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PROPER ADDITIVE CHOICE NUMBER OF PLANAR GRAPHS

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Abstract

A *proper additive coloring* of a graph G is a labeling of its vertices with positive integers such that, for every pair of adjacent vertices, the assigned integers are distinct and the sums of integers assigned to their neighbors are different. The *proper additive choice number* of G is the least integer k such that, whenever each vertex is given a list of at least k available integers, a proper additive coloring can be chosen from the lists.

In this paper, we introduce some applications of Combinatorial Nullstellensatz in the study of proper additive coloring and present upper bounds on the proper additive choice number of planar graphs.

Keywords: proper additive coloring, proper additive chromatic number, proper additive choice number, planar graph, Combinatorial Nullstellensatz, discharging method.

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1. Introduction

Graphs considered in this paper are finite graphs without loops or multiple edges. Let $N_G(v)$ denote the set of neighbors of a vertex v in G. If S is a set of vertices in G, then $N_G(S) = \bigcup_{v \in S} N_G(v)$. The *degree* of a vertex v in G, denoted $d_G(v)$, is the number of neighbors of v in G. A vertex v is a *k-vertex* if $d_G(v) = k$ and a *k^--vertex* if $d_G(v) \leqslant k$. And a neighbor u of a vertex v is a *k-neighbor* of v if $d_G(u) = k$ and a *k^--neighbor* of v if $d_G(u) \leqslant k$. A vertex v is *isolated* if $d_G(v) = 0$. Let $\Delta(G)$ denote the maximum degree of a vertex in G.

A *labeling* of a graph G is a mapping f from its vertex set to positive integers. For an edge xy of G with endpoints x and y, define $\delta_f(xy) = \sum_{z \in N_G(x)} f(z) - \sum_{z \in N_G(y)} f(z)$. A labeling f of G is said to be an *additive coloring* if $\delta_f(xy) \neq 0$ for any edge xy of G. The additive coloring was first introduced by Czerwiński, Grytczuk, and Żelazny [14] as *lucky labeling*. It was noted by Akbari, Ghanbari, Manaviyat, and Zare [2] that an additive coloring for a graph G with vertex set $\{v_1, v_2, \ldots, v_{|G|}\}$ always exists by mapping v_i to 2^i for $1 \leqslant i \leqslant |G|$. The *additive chromatic number* of G, denoted $\chi_{_\Sigma}(G)$, is the least integer k such that G has an additive coloring from the vertex set of G to $\{1, 2, \ldots, k\}$. Note that $\chi_{_\Sigma}(G) = 1$ if G is a graph with no edges. An open conjecture proposed in [14] states that $\chi_{_\Sigma}(G) \leqslant \chi(G)$ for any graph G, where $\chi(G)$ denotes the *chromatic number* of G.

A *list* L of G is a mapping that assigns a finite set of positive integers to each vertex of G. For a positive integer k, a list L is a *k-list* if $|L(v)| \geqslant k$ for each vertex v of G. An additive coloring ϕ of G is called an *additive L-coloring* of G if $\phi(v) \in L(v)$ for each vertex v of G. The *additive choice number* of G, denoted $\mathrm{ch}_\Sigma(G)$, is the least integer k such that G has an additive L-coloring for any k-list L. Obviously, $\chi_{_\Sigma}(G) \leqslant \mathrm{ch}_\Sigma(G)$ for any graph G. And it was proved by Ahadi and Dehghan [1] that the difference between $\chi_{_\Sigma}(G)$ and $\mathrm{ch}_\Sigma(G)$ can be arbitrary large.

The known results about additive coloring of planar graphs are summarized as follows. Let G be a planar graph. It was proved by Bartnicki, Bosek, Czerwiński, Grytczuk, Matecki, and Żelazny [7] that $\chi_\Sigma(G) \leqslant 468$ and $\chi_\Sigma(G) \leqslant 36$ when G is 3-colorable. It was proved that $\mathrm{ch}_\Sigma(G) \leqslant 5\Delta(G) + 1$ by Axenovich, Harant, Przybyło, Soták, Voigt, and Weidelich [6] and $\mathrm{ch}_\Sigma(G) \leqslant 2\Delta(G) + 25$ by Lai and Lih [16]. And it was proved in [14] that $\mathrm{ch}_\Sigma(G) \leqslant 3$ if G is bipartite. The *girth* $g(G)$ of G is defined to be the length of a shortest cycle, where $g(G) = \infty$ if G has no cycle. It was proved by Brandt, Jahanbekam, and White [9] that $\chi_\Sigma(G) \leqslant 4$ if $g(G) \geqslant 10$ and $\mathrm{ch}_\Sigma(G) \leqslant 3$, 8, 9, and 19 if $g(G) \geqslant 26$, 7, 6, and 5, respectively. And it was proved by Brandt, Tenpas, and Yerger [10] that $\mathrm{ch}_\Sigma(G) \leqslant 3$ if $g(G) \geqslant 20$.

A labeling f of G is said to be a *proper additive coloring* if f is an additive coloring and $f(u) \neq f(v)$ for any edge uv of G. The *proper additive chromatic number* of G, denoted $\chi_\sigma(G)$, is the least integer k such that G has a proper additive coloring from the vertex set of G to $\{1, 2, \ldots, k\}$. The proper additive coloring was first studied in [2] and called proper lucky labeling in some papers. A similar application of the Combinatorial Nullstellensatz in [2] or a consequence of Lemma 5 showed that $\mathrm{ch}_\sigma(G)$ exists and is at most $\Delta(G)^2 + \Delta(G) + 1$ for any graph G. The proper additive chromatic number is an upper bound on the additive chromatic number for any graph G. Results of proper additive chromatic number of paths, stars, cycles, wheels, complete bipartite graphs, and some classes of graphs can be found in [5, 18].

Let L be a list of G. A proper additive coloring φ of G is called a *proper additive L-coloring* of G if $\varphi(v) \in L(v)$ for each vertex v of G. The *proper additive choice number* of G, denoted $\mathrm{ch}_\sigma(G)$, is the least integer k such that G has a proper additive L-coloring for any k-list L. Obviously, $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G)$ for any graph G. It was proved that $\mathrm{ch}_\Sigma(T) \leqslant 2$ in [14] and $\mathrm{ch}_\sigma(T) \leqslant 3$ in [2] for every tree T.

In this paper, we study the proper additive coloring and the proper additive choice number of planar graphs by using Combinatorial Nullstellensatz and the discharging method. In Section 2, we introduce some auxiliary lemmas obtained by using Combinatorial Nullstellensatz. Let G be a planar graph. In Section 3, we prove that $\mathrm{ch}_\sigma(G) \leqslant \min\{2\Delta(G) + 30, 5\Delta(G) + 6\}$. In Section 4, we prove that $\mathrm{ch}_\sigma(G) \leqslant 5$ if G is bipartite. In Section 5, we prove that $\mathrm{ch}_\sigma(G) \leqslant 5$, 10, 12, and 28 if $g(G) \geqslant 26$, 7, 6, and 5, respectively. And in Section 6, we prove that $\mathrm{ch}_\sigma(G) \leqslant 8$ if $\Delta(G) \leqslant 6$ and $g(G) \geqslant 10$ and $\mathrm{ch}_\sigma(G) \leqslant 9$ if $\Delta(G) \leqslant 4$ and $g(G) \geqslant 6$.

2. COMBINATORIAL NULLSTELLENSATZ AND AUXILIARY LEMMAS

The following lemma is a well-known fact of the additive coloring and the proper additive coloring. Note that the first equality is the Observation 1.8 in Chartrand, Okamoto, and Zhang [11].

Lemma 1. *If the components of G are $G_1, G_2, \ldots, G_k$, then* $\chi_\Sigma(G) = \max_{1\le j\le k} \chi_\Sigma(G_j)$, $\chi_\sigma(G) = \max_{1\le j\le k} \chi_\sigma(G_j)$, $\mathrm{ch}_\Sigma(G) = \max_{1\le j\le k} \mathrm{ch}_\Sigma(G_j)$, *and* $\mathrm{ch}_\sigma(G) = \max_{1\le j\le k} \mathrm{ch}_\sigma(G_j)$.

The following Combinatorial Nullstellensatz due to Alon [3] is a very useful tool in dealing with problems of combinatorial nature. It has already been used to study the additive coloring in [2, 7, 9, 10, 14].

Theorem 2. *Let $\mathbb{F}$ be a field and $F(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose the degree of F is $\sum_{i=1}^n k_i$, where each k_i is a nonnegative integer and the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in $\mathbb{F}$ is nonzero. If $L_1, L_2, \ldots, L_n$ are subsets of $\mathbb{F}$ with $|L_i| > k_i$ for each i, then there exist $c_1 \in L_1, c_2 \in L_2, \ldots, c_n \in L_n$ such that $F(c_1, c_2, \ldots, c_n) \neq 0$.*

In order to apply Combinatorial Nullstellensatz in the study of proper additive coloring, we introduce the following definitions and Proposition 3. Note that Proposition 3 will be a handy tool in the proofs of this paper.

Assume $S = \{v_1, \ldots, v_t\}$ is a set of vertices in G. Label the set $N_G(S) \setminus S$ by $\{v_{t+1}, \ldots, v_r\}$. Let $E_1(S)$ be the set of edges such that $ab \in E_1(S)$ if and only if exactly one of a and b has a neighbor in S. Let $E_2(S)$ be the set of edges such that $ab \in E_2(S)$ if and only if both a and b have a neighbor in S and the set of neighbors of a in S and the set of neighbors of b in S are different. And let $E_3(S)$ be the set of edges in $E_1(S) \cup E_2(S)$ with at least one endpoint in S. Note that each edge in $E_1(S)$ is in the form $v_i w$ where $1 \leqslant i \leqslant r$ and $w \notin N_G(S)$. And each edge in $E_2(S) \cup E_3(S)$ is in the form $v_i v_j$, $1 \leqslant i < j \leqslant r$.

Let $G \setminus S$ be the subgraph of G obtained by removing all the vertices in S and λ be a proper additive coloring of $G \setminus S$. We are going to define a proper additive coloring f of G by labeling v_i with x_i for each vertex v_i in S such that f extends λ in the sense that f coincides with λ on $G \setminus S$. For convenience, we define that $\lambda(v_i) = 0$ for each vertex v_i in S.

If $v_i w$ is an edge in $E_1(S)$, then the difference between $\delta_f(v_i w)$ and $\delta_\lambda(v_i w)$ is $\sum\limits_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p$. We have $\sum\limits_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p \neq -\delta_\lambda(v_i w) = \sum\limits_{z \in N_G(w)} \lambda(z) - \sum\limits_{z \in N_G(v_i) \setminus S} \lambda(z)$ by the claim of f. If $v_i v_j$ is an edge in $E_2(S)$, then the difference between $\delta_f(v_i v_j)$ and $\delta_\lambda(v_i v_j)$ is $\sum\limits_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p - \sum\limits_{v_q \sim v_j, 1 \leqslant q \leqslant t} x_q$. Since v_i and v_j have different sets of neighbors in S, $\sum\limits_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p - \sum\limits_{v_q \sim v_j, 1 \leqslant q \leqslant t} x_q$ is not zero. And we have $\sum\limits_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p - \sum\limits_{v_q \sim v_j, 1 \leqslant q \leqslant t} x_q \neq -\delta_\lambda(v_i v_j) = \sum\limits_{z \in N_G(v_j) \setminus S} \lambda(z) - \sum\limits_{z \in N_G(v_i) \setminus S} \lambda(z)$ by the claim of f. If e is an edge in $E_3(S)$ and the two endpoints of e are both in S, then $e = v_i v_j$ for some $1 \leqslant i < j \leqslant t$ and we have $x_i - x_j \neq 0$ by the claim of f. If e is an edge in $E_3(S)$ and exactly one endpoint of e is in S, then $e = v_i v_j$ for some $1 \leqslant i \leqslant t < j \leqslant r$ and we have $x_i \neq \lambda(v_j)$ by the claim of f. Note that, for each of the above inequalities, the left-hand side is a polynomial of the variables $x_1, \ldots, x_t$ with degree 1 and the right-hand side is a constant.

Assume that $E_1(S) \cup E_2(S) \cup E_3(S) \neq \emptyset$. The following polynomial is obtained by multiplying the left-hand sides of the above inequalities.

$$P_S(x_1, \ldots, x_t) = \prod_{v_i w \in E_1(S), i \leqslant r} \left(\sum_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p \right) \prod_{v_i v_j \in E_2(S), i < j} \left(\sum_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p - \right.$$

$$\left. \sum_{v_q \sim v_j, 1 \leqslant q \leqslant t} x_q \right) \prod_{v_i v_j \in E_3(S), i < j \leqslant t} (x_i - x_j) \prod_{v_i v_j \in E_3(S), i \leqslant t < j} (x_i) \in \mathbb{Q}[x_1, \ldots, x_t].$$

Since each parenthesis is a polynomial with degree 1, the polynomial P_S is not the zero polynomial and the degree of P_S is $|E_1(S)| + |E_2(S)| + |E_3(S)|$.

A graph G and a set $S = \{v_1, v_2\}$ of vertices in G are given in Figure 1. We have $N_G(S) \setminus S = \{v_3, v_4, v_5, v_6\}$, $E_1(S) = \{e_2\}$, $E_2(S) = \{e_1, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$, and $E_3(S) = \{e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$. Note that $e_{11} \notin E_2(S)$ since the two endpoints of e_{11} have the same neighbors in S. In addition, $P_S(x_1, x_2) = x_1^7 x_2^7 (x_1 + x_2)(x_1 - x_2)^3$.

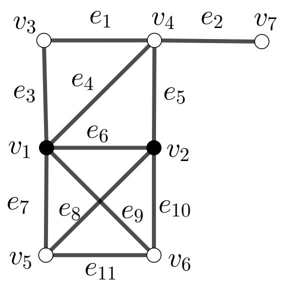

Figure 1. A graph G and a set $S = \{v_1, v_2\}$ of vertices in G.

Proposition 3. *Let $S = \{v_1, \ldots, v_t\}$ be a set of vertices in G and k be a positive integer. If the coefficient of $\prod_{1 \leqslant i \leqslant t} x_i^{d_i}$ in P_S is nonzero, $\sum_{1 \leqslant i \leqslant t} d_i = |E_1(S)| + |E_2(S)| + |E_3(S)|$, and $k > d_i$ for each $1 \leqslant i \leqslant t$, then*

$$\mathrm{ch}_\sigma(G) \leqslant \max\{k, \mathrm{ch}_\sigma(G \setminus S)\}.$$

Proof. Let L be a $\max\{k, \mathrm{ch}_\sigma(G \setminus S)\}$-list of G and λ be a proper additive L-coloring of $G \setminus S$. Define the polynomial

$$P_{S,\lambda}(x_1, \ldots, x_t) = \prod_{v_i w \in E_1(S), i \leqslant r} \left(\sum_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p + \sum_{z \in N_G(v_i) \setminus S} \lambda(z) - \sum_{z \in N_G(w)} \lambda(z) \right)$$

$$\prod_{v_i v_j \in E_2(S), i<j} \left(\sum_{v_p \sim v_i, 1 \leqslant p \leqslant t} x_p - \sum_{v_q \sim v_j, 1 \leqslant q \leqslant t} x_q + \sum_{z \in N_G(v_i) \setminus S} \lambda(z) - \sum_{z \in N_G(v_j) \setminus S} \lambda(z) \right)$$

$$\prod_{v_i v_j \in E_3(S), i<j \leqslant t} (x_i - x_j) \prod_{v_i v_j \in E_3(S), i \leqslant t < j} (x_i - \lambda(v_j)) \in \mathbb{Q}[x_1, \ldots, x_t].$$

The degrees of $P_{S,\lambda}$ and P_S are both $|E_1(S)| + |E_2(S)| + |E_3(S)|$. Since the degree of each parenthesis in $P_{S,\lambda}$ is 1 and $\sum_{1 \leqslant i \leqslant t} d_i$ equals the degree of $P_{S,\lambda}$, the coefficient of $\prod_{1 \leqslant i \leqslant t} x_i^{d_i}$ in $P_{S,\lambda}$ equals the coefficient of $\prod_{1 \leqslant i \leqslant t} x_i^{d_i}$ in P_S. By Theorem 2, there are $c_1 \in L(v_1), c_2 \in L(v_2), \ldots, c_t \in L(v_t)$ so that $P_{S,\lambda}(c_1, c_2, \ldots, c_t) \neq 0$.

Define the labeling f of G such that $f(v) = \lambda(v)$ if $v \notin S$ and $f(v_i) = c_i$ if $1 \leqslant i \leqslant t$. First, we prove that f is an additive L-coloring of G. Let xy be an edge of G.

Case 1. Both x and y have no neighbor in S. Then $\delta_f(xy) = \delta_\lambda(xy) \neq 0$.

Case 2. Only one of x and y has neighbors in S. Assume that x has a neighbor in S. Then $xy \in E_1(S)$ and $x = v_i$ for some $1 \leqslant i \leqslant r$. We have $\delta_f(xy) = \delta_f(v_i y) = \sum_{v_p \sim v_i, 1 \leqslant p \leqslant t} c_p + \sum_{z \in N_G(v_i) \setminus S} \lambda(z) - \sum_{z \in N_G(y)} \lambda(z)$. Since $P_{S,\lambda}(c_1, c_2, \ldots, c_t) \neq 0$, we have $\delta_f(xy) \neq 0$.

Case 3. Both x and y have neighbors in S.

Subcase 3.1. The set of neighbors of x in S and the set of neighbors of y in S are the same. Both x and y are in $G \setminus S$ and $\delta_f(xy) = \delta_\lambda(xy) + \sum_{v_p \sim x, 1 \leqslant p \leqslant t} c_p - \sum_{v_p \sim y, 1 \leqslant p \leqslant t} c_p = \delta_\lambda(xy) \neq 0$.

Subcase 3.2. The set of neighbors of x in S and the set of neighbors of y in S are different. Then $xy \in E_2(S)$ and assume $x = v_i$ and $y = v_j$ with $1 \leqslant i < j \leqslant r$. Then $\delta_f(xy) = \delta_f(v_i v_j) = \sum_{v_p \sim v_i, 1 \leqslant p \leqslant t} c_p - \sum_{v_q \sim v_j, 1 \leqslant q \leqslant t} c_q + \sum_{z \in N_G(v_i) \setminus S} \lambda(z) - \sum_{z \in N_G(v_j) \setminus S} \lambda(z)$. Since $P_{S,\lambda}(c_1, c_2, \ldots, c_t) \neq 0$, we have $\delta_f(xy) \neq 0$.

Therefore, f is an additive L-coloring of G. Now we prove that $f(x) \neq f(y)$ for any edge xy of G.

Case 4. $xy \notin E_3(S)$. Both x and y are in $G \setminus S$. Then $f(x) = \lambda(x) \neq \lambda(y) = f(y)$ by the assumption of λ.

Case 5. $xy \in E_3(S)$.

Subcase 5.1. Only one of x and y is in S. Assume that x is in S. Then $x = v_i$ and $y = v_j$ for some $1 \leqslant i \leqslant t < j$. We have $f(v_i) - f(v_j) = c_i - \lambda(v_j)$. Since $P_{S,\lambda}(c_1, c_2, \ldots, c_t) \neq 0$, we have $f(v_i) - f(v_j) \neq 0$.

Subcase 5.2. Both x and y are in S. Assume that $x = v_i$ and $y = v_j$ for some $1 \leqslant i < j \leqslant t$. Then $f(v_i) - f(v_j) = c_i - c_j$. Since $P_{S,\lambda}(c_1, c_2, \ldots, c_t) \neq 0$, we have $f(v_i) - f(v_j) \neq 0$.

Hence, f is a proper additive L-coloring of G and $\mathrm{ch}_\sigma(G) \leqslant \max\{k, \mathrm{ch}_\sigma(G \setminus S)\}$. ■

The following result of the additive chromatic number and the additive choice number was proved in [16] and can be obtained by rewriting the Lemma 3.1(a) in [9].

Lemma 4. *If G is a graph and v is a vertex of G, then*

$$\chi_\Sigma(G) \leqslant \max\left\{\chi_\Sigma(G \setminus \{v\}), 1 + \sum_{u \in N_G(v)} d_G(u)\right\}$$

and

$$\mathrm{ch}_\Sigma(G) \leqslant \max\left\{\mathrm{ch}_\Sigma(G \setminus \{v\}), 1 + \sum_{u \in N_G(v)} d_G(u)\right\}.$$

By similar arguments of the proofs of Lemma 4 in [9] and [16], we have the following result of the proper additive choice number. Now we give a proof obtained by applying Proposition 3.

Lemma 5. *If G is a graph and v is a vertex of G, then*

$$\mathrm{ch}_\sigma(G) \leqslant \max\left\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1 + d_G(v) + \sum_{u \in N_G(v)} d_G(u)\right\}.$$

Proof. Let S be $\{v\}$. Since the endpoints of any edge in $E_2(S)$ have different sets of neighbors in S and $|S| = 1$, $E_2(S) = \emptyset$. We have $|E_1(S)| \leqslant \sum_{u \in N_G(v)} d_G(u)$, $|E_3(S)| = d_G(v)$, and the polynomial P_S equals $x_1^{|E_1(S)|+|E_3(S)|}$.

By Proposition 3, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{|E_1(S)| + |E_3(S)| + 1, \mathrm{ch}_\sigma(G \setminus \{v\})\} \leqslant \max\{1 + d_G(v) + \sum_{u \in N_G(v)} d_G(u), \mathrm{ch}_\sigma(G \setminus \{v\})\}$. ■

The following result of the additive choice number was rewritten from the Lemma 3.1(b) in [9].

Lemma 6. *Let G be a graph with girth at least 5 and v be a vertex of G. Let R be a set of 1-neighbors of v with $|R| = r$ and $Q = \{v_1, \ldots, v_q\}$ be a set of 2-neighbors of v having a $(k-1)^-$-neighbor other than v, say $v'_1, \ldots, v'_q$, respectively, such that $v'_1, \ldots, v'_q$ are independent. If $r(k-1) + \sum_{v_i \in Q} (k - d_G(v'_i) - 1) \geqslant d_G(v)$, then $\mathrm{ch}_\Sigma(G) \leqslant \max\{\mathrm{ch}_\Sigma(G \setminus (R \cup Q)), k\}$.*

By a similar argument of the proof of Lemma 6 in [9], we have the following result of the proper additive choice number. Now we give a proof obtained by applying Proposition 3.

Lemma 7. *Let G be a graph with girth at least 5 and v be a vertex of G. Let R be a set of 1-neighbors of v with $|R| = r$ and $Q = \{v_1, \ldots, v_q\}$ be a set of 2-neighbors of v having a $(k-3)^-$-neighbor other than v, say $v'_1, \ldots, v'_q$, respectively, such that $v'_1, \ldots, v'_q$ are independent. If $r(k-2) + \sum_{v_i \in Q} (k - d_G(v'_i) - 3) \geqslant d_G(v)$, then $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (R \cup Q)), k\}$.*

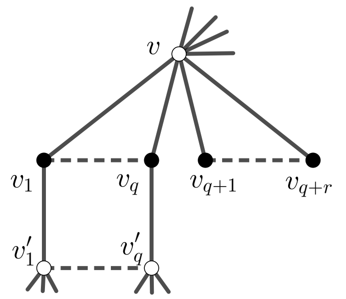

Figure 2. An illustration of Lemma 7.

Proof. Let $R = \{v_{q+1}, \ldots, v_{q+r}\}$ and $S = R \cup Q$. Since $v'_1, \ldots, v'_q$ are independent, $E_2(S) = \emptyset$. We have $|E_1(S)| = d_G(v) + \sum_{v_i \in Q} d_G(v'_i)$, $|E_3(S)| = r + 2q$, and the polynomial P_S equals $(x_1 + \cdots + x_{q+r})^{d_G(v)} \prod_{1 \leqslant i \leqslant q} x_i^{d_G(v'_i)+2} \prod_{q+1 \leqslant j \leqslant q+r} x_j$.

Since $\sum_{v_i \in Q} (k - d_G(v'_i) - 3) + r(k-2) \geqslant d_G(v)$, there exist nonnegative integers $d_1, \ldots, d_{q+r}$ such that $d_1 + \cdots + d_{q+r} = d_G(v)$, $d_i \leqslant k - d_G(v'_i) - 3$ for each $1 \leqslant i \leqslant q$, and $d_j \leqslant k-2$ for each $q+1 \leqslant j \leqslant q+r$. By the multinomial theorem, the coefficient of $\prod_{1 \leqslant i \leqslant q+r} x_i^{d_i}$ in $(x_1 + \cdots + x_{q+r})^{d_G(v)}$ is nonzero. Hence, the coefficient of $\prod_{1 \leqslant i \leqslant q} x_i^{d_i + d_G(v'_i)+2} \prod_{q+1 \leqslant j \leqslant q+r} x_j^{d_j+1}$ in P_S equals the coefficient of $\prod_{1 \leqslant i \leqslant q+r} x_i^{d_i}$ in $(x_1 + \cdots + x_{q+r})^{d_G(v)}$ and is nonzero. Since $d_i + d_G(v'_i) + 2 \leqslant k-1$ for each $1 \leqslant i \leqslant q$ and $d_j + 1 \leqslant k-1$ for each $q+1 \leqslant j \leqslant q+r$, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{k, \mathrm{ch}_\sigma(G \setminus (R \cup Q))\}$ by Proposition 3. ■

3. Planar Graphs

The following lemma was proved by van den Heuvel and McGuinness [15] using the discharging method and used to prove that the additive choice number of a planar graph is at most $2\Delta(G) + 25$ in [16].

Lemma 8. *Let G be a planar graph. Then there exists a vertex v whose k neighbors $v_1, v_2, \ldots, v_k$ are enumerated to satisfy $d_G(v_1) \leqslant \cdots \leqslant d_G(v_k)$ and one of the following statements holds.*

(A1) $k \leqslant 2$;

(A2) $k = 3$ *with* $d_G(v_1) \leqslant 11$;

(A3) $k = 4$ *with* $d_G(v_1) \leqslant 7$ *and* $d_G(v_2) \leqslant 11$*;*

(A4) $k = 5$ *with* $d_G(v_1) \leqslant 6$*,* $d_G(v_2) \leqslant 7$*, and* $d_G(v_3) \leqslant 11$*.*

If G is planar, then G contains a vertex with degree at most 5. By the well-known property and Lemmas 5 and 8, we have the following upper bound on the proper additive choice number of planar graphs.

Theorem 9. *If* G *is a planar graph, then*

$$\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant \min\{2\Delta(G) + 30, 5\Delta(G) + 6\}.$$

Proof. We prove the theorem by induction on the number of vertices of G. The theorem is trivially true for the induction basis of a single vertex graph.

Let v be the vertex characterized in Lemma 8, we have $1 + d_G(v) + d_G(v_1) + \cdots + d_G(v_k) \leqslant 2\Delta(G) + 30$ for each case. Hence, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1 + d_G(v) + \sum_{u \in N_G(v)} d_G(u)\} \leqslant \max\{2\Delta(G \setminus \{v\}) + 30, 2\Delta(G) + 30\} \leqslant 2\Delta(G) + 30$ by Lemma 5.

Let w be a vertex with $d_G(w) \leqslant 5$. By Lemma 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{w\}), 1 + d_G(w) + \sum_{u \in N_G(w)} d_G(u)\} \leqslant \max\{5\Delta(G \setminus \{w\}) + 6, 5\Delta(G) + 6\} \leqslant 5\Delta(G) + 6$. ∎

4. Bipartite Planar Graphs

The following theorem of proper additive choice number of bipartite graphs was proved in [2] by using Combinatorial Nullstellensatz and used to prove that the proper additive choice number of a forest is at most 3.

Theorem 10. *Let* G *be a bipartite graph and* k *be a positive integer such that* G *has an orientation with maximum out-degree at most* k*. Then* $\mathrm{ch}_\sigma(G) \leqslant 2k + 1$*.*

The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum of the average degrees $\frac{2\|H\|}{|H|}$ taken over all the subgraphs H of G. A graph G has an orientation with maximum out-degree at most k if and only if the maximum average degree of G is at most k (cf. [4]). Since the maximum average degree of a bipartite planar graph is less than 2, we have the following corollary.

Corollary 11. *If* G *is a bipartite planar graph, then* $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 5$*.*

5. Planar Graphs with Given Girth

Recall that it was proved in [9] that the additive choice number of a planar graph with girth at least 5 is at most 19. By the proof in [9], we have the following lemma. Note that the constraint $1 \leqslant d \leqslant 9$ in (B1) which we will use in the proof of Theorem 13 is not described in [9] but we can obtain it from the proof. The constraints in (C1) of Lemma 14, (D1) of Lemma 18, and (E1) of Lemma 12 are similar.

Lemma 12. *If G is planar and the girth of G is at least 5, then G contains at least one of the following configurations.*

(B1) *An isolated vertex or a d-vertex with $1 \leqslant d \leqslant 9$ has neighbors with degree sum at most 18.*

(B2) *A d-vertex v with $d \leqslant 18r + (18-t)q_t$, where r is the number of 1-neighbors of v and q_t is the number of 2-neighbors of v having a t^--neighbor other than v for some $t \leqslant 18$.*

(B3) *An induced cycle $v_3v_1v_4v_5v_2$ such that $d_G(v_3) \leqslant 17$, $d_G(v_1) = d_G(v_2) = 2$, $d_G(v_4) \leqslant 7$, and $d_G(v_5) \leqslant 7$.*

By Lemma 12 and the auxiliary lemmas, we have the following theorem for planar graphs with girth at least 5.

Theorem 13. *If G is a planar graph with girth at least 5, then*

$$\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 28.$$

Proof. We prove by using mathematical induction on the number of vertices. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 12, we have three configurations to argue.

For Configuration B1, let v be the characterized vertex. We have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 28$ by the induction hypothesis. By Lemmas 1 and 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1 + d_G(v) + \sum_{u \in N_G(v)} d_G(u)\} \leqslant \max\{28, 1 + 9 + 18\} = 28$.

For Configuration B2, note that $d_G(v) \leqslant 18r + (18-t)q_t \leqslant 26r + (25-t)q_t$ for some $t \leqslant 18$. Let R be the set of the r 1-neighbors of v and Q be the set of the 2-neighbors of v having a t^--neighbor other than v. By the induction hypothesis, we have $\mathrm{ch}_\sigma(G \setminus (R \cup Q)) \leqslant 28$. By Lemma 7, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (R \cup Q)), 28\} = 28$.

For Configuration B3, let $S = \{v_1, v_2\}$. Since the most restrictions on labels occurs when $d_G(v_3) = 17$ and $d_G(v_4) = d_G(v_5) = 7$, we assume that this is the case. By the induction hypothesis, we have $\mathrm{ch}_\sigma(G \setminus \{v_1, v_2\}) \leqslant 28$. Since the coefficient of $x_1^{26}x_2^8$ in $P_{\{v_1,v_2\}} = x_1^8x_2^8(x_1+x_2)^{17}(x_1-x_2)$ is 1, we have $\mathrm{ch}_\sigma(G) \leqslant 28$ by Proposition 3.

Therefore, we have $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 28$. ■

Recall that it was proved in [9] that the additive choice number of a planar graph with girth at least 6 is at most 9. By the proof in [9], we have the following lemma.

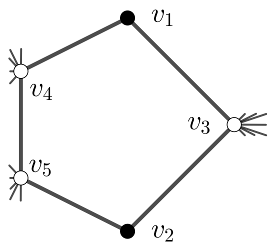

Figure 3. Configuration B3.

Lemma 14. *If the girth of G is at least 6 and the maximum average degree of G is less than 3, then G contains at least one of the following configurations.*

(C1) *An isolated vertex, a 1-vertex that has a 8^--neighbor, or a 2-vertex that has neighbors with degree sum at most 8.*

(C2) *A d-vertex v with $d \leqslant 8r + (8-t)q_t$, where r is the number of 1-neighbors of v and q_t is the number of 2-neighbors of v having a t^--neighbor other than v for some $t \leqslant 8$.*

(C3) *A 6-vertex having six 2-neighbors one of which has a 3^--neighbor.*

(C4) *A 7-vertex having seven 2-neighbors two of which have a 4^--neighbors.*

By Lemma 14 and the auxiliary lemmas, we have the following theorem for graphs with girth at least 6 and maximum average degree less than 3.

Theorem 15. *If the girth of G is at least 6 and the maximum average degree of G is less than 3, then $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 12$.*

Proof. We prove by using mathematical induction on the number of vertices. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 14, we have four configurations to argue. Let v be the characterized vertex in each configuration.

For Configuration C1, we have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 12$ by the induction hypothesis. By Lemmas 1 and 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1 + d_G(v) + \sum_{u \in N_G(v)} d_G(u)\} \leqslant \max\{12, 1+2+8\} = 12$.

For Configuration C2, note that $d_G(v) \leqslant 8r + (8-t)q_t \leqslant 10r + (9-t)q_t$ for some $t \leqslant 8$. Let R be the set of the r 1-neighbors of v and Q be the set of the 2-neighbors of v having a t^--neighbor other than v. By the induction hypothesis, we have $\mathrm{ch}_\sigma(G \setminus (R \cup Q)) \leqslant 12$. By Lemma 7, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (R \cup Q)), 12\} = 12$.

For Configuration C3, let v_2 be a 2-neighbor of v having a 3^--neighbor. Since the most restrictions on labels occurs when v_2 has a 3-neighbor, we assume that this is the case. Relabel v by v_1 and label the vertices in $N_G(\{v_1, v_2\})$ as in Figure 4. By the induction hypothesis, we have $\mathrm{ch}_\Sigma(G \setminus \{v_1, v_2\}) \leqslant 12$. Since the coefficient of $x_1^{10}x_2^{11}$ in $P_{\{v_1,v_2\}} = x_1^{10}x_2^{3}(x_1 - x_2)^8$ is 1, we have $\mathrm{ch}_\sigma(G) \leqslant 12$ by Proposition 3.

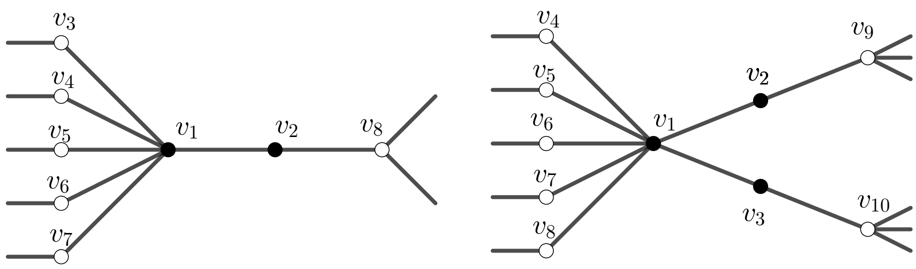

Figure 4. Configurations C3 and C4.

For Configuration C4, let v_2 and v_3 be 2-neighbors of v having a 4^--neighbor. Since the most restrictions on labels occurs when v_2 and v_3 both have a 4-neighbor, we assume that this is the case. Relabel v by v_1 and label the vertices in $N_G(\{v_1, v_2, v_3\})$ as in Figure 4. By the induction hypothesis, we have $\mathrm{ch}_\Sigma(G \setminus \{v_1, v_2, v_3\}) \leqslant 12$. Since the coefficient of $x_1^{10}x_2^{10}x_3^{9}$ in $P_{\{v_1,v_2,v_3\}} = x_1^{10}x_2^{4}x_3^{4}(-x_1 + x_2 + x_3)^7(x_1 - x_2)^2(x_1 - x_3)^2$ is 35, we have $\mathrm{ch}_\sigma(G) \leqslant 12$ by Proposition 3.

Therefore, we have $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 12$. ■

The following proposition is a well-known property of planar graphs.

Proposition 16. *Let G be a planar graph with girth at least g. Then,* $\mathrm{mad}(G) < \frac{2g}{g-2}$.

By Proposition 16, we have $\mathrm{mad}(G) < 3$ if G is a planar graph with girth at least 6. Therefore, we have the following corollary for planar graphs with girth at least 6.

Corollary 17. *If G is a planar graph with girth at least* 6, *then*

$$\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 12.$$

Recall that it was proved in [9] that the additive choice number of a planar graph with girth at least 7 is at most 8. By the proof in [9], we have the following lemma.

Lemma 18. *If the girth of G is at least* 6 *and the maximum average degree of G is less than $\frac{14}{5}$, then G contains at least one of the following configurations.*

(D1) *An isolated vertex, a* 1*-vertex that has a* 7^-*-neighbor, or a* 2*-vertex that has neighbors with degree sum at most* 7*.*

(D2) *A* d*-vertex* v *with* $d \leqslant 7r + (7-t)q_t$*, where* r *is the number of* 1*-neighbors of* v *and* q_t *is the number of* 2*-neighbors of* v *having a* t^-*-neighbor other than* v *for some* $t \leqslant 7$*.*

By Lemma 18 and the auxiliary lemmas, we have the following theorem for graphs with girth at least 6 and maximum average degree less than $\frac{14}{5}$.

Theorem 19. *If the girth of* G *is at least* 6 *and the maximum average degree of* G *is less than* $\frac{14}{5}$*, then* $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 10$*.*

Proof. We prove by using mathematical induction on the number of vertices. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 18, we have two configurations to argue.

For Configuration D1, let v be the characterized vertex. We have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 10$ by the induction hypothesis. By Lemmas 1 and 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1+2+7\} = 10$.

For Configuration D2, note that $d_G(v) \leqslant 7r + (7-t)q_t \leqslant 8r + (7-t)q_t$ for some $t \leqslant 7$. Let R be the set of the r 1-neighbors of v and Q be the set of the 2-neighbors of v having a t^--neighbor other than v. By the induction hypothesis, we have $\mathrm{ch}_\sigma(G \setminus (R \cup Q)) \leqslant 10$. By Lemma 7, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (R \cup Q)), 10\} = 10$.

Therefore, we have $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 10$. ■

By Proposition 16, we have $\mathrm{mad}(G) < \frac{14}{5}$ if G is a planar graph with girth at least 7. Therefore, we have the following corollary for planar graphs with girth at least 7.

Corollary 20. *If* G *is a planar graph with girth at least* 7*, then*

$$\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 10.$$

Let $P(t_2, \ldots, t_{n-1})$ be the path $v_1 v_2 \cdots v_{n-1} v_n$ such that, for each i in $\{2, \ldots, n-1\}$, the vertex v_i has t_i 1-neighbors and $d_G(v_i) = 2 + t_i$. Recall that it was proved in [9] that the additive choice number of a planar graph with girth at least 26 is at most 3. By the proof in [9], we have the following lemma.

Lemma 21. *If* G *is planar and the girth of* G *is at least* 26*, then* G *contains at least one of the following configurations.*

(E1) *An isolated vertex or a* 1*-vertex that has a* 2^-*-neighbor.*

(E2) *A* d*-vertex* v *with* $d \leqslant 2r + (2-t)q_t$*, where* r *is the number of* 1*-neighbors of* v *and* q_t *is the number of* 2*-neighbors of* v *having a* t^-*-neighbor other than* v *for some* $t \leqslant 2$*.*

(E3) *A bipartite component.*

(E4) *A path* $P(t_2, \ldots, t_{n-1})$ *with* $(t_2, \ldots, t_{n-1}) = (1,0,1)$, $(1,1,1)$, $(1,1,0,0)$, $(0,1,0,0)$, *or* $(1,0,0,0)$.

(E5) *A path* $v_6v_3v_1v_2v_4v_5v_7$ *with* $d_G(v_i) = 2$ *for each* $1 \leqslant i \leqslant 5$.

By Lemma 21 and the auxiliary lemmas, we have the following theorem for planar graphs with girth at least 26.

Theorem 22. *If* G *is a planar graph with girth at least* 26, *then*

$$\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 5.$$

Proof. We prove by using mathematical induction on the number of vertices. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 21, we have five configurations to argue.

For Configuration E1, let v be the characterized vertex. We have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 5$ by the induction hypothesis. By Lemmas 1 and 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1+1+2\} \leqslant 5$.

For Configuration E2, note that $d_G(v) \leqslant 2r + (2-t)q_t \leqslant 3r + (2-t)q_t$ for some $t \leqslant 2$. Let R be the set of the r 1-neighbors of v and Q be the set of the 2-neighbors of v having a t^--neighbor other than v. By the induction hypothesis, we have $\mathrm{ch}_\sigma(G \setminus (R \cup Q)) \leqslant 5$. By Lemma 7, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (R \cup Q)), 5\} = 5$.

For Configuration E3, we have $\mathrm{ch}_\sigma(G) \leqslant 5$ by Lemma 1 and Corollary 11.

For Configuration E4, choose an integer i with $2 \leqslant i \leqslant n-1$ and $t_i = 1$. Let u_i be the 1-neighbor of v_i. We have $\mathrm{ch}_\sigma(G \setminus \{u_i\}) \leqslant 5$ by the induction hypothesis. By Lemmas 1 and 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{u_i\}), 1+1+3\} = 5$.

For Configuration E5, let $S = \{v_1, v_2\}$. By the induction hypothesis, we have $\mathrm{ch}_\Sigma(G \setminus \{v_1, v_2\}) \leqslant 5$. Since the coefficient of $x_1^4x_2^4$ in $P_{\{v_1,v_2\}} = x_1^2x_2^2(x_1 - x_2)^4$ is 6, we have $\mathrm{ch}_\sigma(G) \leqslant 5$ by Proposition 3.

Therefore, we have $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 5$. ∎

6. Planar Graphs with Given Maximum Degree and Girth

Let l be a positive integer. An *l-thread* in a graph G is a trail of length $l+1$ in G whose l internal vertices have degree 2 in G. Under the definition, the ends of an l-thread may be the same vertex. It was proved in [8, 12] that the square of a graph with the maximum degree at most 6 and maximum average degree less than $\frac{5}{2}$ is 7-choosable and the following lemma can be obtained by the proofs (cf. [13]).

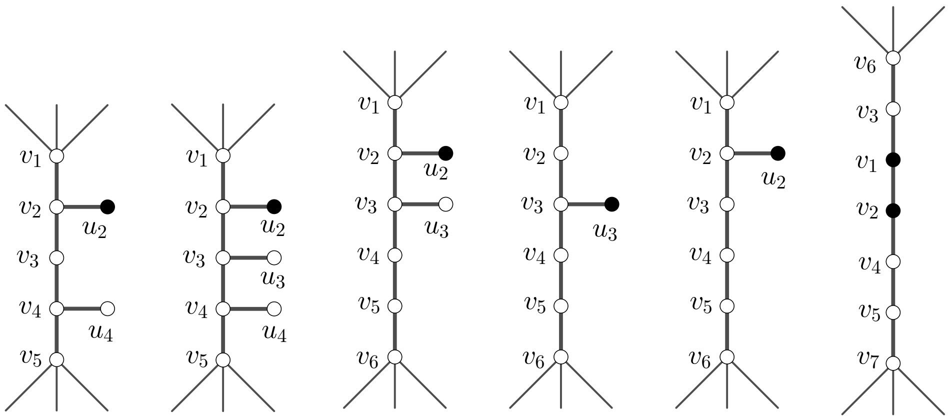

Figure 5. Configurations E4 and E5.

Lemma 23. *Let G be a graph with the maximum degree at most 6 and maximum average degree less than $\frac{5}{2}$. Then G contains at least one of the following configurations.*

(F1) *An isolated vertex v or a 1-vertex.*

(F2) *A 2-thread joining a 5^--vertex and a 4^--vertex.*

(F3) *A cycle of length divisible by 4 composed of 3-threads whose endpoints have degree 6.*

(F4) *A 4-vertex v having four 2-neighbors v_1, v_2, v_3, v_4 such that v_1, v_2, and v_3 have a 2-neighbor and v_4 has a 4^--neighbor other than v.*

(F5) *A 3-vertex v having three 2-neighbors v_1, v_2, v_3 which have a 4^--neighbor other than v.*

(F6) *A 3-vertex u having one 4^--neighbor and two 2-neighbors v, x such that v has a 2-neighbor and x has a 4^--neighbor.*

By Lemma 23 and the auxiliary lemmas, we have the following theorem for planar graphs with maximum degree at most 6 and girth at least 10.

Theorem 24. *If G is a planar graph with maximum degree at most 6 and girth at least 10, then $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 8$.*

Proof. We prove by using mathematical induction on the number of vertices. The theorem is trivially true for the induction basis of a single vertex graph. Since G is a planar graph with girth at least 10, we have $\mathrm{mad}(G) < \frac{5}{2}$ by Proposition 16. By Lemma 23, we have six configurations to argue.

For Configuration F1, let v be the characterized vertex. We have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 8$ by the induction hypothesis. By Lemmas 1 and 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1+1+6\} \leqslant 8$.

For Configuration F2, label the 2-thread by $v_3v_1v_2v_4$. Since the girth of G is at least 10, v_3 and v_4 are distinct and nonadjacent. Since the most restrictions on labels occurs when $d_G(v_3) = 5$ and $d_G(v_4) = 4$, we assume that this is the case. We have $\mathrm{ch}_\sigma(G \setminus \{v_1, v_2\}) \leqslant 8$ by the induction hypothesis. Since the coefficient of $x_1^7x_2^6$ in $P_{\{v_1,v_2\}} = x_1^5x_2^4(x_1 - x_2)^4$ is 6, we have $\mathrm{ch}_\sigma(G) \leqslant 8$ by Proposition 3.

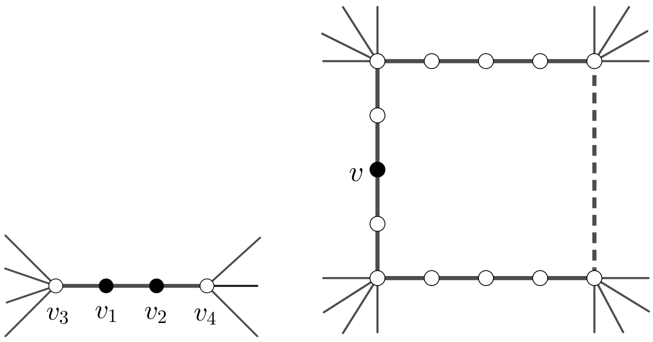

Figure 6. Configurations F2 and F3.

For Configuration F3, let v be the third vertex in a 3-thread of the cycle. We have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 8$ by the induction hypothesis. By Lemma 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1+2+4\} \leqslant 8$.

For Configuration F4, we have $\mathrm{ch}_\sigma(G \setminus \{v_1, v_2, v_3\}) \leqslant 8$ by the induction hypothesis. Since $\{v_1, v_2, v_3\}$ is a set of 2-neighbors of v having a 2-neighbor other than v and $d_G(v) = 4 \leqslant (8-2-3) \times 3$, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (\{v_1, v_2, v_3\}), 8\} = 8$ by Lemma 7.

For Configuration F5, we have $\mathrm{ch}_\sigma(G \setminus \{v_1, v_2, v_3\}) \leqslant 8$ by the induction hypothesis. Since $\{v_1, v_2, v_3\}$ is a set of 2-neighbors of v having a 4^--neighbor other than v and $d_G(v) = 3 \leqslant (8-4-3) \times 3$, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (\{v_1, v_2, v_3\}), 8\} = 8$ by Lemma 7.

For Configuration F6, we have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 8$ by the induction hypothesis. By Lemma 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1+2+5\} \leqslant 8$.

Therefore, we have $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 8$. ■

The following lemma was proved by Ruksasakchai and Wang [17] using the discharging method.

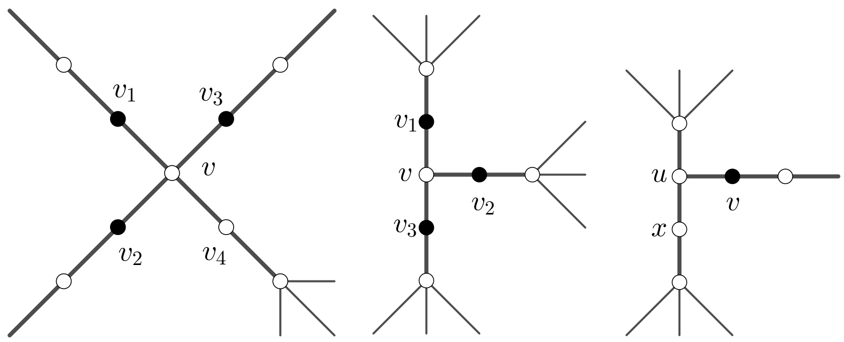

Figure 7. Configurations F4, F5, and F6.

Lemma 25. *Let G be a graph with maximum degree at most 4 and maximum average degree less than 3. Then G contains at least one of the following configurations.*

(G1) *An isolated vertex v or a 1-vertex.*

(G2) *A 2-vertex v having two 3^--neighbors.*

(G3) *A 4^--vertex v having two 2-neighbors v_1, v_2 one of which has a 3^--neighbor other than v.*

(G4) *A 4-vertex v having four 2-neighbors v_1, v_2, v_3, v_4.*

(G5) *A 2-vertex u having two 4-neighbors v and w such that v has three 2-neighbors u, v_1, v_2, and w has two or three 2-neighbors.*

By Lemma 25 and the auxiliary lemmas, we have the following theorem for planar graphs with maximum degree at most 4 and girth at least 6.

Theorem 26. *If G is a planar graph with maximum degree at most 4 and girth at least 6, then $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 9$.*

Proof. We prove by using mathematical induction on the number of vertices. The theorem is trivially true for the induction basis of a single vertex graph. Since G is a planar graph with girth at least 6, we have $\mathrm{mad}(G) < 3$ by Proposition 16. By Lemma 25, we have five configurations to argue.

For Configuration G1, let v be the characterized vertex. We have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 9$ by the induction hypothesis. By Lemmas 1 and 5, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1+1+4\} \leqslant 9$.

For Configuration G2, we have $\mathrm{ch}_\sigma(G \setminus \{v\}) \leqslant 9$ by the induction hypothesis. We have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus \{v\}), 1+2+6\} \leqslant 9$ by Lemma 5.

For Configurations G3, G4, and G5, we have $\mathrm{ch}_\sigma(G \setminus \{v_1, v_2\}) \leqslant 9$ by the induction hypothesis. Since $\{v_1, v_2\}$ is a set of 2-neighbors of v having a 4^--

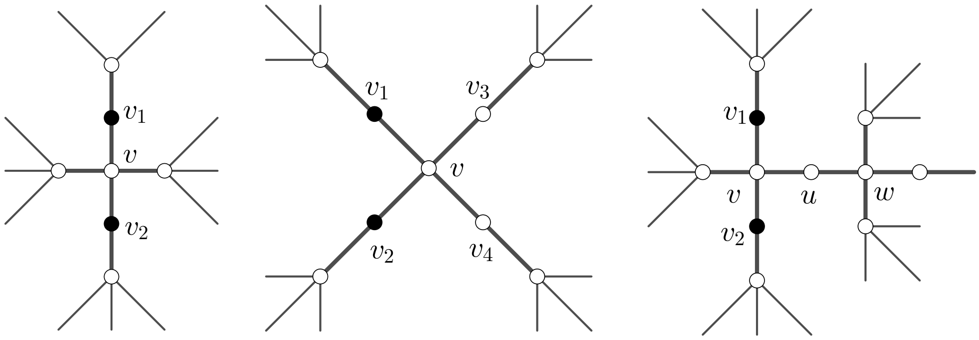

Figure 8. Configurations G3, G4, and G5.

neighbor other than v and $d_G(v) \leqslant 4 \leqslant (9-4-3) \times 2$, we have $\mathrm{ch}_\sigma(G) \leqslant \max\{\mathrm{ch}_\sigma(G \setminus (\{v_1, v_2\}), 9\} = 9$ by Lemma 7.

Therefore, we have $\chi_\sigma(G) \leqslant \mathrm{ch}_\sigma(G) \leqslant 9$. ■

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