenging problem with our current approach, and we leave this as an open direction for further research.

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APPENDIX. CASES 2 & 3 IN THE PROOF OF THEOREM 10

In what follows, we display all possible scenarios for removing one vertex at a time, using symmetries whenever possible to reduce the number of cases to consider. The notation $x_1.x_2.\dots.x_k$, where $x_1 < x_2 < \dots < x_k$, is used to indicate the non-word-representable graph obtained by first removing vertex x_1 , then vertex x_2 , and so on, and finally vertex x_k , with the property that removing any other vertex with label larger than x_k results in a word-representable graph. This can be verified using software [9, 19]. In each case, we give a reason for

the graph being non-word-representable by indicating which of the graphs in $\{B_1, \dots, B_7\}$ is contained in it as an induced subgraph (we list vertices forming such a subgraph). Often, we use shorthand notation by replacing $x.(x+1).\dots y$ with $x-y$; for example, 6.10–14.16.17 represents 6.10.11.12.13.14.16.17.

Note also that, for example, removing 6–14 and 6–13.15 results in isomorphic graphs; therefore, it is sufficient to consider only the removal of 6–14 (the lexicographically smallest case). We frequently and implicitly use similar observations throughout the proof.

Case 2.

- 5–10; contains B_1 formed by 1, 2, 3, 4, 11, 12, 15;
- 5–11; contains B_1 formed by 1, 2, 3, 4, 13, 14, 15;
- 5–12; contains B_1 formed by 1, 2, 3, 4, 13, 14, 15;
- 5–11.13; contains B_2 formed by 1, 2, 3, 4, 12, 14, 15;
- 5–8.10–13; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10–14; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10–15; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10–14.16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10–14.16.17; contains B_5 formed by 1, 2, 3, 4, 9, 15, 18, 19;
- 5–8.10–14.17; contains B_5 formed by 1, 2, 3, 4, 9, 15, 18, 19;
- 5–8.10–13.16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10–13.16–18; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 5–8.10–13.16.17; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 5–8.10–13.17; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 5–8.10–13.17.18; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 5–8.10–12.14; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–8.10–12.14.15; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–8.10–12.14.16; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–8.10–12.14.16.17; contains B_1 formed by 1, 2, 3, 4, 9, 15, 18, 19;
- 5–8.10.11.13.14; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10.11.13–15; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10.11.13–16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10.11.13.14.16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–8.10.11.13.14.16.17; contains B_4 formed by 1, 2, 3, 4, 9, 12, 15, 18;
- 5–8.10.11.13.14.17; contains B_4 formed by 1, 2, 3, 4, 9, 12, 15, 18;
- 5–8.10.11.14; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–8.10.11.14.15; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–8.10.11.14–16; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–8.10.11.14.16; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–8.10.11.14.16.17; contains B_4 formed by 1, 2, 3, 4, 9, 12, 15, 18;
- 5–8.10.11.14.17; contains B_4 formed by 1, 2, 3, 4, 9, 12, 15, 18;

- 5–8.10.12–14; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12–15; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10–14.16; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12–14.16.17; contains B_5 formed by 1, 2, 3, 4, 9, 15, 18, 19;
- 5–8.10.12–14.16.17.18; contains B_6 formed by 1, 2, 3, 4, 9, 11, 15, 19;
- 5–8.10.12–14.16.18; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12–14.18; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12.13.16; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12.13.15.16; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12.13.16–18; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 5–8.10.12.13.16.18; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12.13.17.18; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 5–8.10.12.13.18; contains B_1 formed by 1, 2, 3, 9, 11, 17, 19;
- 5–8.10.12.15.16; contains B_1 formed by 1, 2, 3, 9, 11, 13, 17;
- 5–8.10.15; contains B_1 formed by 1, 2, 3, 9, 11, 13, 17;
- 5–8.10.15.16; contains B_1 formed by 1, 2, 3, 9, 11, 13, 17;
- 5–8.12–14.15; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12–17; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12–18; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12–14.16.17; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12–14.16–18; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12–14.16.18; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12–14.18; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12.13.16.17; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–8.12.13.17.18; contains B_1 formed by 1, 2, 3, 9, 10, 11, 19;
- 5–7.10–16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–7.10–15.18; contains B_6 formed by 1, 2, 3, 4, 8, 9, 16, 19;
- 5–7.10–14.16.17.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10–14.16.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10–14.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10–13.15.16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–7.10–13.15.17.18; contains B_4 formed by 1, 2, 3, 8, 9, 14, 16, 19;
- 5–7.10–13.16–18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10–13.16.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10–13.17.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10–13.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10.15.16; contains B_1 formed by 1, 2, 3, 9, 11, 13, 17;
- 5–7.11–18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 5–7.11–14.16–18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 5–7.11–14.16.18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 5–7.11–14.18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 5–7.11.13–18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;

- 5–7.11.13–15.17.18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.13.14.16–18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.13.14.16.18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.13.14.17.18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.13.14.18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.14–18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.14.16–18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.14.16.18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.11.14.18; contains B_1 formed by 1, 2, 4, 8, 10, 12, 19;
- 5–7.10.11.13.14–16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–7.10.11.13–15.17.18; contains B_1 formed by 1, 2, 4, 8, 12, 16, 19;
- 5–7.10.11.13–15.18; contains B_1 formed by 1, 2, 4, 8, 12, 16, 19;
- 5–7.10.11.13.14.16–18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10.11.13.14.16.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10.11.13.14.17.18; contains B_1 formed by 1, 2, 4, 8, 12, 16, 19;
- 5–7.10.11.13.14.18; contains B_1 formed by 1, 2, 4, 8, 12, 16, 19;
- 5–7.10.11.13.15.16; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5–7.10.11.14.15.16; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5–7.10.11.14.16–18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10.11.14.16.18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 5–7.10.11.14.15.18; contains B_1 formed by 1, 2, 4, 8, 12, 16, 19;
- 5–7.10.11.14.18; contains B_1 formed by 1, 2, 4, 8, 12, 16, 19;
- 5–7.10.11.15.16; contains B_1 formed by 1, 2, 3, 9, 13, 17, 18;
- 5.6.10–17; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10–14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10–14.16.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10–13.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10–13.17.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.13–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.13–16.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.13.14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.13.14.16.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.14.15–17; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.14.15–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.11.14.16.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.6.10.13–18; contains B_1 formed by 1, 3, 4, 7, 11, 12, 19;
- 5.6.10.13–17; contains B_1 formed by 1, 3, 4, 7, 11, 12, 19;
- 5.6.10.13.15–18; contains B_1 formed by 1, 3, 4, 7, 11, 12, 19;
- 5.6.10.15–18; contains B_1 formed by 1, 3, 4, 7, 11, 12, 19;

- 5.10–17.18; contains B_1 formed by 1, 2, 3, 9, 17, 18, 19;
- 5.10–13.16–18; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 5.10–14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.10.11.13–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.10.11.13.14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.10.11.13.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.10.11.14.15–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.10.11.14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 5.10.13–18; contains B_1 formed by 1, 3, 4, 7, 11, 12, 19;
- 5.10.13.15–18; contains B_1 formed by 1, 3, 4, 7, 11, 12, 19;
- 5.10.15–18; contains B_1 formed by 2, 3, 4, 7, 8, 9, 19.

Case 3.

- 6–16; contains B_1 formed by 1, 2, 3, 5, 17, 18, 19;
- 6–14.16; contains B_1 formed by 1, 2, 3, 5, 17, 18, 19;
- 6–11.13–16; contains B_1 formed by 1, 2, 3, 5, 17, 18, 19;
- 6–11.13–15.17; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6–11.13–15.17.18; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6–11.13.14.16; contains B_1 formed by 1, 2, 3, 5, 17, 18, 19;
- 6–11.14–16; contains B_1 formed by 1, 2, 3, 5, 13, 17, 18;
- 6–11.14.16; contains B_1 formed by 2, 3, 4, 5, 13, 15, 17;
- 6–8.10–17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10–16.18; contains B_3 1, 2, 3, 4, 5, 9, 17, 19;
- 6–8.10–15.17; contains B_1 formed by 1, 2, 4, 5, 16, 18, 19;
- 6–8.10–14.16.17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10–14.16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6–8.10–14.17; contains B_1 formed by 1, 2, 4, 5, 16, 18, 19;
- 6–8.10–13.16–18; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 6–8.10–13.16.17.19; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10–13.16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6–8.10.11.13–17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10.11.13–15.17; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6–8.10.11.13–15.17.18; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6–8.10.11.13.14.16.17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10.11.13.14.16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6–8.10.11.14–17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10.11.14.15.17.18; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6–8.10.11.14.16.17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10.12–17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10.12–16.18; contains B_1 formed by 1, 2, 3, 5, 11, 17, 19;
- 6–8.10.12–15.18; contains B_1 formed by 1, 2, 3, 5, 11, 17, 19;
- 6–8.10.12–14.16–18; contains B_4 formed by 2, 3, 4, 5, 9, 11, 15, 19;

- 6–8.10.12–14.16.17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10.12.13.15–17; contains B_3 formed by 1, 2, 3, 4, 5, 9, 18, 19;
- 6–8.10.12.13.15.17; contains B_1 formed by 1, 2, 4, 5, 14, 16, 18;
- 6–8.10.12.13.16–18; contains B_4 formed by 1, 2, 3, 4, 9, 14, 15, 19;
- 6–8.10.12.15–17; contains B_1 formed by 2, 3, 4, 5, 13, 14, 18;
- 6–8.12–17; contains B_1 formed by 1, 2, 3, 5, 10, 11, 19;
- 6–8.12–18; contains B_1 formed by 1, 2, 3, 5, 10, 11, 19;
- 6–8.12–14.16.17; contains B_1 formed by 1, 2, 3, 5, 10, 11, 19;
- 6–8.12–14.16–18; contains B_1 formed by 1, 2, 3, 5, 10, 11, 19;
- 6.7.10–17; contains B_3 formed by 1, 2, 3, 4, 5, 8, 18, 19;
- 6.7.10–16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6.7.10–15.18; contains B_1 formed by 1, 3, 4, 5, 16, 17, 19;
- 6.7.10–14.16–18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 6.7.10–14.16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6.7.10–13.15–17; contains B_3 formed by 1, 2, 3, 4, 5, 8, 18, 19;
- 6.7.10–13.15.17.18; contains B_3 formed by 1, 2, 3, 4, 5, 8, 16, 19;
- 6.7.10–13.16–18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 6.7.10–13.16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6.7.10–12.15–17; contains B_1 formed by 2, 3, 4, 5, 13, 14, 18;
- 6.7.10.11.13–17; contains B_3 formed by 1, 2, 3, 4, 5, 8, 18, 19;
- 6.7.10.11.13–16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6.7.10.11.13–15.17.18; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6.7.10.11.13–15.18; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6.7.10.11.13.14.16–18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 6.7.10.11.13.14.16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6.7.10.11.13.15–17; contains B_3 formed by 1, 2, 3, 4, 5, 8, 18, 19;
- 6.7.10.11.14.15–17; contains B_3 formed by 1, 2, 3, 4, 5, 8, 18, 19;
- 6.7.10.11.14–16.18; contains B_3 formed by 1, 2, 3, 4, 5, 9, 17, 19;
- 6.7.10.11.14.15.17.18; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6.7.10.11.14.15.18; contains B_1 formed by 1, 2, 4, 5, 12, 16, 19;
- 6.7.10.11.14.16–18; contains B_3 formed by 1, 2, 3, 4, 8, 9, 15, 19;
- 6.7.10.11.15–17; contains B_1 formed by 2, 3, 4, 5, 13, 14, 18;
- 6.7.11–18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 6.7.11–14.16–18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 6.7.11–14.16.18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 6.7.11–13.16–18; contains B_2 formed by 2, 3, 4, 8, 9, 10, 19;
- 6.7.11.13–18; contains B_1 formed by 1, 2, 4, 5, 10, 12, 19;
- 6.7.11.13–15.17.18; contains B_1 formed by 1, 2, 4, 5, 10, 12, 19;
- 6.7.11.13.14.16–18; contains B_1 formed by 1, 2, 4, 5, 10, 12, 19;
- 6.7.11.13.14.16.18; contains B_1 formed by 1, 2, 4, 5, 10, 12, 19;
- 6.7.11.14–18; contains B_1 formed by 1, 2, 4, 5, 10, 12, 19;

- 6.10–17; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10–14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10–14.16.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10–13.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.13–17; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.13–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.13.14.16.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.13.14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.14–17; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.14–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.14–16.18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.11.14.16–18; contains B_2 formed by 2, 3, 4, 7, 8, 9, 19;
- 6.10.13–17; contains B_1 formed by 1, 3, 4, 5, 11, 12, 19;
- 6.10.13–18; contains B_1 formed by 1, 3, 4, 5, 11, 12, 19;
- 6.10.13.15–17; contains B_1 formed by 1, 3, 4, 5, 11, 12, 19;
- 6.10.13.15–18; contains B_1 formed by 1, 3, 4, 5, 11, 12, 19.

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