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BOUNDS FOR IRREDUNDANT AND CO-IRREDUNDANT RAMSEY NUMBERS

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Abstract

A set of vertices $X \subseteq V$ in a simple graph $G(V,E)$ is irredundant (CO-irredundant) if each vertex $x \in X$ is either isolated in the induced subgraph $G[X]$ or else has a private neighbor $y \in V \setminus X$ $(y \in V)$ that is adjacent to x and to no other vertex of X. The irredundant Ramsey number $s(t_1, \ldots, t_l)$ and CO-irredundant Ramsey number $s_{\text{CO}}(t_1, \ldots, t_l)$ are respectively the minimum N such that every l-coloring of the edges of the complete graph K_N on N vertices has a monochromatic irredundant set and a monochromatic CO-irredundant set of size t_i for some $1 \leq i \leq l$. In this paper, first, we establish a lower bound for the irredundant Ramsey number $s(t_1, \ldots, t_l)$ using a random and probabilistic method, which extends the lower bound for $s(t,t)$ due to Chen-Hattingh-Rousseau. Second, using Krivelevich's lemma, we give an asymptotic lower bound for the CO-irredundant Ramsey number $s_{\text{CO}}(m,n)$. In the end, we improve the upper bound for $s(3,9)$ such that $24 \leq s(3,9) \leq 26$.

Keywords: irredundant Ramsey number, CO-irredundant Ramsey number, irredundant set.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G, respectively. For any subset $X \subseteq V(G)$, let $G[X]$ denote the subgraph induced by X. Similarly, for any subset $F \subseteq E(G)$, let $G[F]$ denote the subgraph induced by F. A *path* on n vertices is denoted by P_n, and a *cycle* on n vertices is denoted by C_n. The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v. A graph G is called k-*regular* if $d_G(v) = k$ for every $v \in V(G)$. The *join* $G \vee H$ of two disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. The *union* $G \cup H$ of two graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For $V_1, V_2 \subset V$, we denote the number of edges between V_1 and V_2 by $e(V_1, V_2)$. Any undefined concepts or notation can be found in [1].

1.1. Irredundant and CO-irredundant Ramsey numbers

In 1978, Cockayne, Hedetniemi, and Miller [10] introduced the concept of irredundance which is relevant for dominating sets. A set of vertices $X \subseteq V$ in a simple graph $G(V, E)$ is *irredundant* if each vertex $x \in X$ is either isolated in the induced subgraph $G[X]$ or else has a private neighbor $y \in V \setminus X$ that is adjacent to x and to no other vertex of X. Farley and Schacham [14] defined the CO-irredundant set: given a graph $G = (V, E)$, a vertex subset $X \subseteq V$ is called CO-*irredundant* if every vertex $v \in X$ either contains no neighbors in X or else has a private neighbor $y \in V$ that is adjacent to x and to no other vertex of X. The *irredundant Ramsey number* $s(t_1, \dots, t_l)$ (respectively, CO-*irredundant Ramsey number* $s_{\mathrm{CO}}(t_1, \dots, t_l)$) is the minimum N such that every l-coloring of the edges of the complete graph K_N has a monochromatic t_i-element irredundant set (respectively, t_i-element CO-irredundant set, say COIR_{t_i}) for certain $1 \le i \le l$. If $t_1 = t_2 = \cdots = t_l = t$, we denote it by $s(t; l)$ (respectively, $s_{\mathrm{CO}}(t; l)$). The definition of the *Ramsey number* $r(t_1, \dots, t_l)$ differs from $s(t_1, \dots, t_l)$ in that the t_i-element irredundant set is replaced by a t_i-element independent set. The *mixed Ramsey number* $t(m, n)$ is the smallest N for which every red-blue coloring of the edges of K_N yields an m-element irredundant set in the blue subgraph or an n-element independent set in the red subgraph. Note that each independent set is an irredundant set, and each irredundant set is a CO-irredundant set.

Consequently, it follows that

$$s_{\mathrm{CO}}(t_1,\ldots,t_l) \leq s(t_1,\ldots,t_l) \leq r(t_1,\ldots,t_l)$$

and

$$s_{\mathrm{CO}}(m,n) \leq s(m,n) \leq t(m,n) \leq r(m,n).$$

The difficulty of obtaining exact values for irredundant Ramsey numbers is evidently comparable to that of obtaining exact values for classical Ramsey numbers. Brewster, Cockayne, and Mynhardt [2] proposed irredundant Ramsey numbers and established the values $s(3,3)=6$, $s(3,4)=8$, $s(3,5)=12$, while $s(3,6)=15$ was established in [3]. It was furthermore shown in [15] that $18 \leq s(3,7) \leq 19$ and Chen and Rousseau proved that $s(3,7)=18$ in [7], and Cockayne *et al.* in [8] obtained that $s(4,4)=13$. The values $t(3,3)=6, t(3,4)=9, t(3,5)=12$ and $t(3,6)=15$ have been shown in [9, 16]. Burger, Hattingh, and Vuuren [4] proved that $t(3,7)=18$ and $t(3,8)=22$. Burger and Vuuren [5] obtained that $s(3,8)=21$. The following table lists all known Ramsey numbers $s(3,n), t(3,n)$ and $r(3,n)$ for $3 \leq n \leq 9$.

$s(m,n)$	$t(m,n)$	$r(m,n)$
$s(3,3)=6$	$t(3,3)=6$	$r(3,3)=6$
$s(3,4)=8$	$t(3,4)=9$	$r(3,4)=9$
$s(3,5)=12$	$t(3,5)=12$	$r(3,5)=14$
$s(3,6)=15$	$t(3,6)=15$	$r(3,6)=18$
$s(3,7)=18$	$t(3,7)=18$	$r(3,7)=23$
$s(3,8)=21$	$t(3,8)=22$	$r(3,8)=28$
-	-	$r(3,9)=36$

Table 1. Exact known Ramsey numbers.

Chen, Hattingh, and Rousseau [6], Erdős and Hattingh [13], and Krivelevich [17] have obtained several asymptotic bounds for irredundant Ramsey numbers $s(m,n)$ and mixed Ramsey number $t(m,n)$. What's more, problems related to irredundant Turán numbers has been studied in [9]. Furthermore, for $s_{\mathrm{CO}}(m,n)$, several exact values were given by Cockayne, MacGillivray and Simmons in [11]. However, the asymptotic bounds for $s_{\mathrm{CO}}(m,n)$ are not given.

For the 2-coloring of the edges of K_N, we call it red-blue coloring and we call two kinds of monochromatic edge-induced subgraphs the red graph $\langle R\rangle$ and the blue graph $\langle B\rangle$. If Y is an m-element irredundant set in $\langle B\rangle$, then for some $k \leq m$, there exist k vertices of Y that have private neighbors in $\langle B\rangle$ and the remaining $m-k$ vertices of Y in the induced subgraph of the $\langle B\rangle$ are isolated. With the k vertices in Y and their private neighbors, there is an $(m+k)$-element set in which all but $2\binom{k}{2}$ of the $\binom{m+k}{2}$ internal edges are completely determined.

So $\langle R\rangle$ contains one or more of the graphs from the graph family $\{K_m, K_{m-k} + (K_{k,k} - kK_2), K_{m,m} - mK_2 \mid 3 \le k \le m-1\}$ where the graph $K_{k,k} - kK_2$ is obtained by removing k independent edges from the complete bipartite graph $K_{k,k}$. Clearly, $K_{m-k} + (K_{k,k} - kK_2)$ has $m + k$ vertices and $\binom{m+k}{2} - 2\binom{k}{2} - k$ edges.

1.2. Our results

Sawin [23] proved a lower bound for $r(t;l)$ using the random method. We will prove a similar result for $s(t;l)$ in Section 2, which extends the lower bound for $s(t,t)$ due to Chen-Hattingh-Roussea [6].

Theorem 1 (Chen, Hattingh and Roussea [6]). *For all sufficiently large t,*

$$s(t,t) > \sqrt{\frac{t}{3}}2^{t/2}.$$

Theorem 2. *For $l > 2$ and sufficiently large t, we have*

$$s(t;l) \ge \left(\frac{t}{3}\right)^{\frac{3-l}{2}} 2^{\frac{lt-t}{2}}.$$

In Section 3, we are going to establish a lower bound for $s_{\mathrm{CO}}(m,n)$ by Krivelevich's lemma which was used to prove the lower bound for $s(m,n)$.

Theorem 3 (Krivelevich [17]). *For each $m \ge 3$ there is a positive constant c_m such that*

$$s(m,n) > c_m \left(\frac{n}{\log n}\right)^{(m^2-m-1)/[2(m-1)]}.$$

Theorem 4. *For each $m \ge 3$ there is a positive constant c_m such that*

$$s_{\mathrm{CO}}(m,n) > c_m \left(\frac{n}{\log n}\right)^{\rho(\mathcal{H})},$$

where $\mathcal{H}$ is a graph family of $((K_{|W_1|} - tK_2) \vee (K_{|Y_1|,|Y_3|} - |Y_1|K_2)) \vee (K_{|W_2|,|Y_2|}\{Y_2\} - |W_2|K_2)$ for $t \le |W_1| \le m-2$ and $\rho(\mathcal{H}) = \min\{\rho(H_i) \mid 1 \le i \le l\}$ for each $H_i \in \mathcal{H}$.

Burger and Vuuren [5] showed the upper and lower bounds for $s(3,9)$ and $t(3,9)$. We will derive an upper bound for $s(3,9)$ in Section 4.

Theorem 5 (Burger and Vuuren [5]). $24 \le s(3,9) \le t(3,9) \le 27$.

Theorem 6. $24 \le s(3,9) \le 26$.

2. A Lower Bound for $s(t;l)$

In this section, we obtain a lower bound for the irredundant Ramsey number $s(t;l)$.

***Proof of Theorem* 2.** $f(n,s,t) = \min\{b_s(G) : |G| = n,\ b_t(\bar{G}) = 0\}$ represents the minimum number of independent sets of size s in a graph G with n vertices which contains no clique of size t, and is to study the number of complete subgraphs contained in a given graph.

Let $c_{s,t} = \lim_{n\to\infty} f(n,s,t)/\binom{n}{s}$, that is, to be the infimum, over graphs G with no t-clique, of the probability that $\{v_1,\ldots,v_s\}$ is an independent set for the vertices $v_1,\ldots,v_s$ of G chosen independently and uniformly at random. Nikiforov [19] first applied it to classical Ramsey number problem. In fact, we could use the method to obtain a new lower bound of irredundant Ramsey number $s(t;l)$ for sufficiently large t. Let $g(n,s,t)$ be the minimum number of s-element irredundant sets in a graph G with n vertices that contains no $\bar{I}_t$ of order t, where $\bar{I}_t$ denotes the complement graph of a t-element irredundant set in K_n. Let

$$w_{s,t} = \lim_{n\to\infty} \frac{g(n,s,t)}{\binom{n}{2s}}$$

be the infimum over graphs G with no $\bar{I}_t$ of order t of the probability that $\{v_1,\ldots,v_s\}$ is an irredundant set for the vertices $v_1,\ldots,v_s$ of G chosen independently and uniformly at random.

By proving the following Lemmas 9 and 10, we immediately have Theorem 2. Before that, we first show two propositions for convenience of the proofs of those lemmas.

Proposition 7. *Let n,t,k be three positive integers such that $n = \left\lceil 2^{t/2}\sqrt{t/3}\right\rceil$ and $k \geq 3$. Let*

$$b_k = \binom{t}{k}\binom{n-t}{k} k!\, 2^{\binom{t+k}{2} - 2\binom{k}{2}}$$

for $3 \leq k \leq t$. For all sufficiently large value t, we have $\max\{b_k \,|\, 3 \leq k \leq t\} = b_3$ or b_t.

Proof. For $3 \leq k \leq (1-\epsilon)t/2$, since $n = \lceil 2^{t/2}\sqrt{t/3}\rceil$, it follows that

$$\frac{b_{k+1}}{b_k} = \frac{(t-k)(n-t-k)}{k+1} 2^{k-t} \leq (t-k)(n-t-k)2^{k-t-2} \text{ (since } k \geq 3)$$
$$\leq tn2^{k-t-2} \leq t2^{k-t-2}\left(2^{t/2}\sqrt{t/3}+1\right) \leq t2^{k-1-t/2}\sqrt{t/3} < 1.$$

This means that the sequence $b_3,\ldots,b_t$ is decreasing for $3 \leq k \leq (1-\epsilon)t/2$.

If $t/2 < k < t$, then

$$\frac{b_{k+1}}{b_k} = \frac{(t-k)(n-t-k)}{k+1}2^{k-t} \geq \frac{(t-k)\left(2^{t/2}\sqrt{t/3}-t-k\right)}{k+1}2^{k-t}$$
$$\geq \frac{(t-k)\left(2^{t/2}\sqrt{t/4}\right)}{k+1}2^{k-t} \geq \frac{\sqrt{t}(t-k)}{k+1}2^{k-1-t/2},$$

where the second inequality holds, since the sufficiently large t.

If $k = \lceil t/2 \rceil$ or $k = t-1$, then

$$\frac{b_{\lceil t/2 \rceil+1}}{b_{\lceil t/2 \rceil}} > \frac{\lfloor t/2 \rfloor \sqrt{t}}{2\left(\lceil t/2 \rceil + 1\right)} > 1$$

or

$$\frac{b_t}{b_{t-1}} \geq \frac{2^{t/2-2}}{\sqrt{t}} > 1.$$

If $t/2 < k \leq t-3$ and t is sufficiently large, then

$$\frac{b_{k-1}b_{k+1}}{b_k^2} = 2\left(\frac{t-k}{t-k+1}\right)\left(\frac{n-t-k}{n-t-k+1}\right)\left(\frac{k}{k+1}\right)$$
$$> 2\left(1-\frac{1}{1+t/2}\right)\left(1-\frac{1}{2^{t/2}\sqrt{t/4}}\right)\left(1-\frac{2}{t}\right)$$
$$> 1.$$

This means that the sequence $b_3, \ldots, b_t$ increases for $t/2 < k < t$.

Hence, for all sufficiently large value t we have $\max\{b_k \,|\, 3 \leq k \leq t\} = b_3$ or b_t. □

Proposition 8. *Let n, t, k, h be positive integers such that $n = \lceil 2^{t/2}\sqrt{t/3} \rceil$ and $k \geq 3$. Let*

$$a_k = \binom{h}{k}\binom{n-h}{k}k!2^{\binom{h+k}{2}-2\binom{k}{2}}$$

for $3 \leq k \leq h \leq t$. For all sufficiently large value t, we have $\max\{a_k \,|\, 3 \leq k \leq t\}$ $= a_3$ *or* a_t.

Proof. We have

$$\frac{a_{k+1}}{a_k} = \frac{(h-k)(n-h-k)}{k+1}2^{k-h}.$$

If $h > t/2$, then

$$\frac{a_{k+1}}{a_k} < \left(\frac{hn}{4}\right)2^{k-h} = \frac{h}{4}\left(\frac{t}{3}\right)^{1/2}2^{k-h+t/2} \leq \frac{h}{4}\left(\frac{t}{3}\right)^{1/2}2^{\epsilon t} < 1$$

for $3 \le k \le (1-\epsilon)(h-t/2)$ and for the sufficiently large t. And for $h-t/2 < k < h$, it follows that

$$\frac{a_{k+1}}{a_k} = \frac{(h-k)(n-h-k)}{k+1}2^{k-h} \ge \frac{(h-k)\left(2^{t/2}\sqrt{t/3}-t-k\right)}{k+1}2^{k-h}$$
$$\ge \frac{(h-k)\,2^{t/2}\sqrt{t}/2}{k+1}2^{k-h} \ge \frac{(h-k)\sqrt{t}/2}{k+1}2^{k-h+t/2}$$

for the sufficiently large t.

If $k = \lceil h - t/2 \rceil + \mu$ for the positive integer $1 \le \mu \le 4$ or $k = h-1$, then it follows from $h > t/2$ that

$$\frac{a_{\lceil h-t/2\rceil+\mu+1}}{a_{\lceil h-t/2\rceil+\mu}} \ge \frac{(h-k)\sqrt{t}/2}{k+1}2^{k-h+t/2} \ge \frac{(t/2-\mu-1)\sqrt{t}/2}{t+1}2^{\mu+1} = O(t^{1/2}) > 1$$

or

$$\frac{a_h}{a_{h-1}} > \frac{\sqrt{t}2^{t/2-1}}{2h} > 1.$$

If $h - t/2 + 5 < k \le h-2$, then

$$\frac{a_{k-1}a_{k+1}}{a_k^2} = 2\left(1-\frac{1}{h-k+1}\right)\left(1-\frac{1}{n-h-k+1}\right)\left(1-\frac{1}{k+1}\right)$$
$$> 2\left(1-\frac{1}{3}\right)\left(1-\frac{1}{1+2^{t/2}\sqrt{t}/2}\right)\left(1-\frac{1}{5+1}\right)$$
$$> \frac{10}{9}\left(1-\frac{1}{1+2^{t/2}\sqrt{t}/2}\right) > 1,$$

for the sufficiently large t.

So, for $h > t/2$, the sequence $a_3, \dots, a_h$ is decreasing for $3 \le k \le (1-\epsilon)(h-t/2)$ and is increasing for $h - t/2 < k < h$ and the largest one must be a_3 or a_h. □

Lemma 9. *For sufficiently large t, we have*

$$w_{t,t} \le \left(\frac{t}{3}\right)^{\frac{t}{2}} 2^{\frac{-t^2}{2}+o(t^2)}.$$

Proof. Let G be a random graph with $n = \sqrt{t/3}p^{-t/2}$ vertices with $p = 1/2$, where each pair of vertices is connected by an edge with probability p. Let $v_1, \dots, v_t$ be uniformly distributed random variable in $[n]$, independent from each other and from G. For $w_{t,t}$ we have

$$w_{t,t} \le \frac{\mathbb{P}(\{v_1,\dots,v_t\}\text{ is an irredundant set})}{\mathbb{P}(G\text{ contains no }\bar{I}_t)}.$$

For the denominator, we first consider that G contains an $\bar{I}_t$.

$$\text{(1)}\quad \mathbb{P}(G\,\text{contains an}\,\bar{I}_t) \le \binom{n}{t}\left[p^{\binom{t}{2}} + \sum_{k=3}^{t}\binom{t}{k}\binom{n-t}{k}k!p^{\binom{k+t}{2}-2\binom{k}{2}-k}(1-p)^k\right].$$

From (1) we have that

$$\text{(2)}\qquad \mathbb{P}(G\,\text{contains an}\,\bar{I}_t) \le \binom{n}{t}\left[2^{-\binom{t}{2}} + \sum_{k=3}^{t}\binom{t}{k}\binom{n-t}{k}k!2^{-\binom{k+t}{2}+2\binom{k}{2}}\right].$$

Since $b_t = \binom{n-t}{t}t!2^{-t^2} \sim \left(\sqrt{t/3}\right)^t 2^{-t^2/2}$ and $b_3 = \left(\binom{t}{3}\right)\left(\binom{n-t}{t}\right)3!2^{-\binom{t+3}{2}+6} \sim 4/3\left(t\sqrt{t/3}\right)^3 2^{-t}2^{-t^2/2}$ for the sufficiently large t. Then we have $b_3 = o(b_t)$ and $2^{-\binom{t}{2}} = o(b_t)$. From (2), we have

$$\begin{aligned}
\mathbb{P}(G\,\text{contains an}\,\bar{I}_t) &\le \binom{n}{t}\left[2^{-\binom{t}{2}} + \sum_{k=3}^{t}\binom{t}{k}\binom{n-t}{k}k!2^{-\binom{k+t}{2}+2\binom{k}{2}}\right] \\
&\le \binom{n}{t}2t\binom{n-t}{t}t!2^{-t^2}(\text{by Proposition 7}) \\
&\le \binom{n}{t}2tn^t2^{-t^2} \le \left(\frac{ne}{t}\right)^t 2tn^t2^{-t^2} \le 2t\left(\frac{e}{3}\right)^t = o(1),
\end{aligned}$$

which means that $\mathbb{P}(G\,\text{contains no}\,\bar{I}_t) = 1 - o(1)$.

For the numerator taking h to be the size of $\{v_1,\ldots,v_t\}$ and $p = 1/2$, we have

$$\begin{aligned}
&\mathbb{P}(\{v_1,\ldots,v_t\}\,\text{is an irredundant set}\,) \\
&\le \sum_{h=1}^{t}\frac{\left\{{t \atop h}\right\}\binom{n}{h}h!\left((1/2)^{\binom{h}{2}} + \sum_{k=3}^{h}\binom{h}{k}\binom{n-h}{k}k!(1/2)^{\binom{k+h}{2}-2\binom{k}{2}}\right)}{\binom{n}{2t}} \\
&\le \sum_{h=1}^{t}\frac{\left\{{t \atop h}\right\}\left((1/2)^{\binom{h}{2}} + \sum_{k=3}^{h}\binom{h}{k}\binom{n-h}{k}k!(1/2)^{\binom{k+h}{2}-2\binom{k}{2}}(2t)!\right)}{n^{2t-h}} \\
&\le t!\max_{0\le h\le t}\left\{\frac{t\cdot n^t2^{-t^2}}{n^{2t-h}}\right\}\quad \left(\text{by}\ \sum_{h=1}^{t}\left\{{t \atop h}\right\} \le t!\ \text{and Proposition 8}\right) \\
&\le 2^{t\log t}\max_{0\le h\le t}\left\{\frac{t2^{-t^2}}{n^{t-h}}\right\}\quad \left(\text{since}\ t! \le 2^{t\log t}\right) \\
&\le 2^{t\log t}\cdot\left(t2^{-t^2}\right) = 2^{t\log t}\cdot\left(t2^{-t^2}\right).
\end{aligned}$$

Remark. $\left\{ t \atop h \right\}$ are the Stirling numbers of the second kind.

Combining the upper bound for $\mathbb{P}(\{v_1, \dots, v_t\}$ is an irredundant set$)$ with $\mathbb{P}(G$ contains no $\bar{I}_t)$, we conclude that

$$w_{t,t} \leq \frac{\mathbb{P}(\{v_1, \dots, v_t\} \text{ is an irredundant set})}{\mathbb{P}(G \text{ contains no } \bar{I}_t)}$$
$$\leq \frac{\left(\frac{t}{3}\right)^{\frac{t}{2}} 2^{\frac{-t^2}{2}+o(t^2)}}{1-o(1)} = \left(\frac{t}{3}\right)^{\frac{t}{2}} 2^{\frac{-t^2}{2}+o(t^2)}.$$

□

Lemma 10. *For $l > 3$ and sufficiently large t, we have*

$$s(t;l) \geq w_{t,t}^{-\frac{l-2}{t}} 2^{\frac{t}{2}} \sqrt{\frac{t}{3}}.$$

Proof. We first fix a sufficient large graph G satisfying two conditions:

- there is no $\bar{I}_t$ in G;
- the probability that t random vertices of G form an irredundant set is at most $w_{t,t} + \epsilon$ for ϵ $(= \epsilon(l,t)) > 0$.

Let $N = \left\lfloor w_{t,t}^{-\frac{l-2}{t}} 2^{\frac{t}{2}} \sqrt{\frac{t}{3}} \right\rfloor$. We construct an l-edge-coloring of K_N without a monochromatic $\bar{I}_t$ as follows. Let $f_1, \dots, f_{q-2}$ be $q-2$ uniform, independent and random functions from the vertex set of K_N to the vertex set of G, all independent of one another. In fact, the injectivity can be ensured. Indeed, we can choose G as large as we need in the beginning of proof, and if we make G absolutely enormous then the probability of mapping two vertices in K_N to the same vertex of G is virtually 0.

As above mapping rules, for $v_1, v_2 \in V(K_N)$, if there are some functions f_i for $i \in [q-2]$ such that $f_i(v_1), f_i(v_2)$ form an edge in G, then we define the color $\chi(v_1v_2) = \min\{i \,|\, f_i(v_1)f_i(v_2) \in E(G) \text{ for } i \in [q-2]\}$, which means that we color the edge v_1v_2 by the minimum subscript of all the functions satisfying $f_i(v_1)f_i(v_2) \in E(G)$ for $i \in [q-2]$. If there is no such i such that $f_i(v_1), f_i(v_2)$ form an edge in G, then we color randomly this edge v_1v_2 with $q-1$ or q, each with probability $1/2$, independently for each remaining edges.

Since G contains no $\bar{I}_t$, it follows that there exists no set of t vertices sent to an $\bar{I}_t$ by any f_i, $1 \leq i \leq l-2$. Thus, there is no a monochromatic $\bar{I}_t$ for color i. So it suffices to prove that the probability that there is an $\bar{I}_t$ for the remaining two colors is less than 1. By the rule of l-coloring, for T to be an $\bar{I}_t$, it contains no edges of color i for each $i \in [l-2]$, and hence the probability of occurrence is at most $(w_{t,t}+\epsilon)^{l-2}$. If, for example, T is an irredundant set with color $l-1$, then for some subset $\{x_1, \dots, x_k\} \subset V(K_N) \setminus T$, the private neighbors of the vertices $y_1,\dots,y_k$ are not isolated in T. So there are exactly $2\binom{k}{2}$ edges such that their colors are non-deterministic, and the coloring of the remaining $\binom{t+k}{2}-2\binom{k}{2}$ edges is completely determined. Let X_T denote the indicator random variable that takes the value i if T is an $\bar{I}_t$ with a monochromatic color $l-1$ or color l. We denote the size of T by t, and let X be the random variable that counts the number of $\bar{I}_t$ of color $l-1$ or l. For a fixed set T with k non-isolated vertices, the choices for the set of non-isolated vertices, the private neighbors, and the matching between these two sets are $\binom{t}{k}$, $\binom{N-t}{k}$, and $k!$, respectively. So we have for sufficiently large t that

$$\mathbb{E}(X)=\sum_T E(X_T)\le 2\binom{N}{t}\left[2^{-\binom{t}{2}}+\sum_{k=3}^{t}\binom{t}{k}\binom{N-t}{k}k!2^{\binom{t+k}{2}-2\binom{k}{2}}\right](w_{t,t}+\epsilon)^{l-2}.$$

Let

$$q_k=\binom{t}{k}\binom{N-t}{k}k!2^{\binom{t+k}{2}-2\binom{k}{2}}$$

for $3\le k<t$. Naturally,

$$\begin{aligned}\frac{q_{k+1}}{q_k}&=\frac{(t-k)(N-t-k)}{k+1}2^{k-t}\ge\frac{w_{t,t}^{-\frac{l-2}{t}}2^{\frac{t}{2}}\sqrt{\frac{t}{3}}-t-k-1}{t}2^{k-t}\\ &\ge\frac{\left(\left(\frac{t}{3}\right)^{\frac{t}{2}}2^{\frac{-t^2}{2}}\right)^{-\frac{l-2}{t}}2^{\frac{t}{2}}\sqrt{\frac{t}{3}}-t-k-1}{t}2^{k-t}\quad\text{(by Lemma 9)}\\ &\ge\frac{2^{k+t/2}\sqrt{3}-t-k-1}{t\sqrt{t}}\quad(\text{since } l\ge 4)>1.\end{aligned}$$

It is easily seen that the largest term of the sum is q_t. Then $q_t=\binom{n-t}{t}t!2^{-t^2}\sim\left(\sqrt{t/3}\right)^t2^{-t^2/2}$, $q_3=\left(\binom{t}{3}\right)\left(\binom{n-t}{t}\right)3!2^{-\binom{t+3}{2}+6}\sim 4/3\left(t\sqrt{t/3}\right)^3 2^{-t}2^{-t^2/2}$ for sufficiently large t and $2^{-\binom{t}{2}}=o(q_t)$.

$$\begin{aligned}\mathbb{E}(X)=&\sum_T E(X_T)\le 2\binom{N}{t}\left[2^{-\binom{t}{2}}+\sum_{k=3}^{t}\binom{t}{k}\binom{N-t}{k}k!2^{\binom{t+k}{2}-2\binom{k}{2}}\right](w_{t,t}+\epsilon)^{l-2}.\\ &\le 2t\binom{N}{t}\binom{N-t}{t}t!2^{-t^2}(w_{t,t}+\epsilon)^{l-2}\\ &<2t\binom{N}{t}N^t2^{-t^2}w_{t,t}^{l-2}<\left(\frac{Ne}{t}\right)^t2tN^t2^{-t^2}w_{t,t}^{l-2}\\ &\le 2t\left(\frac{e}{3}\right)^t<<1\ (t\to\infty).\end{aligned}$$

Clearly, the probability that there is no monochromatic $\bar{I}_t$ in K_N is more than 0. □ ■

3. A Lower Bound for $s_{\mathrm{CO}}(m, n)$

In this section, we first prove a lower bound for $s_{\mathrm{CO}}(m, n)$ analogous to that of Krivelevich [17] for $s(m, n)$. Let us first rephrase the definition of CO-irredundant set. Let $G = (V, E)$ be a simple graph. Let $N[v] = \{v\} \cup N(v)$ for each $v \in V$. The open neighborhood $N(X)$ (respectively, closed neighborhood $N[X]$) of a subset X of V is defined by $N(X) = \cup_{x \in X} N(x)$ (respectively, $N[X] = \cup_{x \in X} N[x]$). A set X is called CO-*irredundant* if, for each vertex $x \in X$, the private neighborhood of x relative to X satisfies that $PN(x, X) = N[x] - N(X - \{x\}) \neq \emptyset$. Note that $y \in PN(x, X)$ if, and only if, it satisfies one of the following three cases:

(i) $y = x$ and x is an isolated vertex in $G[X]$, call y a *private neighbor* of X;

(ii) $y \in V \setminus X$ and $N(v) \cap X = \{x\}$, call y an *internal private neighbor* of X;

(iii) $y \in X$ and $N(v) \cap X = \{x\}$, call y an *external private neighbor* of X.

For sets S and T, we denote by $K_{|S|}$ the complete graph on vertex set S, and denote by $K_{|S|,|T|}$ the complete bipartite graph on vertex set S and T. The *completion* of $S \subset V(G)$ in G, denoted by $G\{S\}$, is the graph $G \cup K_{|S|}$. We denote a k-*matching* by kK_2, and then $G - kK_2$ means that we delete a k-matching in G. We first discuss that if $\langle B \rangle$ contains an m-element CO-irredundant set, then what subgraph structure $\langle R \rangle$ contains. We now give an example of 12-element CO-irredundant sets in $\langle B \rangle$; see Figure 1.

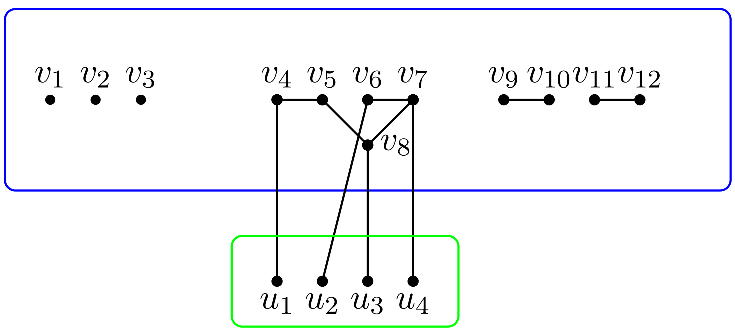

Figure 1. A CO-irredundant set COIR_{12}.

Example 11. Let W_1, W_2, Y_1, Y_2, Y_3 represent the vertex set $\{v_1, v_2, v_3, v_9, v_{10}, v_{11}, v_{12}\}$, $\{v_5, v_7\}$, $\{v_4, v_6, v_8\}$, $\{v_4, v_6\}$, $\{u_1, u_2, u_3, u_4\}$, respectively. Thus, $\langle R \rangle$

must contains the graph $((K_{|W_1|} - 2K_2) \vee (K_{|Y_1|,|Y_3|} - |Y_1|K_2)) \vee (K_{|W_2|,|Y_2|}\{Y_2\} - |W_2|K_2)$.

So we have the following lemma.

Lemma 12. *Let $\langle R\rangle$ and $\langle B\rangle$ be the subgraph induced by red and blue edges in a red/blue-colored complete graph, respectively. If $\langle B\rangle$ contains an m-element* CO*-irredundant set, then $\langle R\rangle$ contains K_m or one graph from $((K_{|W_1|} - tK_2) \vee (K_{|Y_1|,|Y_3|} - |Y_1|K_2)) \vee (K_{|W_2|,|Y_2|}\{Y_2\} - |W_2|K_2)$ for $t \leq |W_1| \leq m-2$ as its subgraph.*

Proof. Let $X = \{v_1, v_2, \ldots, v_m\}$ be an m-element CO-irredundant set in $\langle B\rangle$. Clearly, if X is an indpendent set in $\langle B\rangle$, then $\langle R\rangle$ contains a complete graph K_m.

We use the symbols of Figure 1. Since W_1 is the vertex-set induced by all isolated vertices and t isolated edges in COIR_m of $\langle B\rangle$, there is a subgraph $K_{|W_1|} - tK_2$ for $t \leq |W_1| \leq m-2$ in $\langle R\rangle$. Since the subgraph induced by the edges between Y_1 and Y_3 is made up of isolated edges, then there is a subgraph $K_{|Y_1|,|Y_3|} - |Y_1|K_2$ in $\langle R\rangle$. Let Y_2 be the internal private neighbor set. Since each pair of vertices is not connected by an edge in Y_2, then there is a $K_{|Y_2|}$ in $\langle R\rangle$. Then all edges between W_2 and Y_2 are isolated in $\langle B\rangle$. Moreover, there must be a subgraph $K_{|W_2|,|Y_2|}\{Y_2\} - |W_2|K_2$ in $\langle R\rangle$.

By the definition of CO-irredundant set, we have that $\langle R\rangle$ contains the subgraph $((K_{|W_1|} - tK_2) \vee (K_{|Y_1|,|Y_3|} - |Y_1|K_2)) \vee (K_{|W_2|,|Y_2|}\{Y_2\} - |W_2|K_2)$. ■

For a fixed graph H, let $\rho(H) = \frac{|E(H)|-1}{|V(H)|-2}$. For a collection of fixed graphs $\mathcal{H} = \{H_1, H_2, \ldots, H_l\}$, we set $\rho(\mathcal{H}) = \min\{\rho(H_i) \mid 1 \leq i \leq l\}$. Let G be a graph. For every two disjoint subsets of $S, T \subseteq V(G)$, we denote by $e(S,T)$ the number of edges between S and T.

Theorem 13 (Krivelevich [17])**.** *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be a family of graphs with density $\rho(\mathcal{H}) > 0$. Then there exists a constant $c = c(\mathcal{H})$ such that for every sufficiently large integer N, there exists a graph G_0 of order N satisfying the following.*

- *G_0 is $\mathcal{H}$-free;*
- *G_0 has no independent set of size $n = \lceil cN^{1/\rho(\mathcal{H})} \ln N \rceil$;*
- *for every two disjoint subsets of $S, T \subseteq V(G)$ with $|S| = |T| = n$, we have $e(S,T) > n$.*

In the following, we use Theorem 13 to prove our main theorem in this subsection.

***Proof of Theorem* 4.** For the sake of convenience, we prove there is a positive constant c'_m such that $s_{\mathrm{CO}}(m, 2n-1) > c'_m \left(\frac{n}{\log n}\right)^{\rho(\mathcal{H})}$ for each $m \geq 3$.

Consider an $\mathcal{H}$-free graph G_0 on N vertices from Theorem 13. We denote G_0 by $\langle R\rangle$ and denote $\overline{G_0}$ by $\langle B\rangle$. Then we have the following.

Claim 14. *$\langle B\rangle$ does not contain an m-element* CO*-irredundant set.*

Proof. Assume, to the contrary, that $\langle B\rangle$ contains an COIR_m. By Lemma 12, $\langle R\rangle$ contains K_m or one graph from $\{((K_{|W_1|} - tK_2) \vee (K_{|Y_1|,|Y_3|} - |Y_1|K_2)) \vee (K_{|W_2|,|Y_2|}\{Y_2\} - |W_2|K_2)|\ t \leq |W_1| \leq m-2\}$. We apply Theorem 13 to $\mathcal{H} = \{((K_{|W_1|} - tK_2) \vee (K_{|Y_1|,|Y_3|} - |Y_1|K_2)) \vee (K_{|W_2|,|Y_2|}\{Y_2\} - |W_2|K_2)|\ t \leq |W_1| \leq m-2\}$, a contradiction. □

Claim 15. *$\langle R\rangle$ does not contain an $(2n-1)$-element* CO*-irredundant set.*

Proof. Assume, to the contrary, that $\langle R\rangle$ contains a CO-irredundant set, say D, of size $2n-1$.

By Lemma 13, there exists an independent set, say D', of size at most $n-2$. Fix the independent set D' of size $n-2$. For the remaining $n+1$ vertices, there is at most one leaf in the subgraph induced by the remaining $n+1$ vertices. Otherwise, $\langle R\rangle$ contains an independent set of size n, a contradiction.

It means that there is at most one vertex containing an internal private neighbor. Let v be a vertex adjacent to the leaf. Each of the remaining n vertices (except v), say D', must be adjacent to an external private neighbor. We denote by W the set of external private neighbors. Therefore, $e(D', W) = n$, which contradicts Theorem 13. □

From Claims 14 and 15, $\langle B\rangle$ does not contain a CO-irredundant set of size m or $\langle R\rangle$ does a CO-irredundant set of size $2n-1$. ■

4. An Upper Bound for $s(3,9)$

An $s(m,n,p)$ (respectively, $(r(m,n,p))$) coloring is a red-blue edge-coloring (R,B) of the complete graph K_p which satisfies neither of the two conditions of the irredundant Ramsey number $s(m,n)$ (respectively, $r(m,n)$), thus showing that $s(m,n) > p$ (respectively, Ramsey number $r(m,n) > p$).

Lemma 16 [5]**.** *For two integers $m, n \geq 2$ let $x \in \{s, t\}$ be a Ramsey number. Then we have*

- *$\Delta(R) < x(m-1,n)$ and $\Delta(B) < x(m,n-1)$ in any $x(m,n,p)$ coloring (R,B).*

- *$\delta(R) \geq p - x(m, n-1)$ and $\delta(B) \geq p - x(m-1, n)$ in any $x(m, n, p)$ coloring (R, B).*

We will use the following two theorems in our proof.

Theorem 17 (Brewster, Cockayne, and Mynhardt [2]). *The blue subgraph of a red-blue edge coloring of a complete graph contains an irredundant set of cardinality 3 if and only if the red subgraph contains a 3-cycle or an induced 6-cycle.*

Theorem 18 (Hattingh [15]). *Let (R, B) be a red-blue coloring of the edges of a complete graph K_N in which $\langle B \rangle$ contains no 3-element irredundant set. For each vertex $v \in V(K_N)$, we partition $N_B(v)$ into two parts D_2 and $D_{>2}$, where each vertex u of $N_B(v)$ belongs to D_2 if the distance between u and v in $\langle R \rangle$ is two, otherwise it belongs to $D_{>2}$. Let X be any subset of $N_B(v)$ that contains at most one vertex from $D_{>2}$. Then $\langle X \rangle_R$ is bipartite.*

For the sake of simplicity, we define some notations. A graph G is called an *(m, n)-graph*, if it neither contains an m-irredundant set in G nor an n-irredundant set in $\overline{G}$. Denote the distance in G from u to v by $d(u, v)$; let

$$D_i = \{u \mid d(u, v) = i\} \text{ and } D_{>i} = \{u \mid d(u, v) > i\}.$$

Let (X, Y) be a bipartition of $G[D_2]$ and let (X_i, Y_i) be bipartitions of the components of $G[D_2]$ for $i = 1, 2, \ldots, k$. We denote the number of connected components of (X, Y) by C. Without loss of generality, assume that $D_2 = X \cup Y$, $X = \bigcup X_i$, $Y = \bigcup Y_i$ and $|X_i| \geq |Y_i|$ for $i = 1, 2, \ldots, C$. We say there is an UU'-edge if $N(U) \cap U' \neq \emptyset$ for two disjoint vertex sets U, U' of G.

***Proof of Theorem* 6.** Suppose, to the contrary, that G is a $(3, 9)$-graph of order 26. Let G be the red subgraph. Then it follows

$$5 \leq \delta(G) \leq \Delta(G) < 9$$

by Lemma 16. Let v be a vertex of G with $d(v) = \Delta(G)$. Hence, $d(v) = 5, 6, 7,$ or 8. Since the order of G is 26 and $d(v) = \Delta(G) \leq 8$, it follows that $|D_{\geq 2}| \geq 26 - 8 - 1 = 17$. We get $|D_2| \leq 13$ by Theorem 18. Otherwise, there will be an independent set of size 8 in $G[V_{>1}]$. As G is a $(3, 9)$-graph and $N(v)$ is an independent set of size $\Delta(G)$, it can be deduced that $G[D_{>2}]$ is a $(3, 9 - \Delta(G))$-graph. Otherwise, there is an independent set of size 9 in G, which consists of $N(v)$ and $9 - \Delta(G)$ vertices of $G[D_{>2}]$.

Now we show that G is a 5-regular graph. If $d(v) = 6, 7,$ or 8, then $G[D_{>2}]$ contains at least 6, 5, and 4 vertices, respectively. But $G[D_{>2}]$ is a $(3, 3)$-graph, a $(3, 2)$-graph, or a $(3, 1)$-graph ($s(3, 1) = 1, s(3, 2) = 3, s(3, 3) = 6$), respectively, a contradiction. Therefore, $d(v) \leq 5$. By $\delta(G) \geq 5$, it follows that $d(v) = 5$.

We claim that $|D_2| = 13$, $|D_3| = 7$ and $D_{>3} = \emptyset$. Since $d(v) = 5$ and $s(3,4) = 8$, it follows that $G[D_{>2}]$ is a $(3,4)$-graph and $|D_{>2}| \leq 7$. By simple calculation, $|D_{>1}| = 20$ and $|D_{>2}| \leq 7$ and $|D_2| \leq 13$, we deduce $|D_2| = 13$ and $|D_{>2}| = 7$. We claim that $D_{>3} = \emptyset$. Otherwise, let $w \in D_{>3}$. It illustrates that v, w and seven vertices in D_2 form an independent set. For convenience, we first illustrate some arguments.

Claim 19. *The bipartition (X, Y) satisfies $|X| = 7, |Y| = 6$ and $D_3 \subset N(X)$. For every nonadjacent pair $W = \{w_i, w_j\} \subset D_3$ there both exist a WX edge and a WY edge.*

Proof. It holds since $|D_2| = 13$ and the independent set of $G[D_{>1}]$ is at most 7. □

Claim 20. *For any vertex $w \in D_3$ and any connected component (X_i, Y_i) of (X, Y), either $N(w) \cap X_i = \emptyset$ or $N(w) \cap Y_i = \emptyset$.*

Proof. Since $G[D_2 \cup w]$ is bipartite, the Claim 20 is governed by Lemma 18. □

Claim 21. *For each nonadjacent pair $W = \{w_i, w_j\} \subset D_3$, there is a connected component (X_t, Y_t) of (X, Y) such that there exists a WX_t edge and a WY_t edge in G. We call it property $P(t)$.*

Proof. Suppose that there is no connected component (X_t, Y_t) of (X, Y) such that there exists a WX_t edge and a WY_t edge in G, there is an independent set of size 8 in $G[D_2 \cup W]$. □

It satisfies that $|X| = 7$ and $|Y| = 6$ by Claim 19. Without loss of generality, assume that $|X_1| = |Y_1| + 1$ and $|X_i| = |Y_i|$ for $i \neq 1$. Otherwise, $|X| \geq 8$ which induces an independent set of size at least 8. So $D_3 \subset N(X)$ by Claim 19. It furthermore follows that $D_3 \subset N(X_1)$. Assume that $w \in D_3$ and $w \notin N(X_1)$. There exists an independent set of size 8 in $G[D_2 \cup w]$ by Claim 20.

We now discuss several cases according to the structure of the connected components of $G[D_2]$.

Case 22. $|X| = |X_1| = 7$ and $|Y| = |Y_1| = 6$. Because there is no D_3Y_1-edge by Claim 20, $G[Y \cup D_3]$ contains an independent set with size 8, a contradiction.

Case 23. $1 \leq |X_1| \leq 2$. If $|X_1| = 1$, then $D_3 \subset N(X_1)$ and $d(x) = 7$, which contradicts $\Delta(G) = 5$. If $|X_1| = 2$, then at least 4 vertices of D_3 are adjacent to one vertex $x_1 \in X_1$. Furthermore, $d(x_1)$ is no less than 6, which contradicts $\Delta(G) = 5$.

Case 24. $3 \leq |X_1| \leq 6$. Since $G[D_3]$ is a $(3,4)$-graph, it satisfies that G contains no K_3, an independent set of size 4, or induced 6-cycle by Theorem 17. If $G[D_3]$ contains a vertex with degree at least 4, then it yields a K_3 or an independent set of of size 4. So we have $\Delta(G[D_3]) \leq 3$ and there are exactly the following eight graphs of $\mathcal{F}$ in Figure 2.

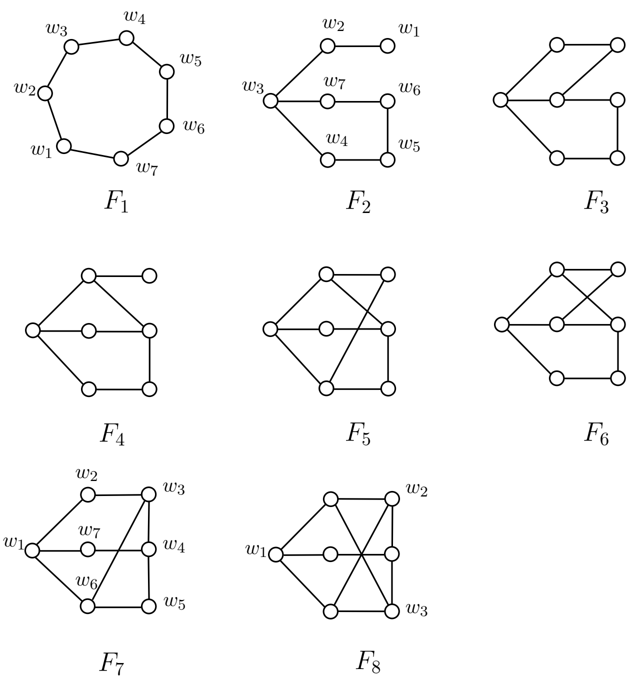

Figure 2. Graphs for $\mathcal{F}$.

By Claim 19, we have $D_3 \subset N(X)$ and $D_3 \cap N(Y_1) = \emptyset$. Now we show that (X, Y) contains at least three connected components. Suppose that the number of connected components of (X, Y) is two, say (X_1, Y_1) and (X_2, Y_2). Let $\bar{D}$ denote all the pairs of nonadjacent vertices in D_3. Thus $(\bar{D} \cap N(X_2)) \cup (\bar{D} \cap N(Y_2)) = \bar{D}$ with $\bar{D} \cap N(X_2) \neq \emptyset$ and $\bar{D} \cap N(Y_2) \neq \emptyset$, which contradicts Claim 20. In the following, we consider the remaining eight graphs of $\mathcal{F}$ in Figure 2.

Subcase 1. $G[D_3] = F_1$. As $G[D_3]$ contains 14 pairs of nonadjacent vertices and $G[D_2]$ contains at most 5 connected components, 3 pairs of nonadjacent vertices in $G[D_3]$ have property $P(t)$ for the same (X_t, Y_t) by Claim 21. Assume that the 3 pairs of nonadjacent vertices in $G[D_3]$ are $\{w_1, w_3\}, \{w_i, w_j\}$ and $\{w_r, w_s\}$. $t \neq 1$ since $D_3 \subset N(X_1)$. Otherwise, it contradicts Theorem 18. Suppose $X_t = \{x_1\}$ and $Y_t = \{y_1\}$. Without loss of generality, let $w_1x_1, w_3y_1 \in E(G)$. We have $N(w_2) \cap X_t = \emptyset$ and $N(w_2) \cap Y_t = \emptyset$ since G contains no triangle. Also, if $\{w_1, w_4\} = \{w_i, w_j\}$, then $w_4y_1 \in E(G)$ and $\{w_3, w_4, y_1\}$ is a triangle, a contradiction. Similarly, $\{w_3, w_7\} \neq \{w_i, w_j\}$. Suppose $\{w_4, w_7\} = \{w_i, w_j\}$. Then one vertex of w_5, w_6 is adjacent to one vertex of (X_t, Y_t), so it induces a triangle. Assume that the three pairs of nonadjacent vertices in $G[D_3]$ are $\{w_1, w_4\}, \{w_2, w_5\}$ and $\{w_3, w_7\}$. Without loss of generality, let $\{y_1w_2, y_1w_4, y_1w_7, x_1w_1, x_1w_3, x_1w_5\} \subset E(G)$. Because of three pairs of nonadjacent vertices $\{w_2, w_4\}$, $\{w_2, w_7\}$ and $\{w_4, w_7\}$ also satisfy Claim 21 respectively, it means that one vertex of w_2, w_4, w_7 has degree at least 6, which contradicts $\Delta(G) = 5$. Assume that $\{w_1, w_4\}, \{w_4, w_6\}$ and $\{w_2, w_6\}$ satisfy Claim 21. Let $\{x_1w_1, x_1w_6, y_1w_2, y_1w_4\} \subset E(G)$. If $\{w_3x_1, w_7y_1\} \cap E(G) \neq \emptyset$, then $\{w_1, w_3\}, \{w_3, w_6\}$ and $\{w_1, w_6\}$ or $\{w_2, w_4\}, \{w_2, w_7\}$ and $\{w_4, w_7\}$ also satisfy Claim 21, which yields a vertex with degree at least 6 from w_1, w_3, w_6 or w_2, w_4, w_7. If $\{w_3x_1, w_7y_1\} \cap E(G) = \emptyset$, then the pair of vertices $\{w_3, w_5\}$ satisfy Claim 21. Furthermore, the three pairs of vertices $\{w_1, w_3\}, \{w_3, w_6\}, \{w_1, w_6\}$ also satisfy Claim 21, which yields a vertex of degree at least 6.

So it means that $|X_t| = |Y_t| \geq 2$. Suppose $|X_t| = |Y_t| = 2$ and $C = 4$. Since at most two pairs of nonadjacent vertices satisfy Claim 21 for the same (X_t, Y_t) with $|X_t| = |Y_t| = 1$, there are at least 10 pairs of nonadjacent vertices in $G[D_3]$ connecting to the same (X_t, Y_t) by Claim 21. Without loss of generality, if the 10 pairs of nonadjacent vertices, $\{w_1, w_i\}$ for $i = 3, 4, 5, 6$, $\{w_2, w_i\}$ for $i = 4, 5, 6$ and $\{w_7, w_i\}$ for $i = 3, 4, 5$, satisfy Claim 21 for the same (X_t, Y_t) and $w_1x_1 \in E(G)$, then the pairs of $\{w_3, w_5\}, \{w_3, w_6\}$ are connected to one component with size 2 of (X, Y), which will induce a triangle. Otherwise, a vertex with degree 6 is contained in G. So for $C = 3$ $|(X_i, Y_i)| \geq 4$ for $i = 1, 2, 3$. Seven pairs of nonadjacent vertices, $\{w_1, w_i\}$ (for $i = 3, 4, 5, 6$), $\{w_7, w_j\}$ (for $j = 3, 4, 5$), satisfy Claim 21 for the same (X_t, Y_t). Let $X_t = \{x_1, x_2\}$ and $Y_t = \{y_1, y_2\}$ with $\{x_1y_1, x_2y_1, x_2y_2, y_1w_3, y_1w_5, y_2w_4, y_2w_6, x_1w_1, x_2w_7\} \subset E(G)$. Thus, $w_1x_1y_1w_5w_6w_7x_1$ is an induced 6-cycle. If $|X_t| = |Y_t| = 3$ or 4, then it will contain a triangle or a vertex with degree 6.

Subcase 2. $G[D_3] = F_2, F_3, F_4, F_5$ or F_6. Firstly, $G[D_3]$ must contain at least 12 pairs of nonadjacent vertices and $G[D_2]$ contains at most 5 connected components. Suppose that $C = 5$, namely, four connected components of G_2 are both of size 2. Since $D_3 \subset N(X_1)$, 3 pairs of nonadjacent vertices contained in $G[D_3]$ satisfy property $P(t)$ for (X_t, Y_t) by Claim 21 at the same time. Assume that $X_t = \{x_1\}, Y_t = \{y_1\}$. We can easily check that $\{w_3, w_i\}$ for $i = 1, 5, 6$ are three pairs of nonadjacent vertices by Figure 2. Without loss of generality, let $\{w_3y_1, w_1x_1, w_5x_1, w_6x_1\} \subset E(G)$. Clearly, there is a triangle $w_5w_6x_1w_5$ in G. Thus the pair of vertices $\{w_3, w_5\}$ are connected to another connected component of $G[D_2]$ except (X_1, Y_1). Based on $D_3 \subset N(X_1)$, $d(w_3) \geq 6$, a contradiction. Then a connected component of $G(D_2)$ has size 4 and $C \leq 4$. Let $X_t = \{x_1, x_2\}$ and $Y_t = \{y_1, y_2\}$. We may let $\{w_6x_2, w_5x_1\} \subset E(G)$. Clearly, an induced 6-cycle $w_3w_4w_5x_1w_1w_2w_3$ occurs. It implies that the of size of a component of $G(D_2)$ is 6 with $C \leq 3$. Suppose $C = 3$, then a component of $G[D_2]$ has size two. Let $X_t = \{x_1, x_2, x_3\}$, $Y_t = \{y_1, y_2, y_3\}$. Let $\{w_3y_1, w_1x_1, w_5x_2, w_6x_3\} \subset E(G)$. Therefore, the pairs of nonadjacent vertices both $\{w_1, w_5\}$ and $\{w_1, w_6\}$ need to be adjacent to each partition of the third connected component of (X, Y), but then they will induce a triangle. Suppose $C = 2$. Since $D_3 \subset N(X_1)$, then $D_3 \cap N(Y_1) = \emptyset$. Clearly, all the pairs of nonadjacent vertices of D_3 are connected to each partition of (X_t, Y_t), however, a vertex of D_3 will be adjacent to each partition of (X_t, Y_t), which contradicts Claim 20.

Subcase 3. $G[D_3] = F_7$ or F_8. Suppose $G[D_3] = F_7$. Since $\{w_1, w_3\}$ satisfies property $P(t)$ by Claim 21, then, without loss of generality, let $\{w_1x_1, w_3y_1\} \subset E(G)$ for $x_1 \in X_t$, $y_1 \in Y_t$. Suppose $|X_t| = |Y_t| = 1$. Then for the pair of nonadjacent vertices $\{w_1, w_5\}$ it follows that $w_5y_1 \notin E(G)$ (otherwise, $K_3 \subset G$) and it has property $P(t')$ by Claim 21. Thus, $d(w_1) \geq 6$ since $D_3 \subset N(X_1)$, a contradiction. Suppose $|X_t| = |Y_t| \geq 2$. For the pair of $\{w_1, w_5\}$ if $w_5y_2 \in E(G)$ for $y_2 \in Y_t$, then for the pair of $\{w_3, w_5\}$ it has property $P(t'')$ by Claim 21, and then $d(w_3) \geq 6$.

Suppose $G[D_3] = F_8$. Since the pairs of nonadjacent vertices $\{w_1, w_2\}$, $\{w_1, w_3\}$, $\{w_2, w_3\}$ satisfy Claim 21, similarly we could deduce that there will be a vertex of $\{w_1, w_2, w_3\}$ with degree at least 6. This completes the proof of Theorem 6. ∎

5. Concluding Remarks

In the end, we would like to say that we tried to improve the lower bound of $s(3, 9)$, but it cannot be realized for us. Based on that, we think that $s(3, 9)$ should be 24.

Problem 25. $s(3, 9) = 24$?

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