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CORRIGENDUM TO ”ON A 3-COLORING OF PLANE GRAPHS WITHOUT MONOCHROMATIC FACIAL 3-PATH”

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Abstract

The purpose of this corrigendum is to rectify a gap in the proof of the main theorem in the paper ”On a 3-coloring of plane graphs without monochromatic facial 3-path” published in Discussiones Mathematicae Graph Theory, https://doi.org/10.7151/dmgt.2584.

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1. Introduction

For the notation and definitions see the original paper [1]. There is stated the following main result of it.

Theorem 3. *Every plane graph admits a* 3*-coloring in which any monochromatic facial path has at most two vertices. Moreover, the bound* 2 *is optimal for* 4*-chromatic plane graphs.*

The error in the proof of this theorem concerns Claim 10, Case 2, in which we wrongly state that the counterexample G to Theorem 3 does not contain any 3-vertex. For our corrected proof of Theorem 3 it is enough to replace the assertion of Claim 10 with the new one.

Claim 10. *Every* 3*-vertex of* G *is contained in a separating* 4*-cycle.*

This statement is correctly proved in Case 1 of the original proof of Claim 10. All other Claims for the proof, up to Claim 14, are proved correctly. Taking them into account together with the new assertion of Claim 10, we amplify the statement of the original Claim 14 and the original proof of the Theorem 3 as mentioned in the next section.

2. Proof of the Main Result

By (original) Claim 4, we see that G is normal plane map. The basic information on normal plane maps is provided by the classical Lebesgue Theorem [2]. From it, (new) Claims 10 and (original) Claims 11, 12, and 13, we have the following.

Claim 14. *G is a normal plane map without adjacent 3-faces containing 3-vertices and/or $(3,4,a,b)$-vertices for $(a,b) \in \{(4,5),(5,4)\}$ but no other necessary k-vertices, $4 \le k \le 5$, required by Lebesgue's theorem.*

By Claim 10 we know that every 3-vertex of G is contained in a separating 4-cycle. For 4-vertices we have the following.

Claim 15. *Every $(3,4,5,4)$-vertex and every $(3,4,4,5)$-vertex in G is contained in a separating 4-cycle.*

Proof. See the original paper. ■

Let the 3-vertex (respectively, 4-vertex) u be contained in a separating 4-cycle. Let the vertices v and y be the neighbors of u on this cycle. Observe that there can be several facial 3-paths between v and y omitting u. These facial 3-paths together with the facial 3-path vuy form separating 4-cycles. We are interesting for such a separating 4-cycle $C^* = uvmy$ for which the subgraph $B^* = C^* \cup \mathrm{int}_G(C^*)$ does not contain as a proper subgraph any other subgraph $B' = C' \cup \mathrm{int}_G(C')$ for a separating 4-cycle $C' = uvoy$ through the edges uv and uy. This separating 4-cycle C^* is called the *light-separating* 4*-cycle* for the triple $(u;\{v,y\})$.

Note that the path vuy appears in one or two light-separating 4-cycles. In the later case, these two separating cycles have disjoint interiors. Observe, that there can be other light-separating 4-cycles through u that use another pairs of vertices adjacent to u.

From now on, let C^* denote the *lightest*-separating 4-cycle, that is, the light-separating 4-cycle with the minimum number of vertices and then with the minimum number of edges in the subgraph $B^* = C^* \cup \mathrm{int}_G(C^*)$ among all light-separating 4-cycles on all 3-vertices, all $(3,4,4,5)$-vertices and all $(3,4,5,4)$-vertices of G. Without loss of generality, let $C^* = uvmy$.

Now we are interested what is inside $B = B^*$ of C^*. To find out this, consider it. Observe that $\deg_B(v) \ge 3$, $\deg_B(y) \ge 3$, $\deg_B(m) \ge 2$, and $2 \le \deg_B(u) \le 4$. We distinguish two cases.

Case 1. Let $\deg_B(u) = 2$. Suppress the vertex u. Denote by D the resulting graph. Construct an auxiliary graph H as follows. The construction starts with a plane graph W of a 6-sided double-wheel, all faces of which are 3-faces $[v,y,m]$ with $\deg_W(m) = 6$, $\deg_W(v) = \deg_W(y) = 4$. We insert the graph D into each

3-face of W so that the boundary cycle vmy of each face of W is identified to the outer cycle vmy of D. Observe that the normal plane map H obtained has the following properties: $\deg_H(m) \geq 6$, $\deg_H(v) \geq 8$, $\deg_H(y) \geq 8$, and any vertex that could appear in H by Lebesgue's theorem [2] has to be present inside of a copy of B^* in H and therefore in G.

Applying Claims 5, 10, and Claim 14 on H it follows that $\delta^*(H) \geq 3$. Observe that H does not contain $(3,4,3,4^+)$-vertices (by Claim 11), $(3,4,4,4)$-vertices (by Claim 12), and $(3,5^+,3,5^+)$-vertices (by Claim 13). This means that H must contain 3-vertices and/or $(3,4,5,4)$-vertices and/or $(3,4,4,5)$-vertices. All these vertices have to be present inside C^* of G by Claim 14. The graph B contains inside the 3-vertices and/or the $(3,4,5,4)$-vertices and/or the $(3,4,4,5)$-vertices. A light-separating 4-cycle $\bar{C}$ through such a 3-vertex or 4-vertex inside B in G bounds (according Claims 10 or 15) less vertices than the lightest-separating 4-cycle C^* in the interior of C^*. A contradiction.

Case 2. Let $\deg_B(u) \geq 3$. To learn a structure of B we construct an auxiliary graph H. Consider the subgraph B. Observe that $\deg_B(v) \geq 3$, $\deg_B(y) \geq 3$, $3 \leq \deg_B(u) \leq 4$, and $\deg_B(m) \geq 2$.

The construction of H starts with a normal plane map $A = A_6$ the dual plane graph to the 6-side anti-prism all faces of which are 4-faces $[u,v,m,y]$ with $\deg_A(u) = \deg_A(v) = \deg_A(y) = 3$ and $\deg_A(m) = 6$. We insert in each 4-face of A the subgraph B so that the boundary cycle $uvmy$ of it is identified to the boundary cycle $uvmy$ of B. Observe that the normal plane map H so obtained has the following properties: Each of the vertices u, v, m, and y has in H the degree at least 6. To get a contradiction in this case we proceed in the rest of this proof in the same way as in Case 1 above.

This finishes the proof of Theorem 3.

Apology

The author would like to apologize for any inconvenience caused.

References

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