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BURNING DISJOINT UNION OF SPIDER AND PATH

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Abstract

For a connected graph G of order n, the graph burning problem models the spread of influence in networks, with the burning number conjecture stating $b(G) \leq \lceil \sqrt{n}\, \rceil$. In 2020, Tan and Teh proposed a stronger conjecture for trees: Every tree with n leaves and order at most $m^2 + n - 2$ is m-burnable with $m > n$. This conjecture has been proven to hold for spiders and double spiders. However, the burning behaviour of disconnected graphs remains unknown. Motivated by this gap, we consider a disjoint union graph T which consists of a spider with n leaves and a path. In this paper, we show a similar result such that if T has order at most $m^2 + n - 2$ where $m \geq n+3$, then T is m-burnable with some exceptional cases. These results provide some insights into how the disjoint structure of a graph influences its burning behaviour.

Keywords: burning number, discrete graph algorithm, disjoint union graph, social network.

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1. Introduction

A combinatorial framework to model the spread of influence or contagion in a network over discrete time steps is laid out by the graph burning problem. The goal is to determine the burning number of a graph G, denoted as $b(G)$, which is the minimum number of time steps required to completely burn all vertices of G. In each time step $t \in \{1, 2, 3, \ldots, k\}$, one vertex x_t is selected to be burned, and fire spreads from every burned vertex to its neighbours in the subsequent round. The process continues until all vertices of G are burned. The sequence $(x_1, x_2, \ldots, x_k)$ where x_t denotes the vertex burned at time t, is called a burning sequence of G.

One of the most well-known burning number conjectures, proposed by Bonato *et al.* [4, 7], states that for every undirected connected graph G of order n, $b(G) \leq \lceil \sqrt{n}\, \rceil$. The burning number conjecture had been studied extensively (see [1, 3, 16]), but it still remains open in general. Recently, the result was proven to hold asymptotically by Norin and Turcotte [22].

Although the conjecture has not been proven in general, researchers have investigated the burning number of many specific classes of graphs, including circulant graphs [10], grid and interval graphs [6, 12], point in plane [15], t-unicyclic graphs [27], generalised Petersen graphs [23], fence graphs [5], theta graphs [19], caterpillars [13, 17], Cartesian product and the strong product of graphs [20], spiders, path forests [8, 9, 18, 26], homeomorphically irreducible trees [21] and as well as directed trees [14]. Additionally, another perspective comes from adversarial graph burning, where a game-theoretic approach is applied to analyze burning densities in dynamic graph sequences [11].

A *spider* is a tree that has exactly one branch vertex adjacent to multiple paths called *arms*. An n-*spider* is a tree with exactly one vertex of degree n, where such a vertex is called the *head* of the spider. The *arm length* refers to the distance along the arm from the head to the leaf, which is equal to the number of vertices on the arm excluding the head. Note that the burning number of a connected graph is the minimum burning number of its spanning trees. Hence, solving the burning number conjecture of trees would imply that the conjecture is solved for all connected graphs.

A graph G is m-*burnable* if $b(G) \leq m$. Suppose G is m-burnable. Then the *neighbourhood associated with a burning source* x_r in G, denoted as $N_{m-r}[x_r]$, is the set of vertices within $m - r$ distance from x_r. An *optimal burning sequence* of G is a burning sequence that completely burns G. Note that the optimal burning sequence of G might not be unique, and the burning number of G is the length of the shortest optimal burning sequence of G.

In 2020, Tan and Teh [24] proposed a tight upper bound on the order of a spider to guarantee that it is m-burnable (see Theorem 1). It is surprising that the tight bounds depend simply on the number of arms. They speculated from their findings that a tree's burning number is determined partially by the number of its leaves. Then they proposed a slightly stronger conjecture.

Conjecture 1. *Every tree with* n *leaves of order at most* m^2+n-2 *is* m*-burnable for all* $m > n$.

Extending this research, they also studied the double spiders [25], which are formed by joining two spiders at their branch vertices with an edge. They showed that double spiders follow Conjecture 1.

However, in spite of this progress, the burning behaviour of disconnected graphs have received relatively little study. In particular, little has been discovered about the burning behaviour of disconnected graphs. Let T be a disjoint graph consisting of an n-spider and an independent path where $n \geq 3$. In this paper, we show that T satisfying the bound in Conjecture 1 such that if the order of T is at most $m^2 + n - 2$ where $m \geq n + 3$, then T is m-burnable with only some exceptional cases, see Theorem 13.

Spiders are particularly interesting because their simple structure makes them a useful starting point to study the graph burning of trees. In addition to that, a spider is chosen as part of the disjoint graph due to a result in [2], which states that there exists an optimal burning sequence with m burning sources of an n-spider with at least three arms when the first burning source is placed at most $m - 1$ distance away from the head. This suggests that after removing the neighbourhood associated with the first burning source of T, the resulting graph forms a path forest of at most $n + 1$ independent paths. In this paper, we will repeatedly use Theorem 6 to identify the exceptional graphs that are not m-burnable.

It is noticeable that having the head of the spider in T being burned within the associated neighbourhood of the first burning source does not necessarily guarantee that T can be burned with the least burning sources, see Figure 1. The graph illustrated in Figure 1 is an example where the graph is not 6-burnable if the head of the spider is within the associated neighbourhood of the first burning source.

In the conclusion, we address the exceptional cases where T is not m-burnable when $m = n + 2$. Although we are unable to characterise these cases in a general way, these cases provide some new insights into the burning behaviour of disjoint graphs and highlight the influence of structural diversity in determining burning numbers of the graph.

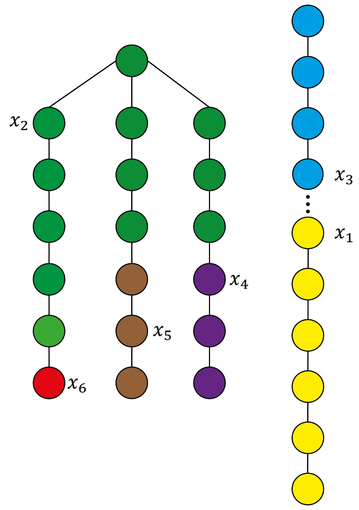

Figure 1. A disjoint graph of a 3-spider and a path of order 18.

2. Preliminaries

Here we present some known results that will be used in our results later.

Theorem 1 [24]**.** *Let $m > n \geq 2$. Every n-spider of order at most $m^2 + n - 2$ is m-burnable. Furthermore, if l is the length of the shortest arm, then the graph can be burned in m rounds in such a way that after the head of the spider is burned, there are still at least* $\min\{m-2, l\}$ *rounds.*

Theorem 1 shows that Conjecture 1 holds for an n-spider. Having the results on the number of rounds left after the head is burned will be useful when we investigate spiders with short arms. A *path forest* is a disjoint union of independent paths. An *n-path forest* also refers to a path forest with n paths. Theorem 2 addresses the problem related to a 2-path forest.

Theorem 2 [24]**.** *Let $m \geq 2$. If T is a path forest with two paths and $|T| \leq m^2$, then T is m-burnable unless the path orders of T are $m^2 - 2$ and 2.*

A tight bound on the order of an m-burnable path forest is shown by Das *et al.* (see Theorem 3) in 2018, and this bound is then improved with just a unique exceptional path forest by Tan and Teh (see Theorem 4) in 2020.

Theorem 3 [9]**.** *Let $m \geq n \geq 2$ and suppose T is a path forest with n paths such that all but possibly one of the paths have order at most m. If $|T| \leq m^2 - (n-1)^2$, then T is m-burnable.*

Theorem 4 [24]**.** *Let $m \geq n \geq 2$ and suppose T is a path forest with n paths. If*

$$|T| \leq m^2 - (n-1)^2 + 1,$$

then T is m-burnable unless $|T| = m^2 - (n-1)^2 + 1$ and T is the unique path forest with path orders $m^2 - n^2 + 2, \underbrace{2, 2, \ldots, 2}_{n-1 \text{ times}}$.

In 2023, Tan and Teh further improved the upper bound on the order of an n-path forest and identified the set of all path forests that are not m-burnable, where $m \geq n \geq 3$. Theorem 6 is a strengthened result of Theorem 4. First, we provide Definition 5.

Definition 5 [25]**.** Let $\mathcal{T}_{n,m}$ define the set of all path forests with n paths where the path orders $l_1, l_2, l_3, \ldots, l_n$ are such that

1. $l_1 = m^2 - (n-1)^2 + 1$ and $l_2 = l_3 = \cdots = l_n = 1$, or
2. $l_1 = m^2 - n^2 + 2$ and $2 \leq l_2, l_3, \ldots, l_n \leq 3$, or
3. $l_1 = m^2 - (n-1)(n+3) + 1$ and $l_2 = l_3 = \cdots = l_n = 5$.

If $m = n$, only the first case is applicable.

For $m \geq n \geq 3$, Theorem 6 indicates that a path forest T with n-paths is m-burnable with some exceptional cases as listed in $\mathcal{T}_{n,m}$. The notation t_T denotes the number of paths of order two of a path forest T when the shortest path of T has order two, or else $t_T = 0$.

Theorem 6 [25]**.** *Suppose $m \geq n \geq 3$ and let T be a path forest with n paths. If*

$$|T| \leq m^2 - (n-1)(n-2) + 1 - t_T,$$

then T is m-burnable unless $|T| = m^2 - (n-1)(n-2) + 1 - t_T$ and $T \in \mathcal{T}_{n,m}$.

In the next section, we define T and T' as follows, unless stated otherwise. Let T be a disjoint union of an n-spider and an independent path where $T = (l_1, l_2, l_3, \ldots, l_n \mid l)$. The order of the independent path is indicated after the vertical bar. Then let $T' = (l'_1, l'_2, l'_3, \ldots, l'_r; l)$ with $r \leq n$ be the path forest obtained by removing the associated neighbourhood of some burning source that burns the head of the spider of T. The order of the independent path is indicated after the semicolon in T'.

3. RESULTS

In this section, our first result focuses on the spider of T with most of the arms that are relatively short in T. Note that a path with at most m^2 vertices is m-burnable.

Lemma 7. *Let $n \geq 3$ and $m \geq n+1$. Suppose T is a disjoint union of an n-spider with arm lengths $l_1 \geq l_2 \geq \cdots \geq l_n$ and a path. If $|T| \leq m^2+n-2$ and $l_2 < m$, then T is m-burnable with a burning sequence such that after the head of the spider of T is burned, there are at least l_n rounds left unless $|T| = m^2+n-2$, the independent path of T has order two, and $l_3 = \cdots = l_n = 1$.*

Proof. Suppose $|T| = m^2+n-2$ and the diameter of the spider of T is D. First, let $D < 2m-1$ and D is even. Consider $D+1 = 2h-1$. Then the spider can be completely burned by placing x_{m-h+1} at the center of the longest path in the spider. Thus, the independent path has order at most m^2-2h+1. Clearly, this path can be burned by the remaining burning sources. Similarly, suppose D is an odd number. Let $D = 2h-1$. Then x_m is placed at the leaf of the longest arm and x_{m-h+1} is placed at distance h away from x_m on the longest arm (possibly the head). It can be verified that the independent path with order at most m^2-2h can be burned by the remaining burning sources.

Next, we consider $D \geq 2m-1$. Then, by placing the first burning source x_1 at the $(m-1-l_2)$-th vertex on the longest arm, all arms except the longest arm are completely burned by the first burning source in m rounds. The remaining unburned vertices form a 2-path forest, say T' with order at most $(m-1)^2$ and thus it is $(m-1)$-burnable by Theorem 2 unless the path order of T' is $((m-1)^2-2, 2)$.

If the independent path of T is of order two, then T is not m-burnable, which is the exceptional case as given in the lemma. Now, we consider the case where the independent path of T is of order $(m-1)^2-2$. It is clear that T contains an n-spider with arm length $l_3 = l_4 = \cdots = l_n = 1$. Otherwise, if $l_3 \geq 2$, then $l_2 \geq 2$ and $T' \leq m^2+n-2-(2m-1)-(n-1) = (m-1)^2-1$. Thus T' is $(m-1)$-burnable. So, we have $T = (2m+1-l_2, l_2, \underbrace{1,\ldots,1}_{n-2 \text{ times}} \mid (m-1)^2-2)$. Here, the first burning source is placed at the independent path, while the second burning source x_2 is placed at the $(m-2-l_2)$-th vertex on the longest arm when $l_2 \leq m-2$ and on the vertex adjacent to the head on the second longest arm when $l_2 = m-1$. Consequently, all arms except the longest arm are completely burned by the second burning source in m rounds. Now, the remaining unburned vertices form the path forest $T'' = (4; (m-2)^2-4)$, which is $(m-2)$-burnable by Theorem 2.

Lastly, note that after the head of the spider is burned, there are at least l_n rounds left. Hence, the clause is fulfilled. ∎

Lemma 8. *Let $m \geq 4$. Suppose T is a disjoint union of a 2-spider and an independent path of order two such that the shortest arm of the spider of T has length at least two and $|T| = m^2-1$. Then there exists a burning sequence with m burning sources such that after the head of the spider of T is burned, there are* *at least two rounds left unless $m = 4$ and $T = (6, 6 \mid 2)$.*

Proof. To avoid repetitiveness, in the proof, we say that T is properly m-burnable if T is m-burnable in such a way that after the head of the spider is burned, there are at least two rounds left. We argue by induction. For the base case $m = 4$, there are only a few possibilities for T, namely

$$(10, 2 \mid 2), (9, 3 \mid 2), (8, 4 \mid 2), (7, 5 \mid 2), (6, 6 \mid 2).$$

It is easy to verify that the first four are properly 4-burnable and the last one is not properly 4-burnable. For the induction step, suppose $m \geq 5$ and T is as given. Then l_1 is at least $2m + 1$, and so unless $m = 5$ and T is $(15, 6 \mid 2)$, we can use the first burning source to burn the last $2m - 1$ vertices from the first arm of T and we can apply the induction hypothesis to see that T' formed by the remaining unburned vertices is properly $(m-1)$-burnable, and thus T is properly m-burnable. Hence, it remains to verify that $(15, 6 \mid 2)$ is properly 5-burnable by placing the first burning source adjacent to the head on the shortest arm. ■

Lemma 9. *Let $n \geq 3$ and $m \geq n + 3$. Suppose T is a disjoint union of an n-spider with arm lengths $l_1 \geq l_2 \geq \cdots \geq l_n$ and a path. If $|T| \leq m^2 + n - 2$ and $l_3 < m$, then T is m-burnable with a burning sequence such that after the head of the spider of T is burned, there are at least $\min\{m - 3, l_n\}$ rounds left unless $|T| = m^2 + n - 2$ with the independent path of T having order two, and $l_3 = \cdots = l_n = 1$.*

Proof. We may suppose $|T| = m^2 + n - 2$ and $l_3 < m$. By Lemma 7, it suffices to consider the case $l_2 \geq m$. Now, we place the first burning source at the head of the spider of T. Consider the remaining unburned vertices by the first burning source in m rounds, which form a 3-path forest T'.

Note that $|T'| \leq m^2 + n - 2 - (2m - 1 + l_3 + n - 3) = (m-1)^2 + 1 - l_3$. Then T' is $(m-1)$-burnable by Theorem 6 if $|T'| \leq (m-1)^2 + 1 - l_3 \leq (m-1)^2 - 1 - t_{T'}$ and $T' \notin \mathcal{T}_{3,m-1}$. If $t_{T'} = 3$, then T' is the path forest consisting of three paths, each of order two. This path forest T' is $(m-1)$-burnable, as $m - 1 \geq 4$. Therefore, for the remaining cases, we only need to consider $t_{T'} \leq 2$.

Suppose $l_3 \geq 4$, T' is $(m-1)$-burnable by Theorem 6 unless $t_{T'} = 2$ and $l_3 = 4$. Hence, to burn the T' with $t_{T'} = 2$ and $l_3 = 4$, we relocate the first burning source of T adjacent to the head on the second arm. Clearly, all arms beside the first and the second arms are completely burned by x_1 since $m \geq 6$. Thus, the remaining unburned vertices by the first burning source in m rounds form T'', which is a 3-path forest. T'' is $(m-1)$-burnable by Theorem 6 since the shortest path is of order one ($t_{T''} = 0$), $|T''| \leq (m-1)^2 - 3$ and $T'' \notin \mathcal{T}_{3,m-1}$.

Consider $l_3 \in \{2, 3\}$ and we suppose $m \leq l_2 \leq m + 1$. Now, the first burning source of T is relocated to the $(l_2 - m + 1)$-th vertex on the second arm. Note that all arms beside the first arm are completely burned by x_1 since $m \geq 6$. Then the remaining unburned vertices by the first burning source in m rounds form a 2-path forest T''. Note that $|T''| \leq m^2+n-2-(2m-1+2+n-3) = (m-1)^2-1$. By Theorem 2, T'' is $(m-1)$-burnable.

Next, we consider the case $l_2 \geq m+2$. This implies $t_{T'} \leq 1$. Suppose $l_3 = 3$. By Theorem 6, T' is $(m-1)$-burnable unless $|T'| = (m-1)^2 - 1 - t_{T'} = (m-1)^2 + 1 - l_3$ and $T' \in \mathcal{T}_{3,m-1}$. This case occurs only if $t_{T'} = 1$, which implies that the independent path is of order two. In this case, by Theorem 6, T' is $(m-1)$-burnable unless $l_2 - (m-1) = 3$. To burn T', we relocate the first burning source of T two vertices away from the head on the second arm. Clearly, the second path of T' is a path of order one, and the third arm onwards are completely burned by x_1 since $m \geq 6$. Then the new subgraph T'' is $(m-1)$-burnable by Theorem 6.

Suppose $l_3 = 2$ and $l_2 \geq m+2$. We further suppose the independent path of T is of order two. Consider the 2-path forest formed by deleting all arms with length at most two of T, say T^* with $|T^*| \leq m^2 + n - 2 - 2 - (n-3) = m^2 - 1$. By Lemma 8, T^* is m-burnable with a burning sequence such that after the head of the spider of T^* is burned, there are at least two rounds left. It follows that T is m-burnable. Now, we suppose the independent path is not of order two. Then $t_{T'} = 0$ since the order of the shortest path is never two as $l_2 \geq m+2$. Thus T' is $(m-1)$-burnable by Theorem 6 unless T' is either $((m-1)^2 - 11, 5, 5)$ or $((m-1)^2 - 7, 3, 3)$. We consider their respective T and relocate the first burning source to a distance of two away from the head on the second arm. The unburned vertices by the first burning source in m rounds form the path forests, either $(7, 3; (m-1)^2 - 11)$, $((m-1)^2 - 9, 3; 5)$, $(5, 1; (m-1)^2 - 7)$ or $((m-1)^2 - 5, 1; 3)$. Clearly, these path forests are $(m-1)$-burnable.

Suppose $l_3 = 1$. We consider the 2-path forest formed by deleting all length-one arms of T. By Theorem 2, it is m-burnable with a burning sequence φ unless the independent path of T has order two. This is the exceptional case as given in the lemma. By using φ, if the head of T is not burned in the last round, T is m-burnable. If the head of T is burned in the last round in φ, then we can easily modify the burning sequence φ so that the head is not burned in the last round.

Hence, our proof is complete. ■

Lemma 10. *Let $n \geq 3$ and $m = n+3$. Suppose T is a disjoint union of an n-spider with arm lengths $l_1 \geq l_2 \geq \cdots \geq l_n$ and a path of order l such that $l_n \geq m+2$. If $|T| \leq m^2+n-2$, then T is m-burnable. Furthermore, the graph T can be burned in m rounds such that after the head is burned, there are at least $m-3$ rounds left.*

Proof. We may suppose $|T| = m^2 + n - 2$. We put the first burning source at the head of the spider. Then the remaining unburned vertices by the first burning source in m rounds form a path forest T' with path orders $l'_1 \geq l'_2 \geq \cdots \geq l'_n$ and l, where $l'_i = l_i - (m-1)$ for $1 \leq i \leq n$. Let L' denote the length of the longest path of T'. Clearly, $L' = \max\{l'_1, l\}$ and $l'_n \geq 3$. Since $|T'| = (n+3)^2 + n - 2 - n(n+2) - 1 = 5n + 6$, it follows that $L' \geq 6$.

Case 1. $6 \leq L' \leq 2n+3$. The second burning source is placed at the longest path of T'. Hence, the longest arm is completely burned in m rounds. The remaining unburned vertices form an n-path forest T'' of order at most $5n$. It suffices to show that T'' is $(n+1)$-burnable.

Suppose $l \neq 2$. Notice that the shortest path of T'' cannot have length two (thus $t_{T''} = 0$). In this case, T'' is $(n+1)$-burnable by Theorem 6 unless $|T''| = 5n$ and $T'' \in \mathcal{T}_{n,n+1}$. This implies that $L' = 6$. The only possible T'' is an n-path forest with each path of order 5, which is $(5, 5, \ldots, 5, 5)$. The path forest $(2n+3, 3, 3, \ldots, 3, 3) \in \mathcal{T}_{n,n+1}$ is not a possible T'' because $2n+3$ is at least 9 which contradicts $L' = 6$.

When $T'' = (5, 5, \ldots, 5, 5)$, it implies that T' is either $(6, 5, \ldots, 5; 5)$ or $(5, 5, \cdot, 5; 6)$. This leads to $T = (n+8, n+7, \cdot, n+7 \mid 5)$ or $T = (n+7, n+7, \ldots, n+7 \mid 6)$, respectively. Both T can be burned by relocating x_1 to the vertex at a distance of two away from the head on the shortest arm. The remaining unburned vertices by the newly relocated x'_1 in m rounds forms an $(n+1)$-path forest with order $(8, 7, \ldots, 7, 3; 5)$ (respectively $(7, \ldots, 7, 3; 6)$) which is clearly $(n+2)$-burnable.

Suppose $l = 2$ and we further suppose $L' \geq 7$. Then T'' is an n-path forest with order at most $5n - 1$ and T'' is $(n+1)$-burnable by Theorem 6 unless $T'' = \{2n+3, 3, \ldots, 3; 2\}$ when $L' = 7$. However, this is not a possible T'' since $2n+3 \geq 9$, which contradicts $L' = 7$.

Lastly, we suppose $L' = 6$. This implies $T' = (6, l'_2, l'_3, \ldots, l'_n; 2)$ where $l'_i \leq 6$ for $2 \leq i \leq n$ and $T = (n+8, l_2, l_3, \ldots, l_n \mid 2)$ with $l_n \leq l_{n-1} \leq \cdots \leq l_2 \leq n+8$. Now, to completely burn T, we relocate x_1 adjacent to the head on the shortest arm. Then the remaining unburned vertices by the newly relocated x'_1 in m rounds forms an $(n+1)$-path forest with order $(l''_1, l''_2, \ldots, l''_n; 2)$ with $l''_1 = 7$, $l''_n \leq 5$ and $l''_i \leq 7$ for $2 \leq i \leq n-1$ is $(n+2)$-burnable.

Case 2. $L' \geq 2n+4$. First, we suppose $L' = 2n+4$. Let $2n+4 = l^*_1 \geq l^*_2 \geq l^*_3 \geq \cdots \geq l^*_{n+1}$ denote the path orders of T'. Since $l^*_2 + l^*_3 + \cdots + l^*_{n+1} \leq 3n+2$, it can be deduced that $l^*_{n+1} \leq 3$ and $l^*_n \leq 5$ (the latter holds as $n \geq 3$). Now, as $l'_n = l_n - (m-1) \geq 3$, it implies that $l^*_n \geq 3$ and thus $l^*_2 \leq 7$ (in fact, $l^*_2 = 7$ is possible only when $n = 3$). It is now clear that T' is $(n+2)$-burnable as $2n+4 = (2n+3)+1$.

Next, we suppose $L' \geq 2n+5$. Consider the independent path has order one. When $2n+5 \leq L' = l'_1 \leq 2n+6$. Note that $l'_2 + l'_3 + \cdots + l'_n \leq 3n$ and $l'_n \geq 3$. It is easy to verify that T' is $(n+2)$-burnable since $l'_1 \leq (2n+3)+3$. Suppose

$L' \geq 2n+7$. Then, T' must be $(2n+8, 3, 3, \ldots, 3; 1)$ or $(2n+7, 4, 3, \ldots, 3; 1)$ which are clearly $(n+2)$-burnable.

Next, we consider the case where the independent path has at least order two. Then T' is one of the following:

$$(2n+7, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 2),\ (\underbrace{3,\ldots,3}_{n \text{ times}}; 2n+6),\ (2n+6, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 3),\ (2n+6, 4, \underbrace{3,\ldots,3}_{n-2 \text{ times}}; 2),$$

$$(4, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 2n+5),\ (2n+5, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 4),\ (2n+5, 4, \underbrace{3,\ldots,3}_{n-2 \text{ times}}; 3),$$

$$(2n+5, 4, 4, \underbrace{3,\ldots,3}_{n-3 \text{ times}}; 2),\ (2n+5, 5, \underbrace{3,\ldots,3}_{n-2 \text{ times}}; 2).$$

For the first five cases:
$(2n+7, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 2)$, $(\underbrace{3,\ldots,3}_{n \text{ times}}; 2n+6)$, $(2n+6, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 3)$, $(2n+6, 4, \underbrace{3,\ldots,3}_{n-2 \text{ times}}; 2)$, and $(4, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 2n+5)$, we relocate x_1 distance two away from the head on the shortest arm. Then the remaining unburned vertices by the first burning source in m rounds forms the path forest $(2n+9, \underbrace{5,\ldots,5}_{n-2 \text{ times}}, 1; 2)$, $(\underbrace{5,\ldots,5}_{n-1 \text{ times}}, 1; 2n+6)$, $(2n+8, \underbrace{5,\ldots,5}_{n-2 \text{ times}}, 1; 3)$, $(2n+8, 6, \underbrace{5,\ldots,5}_{n-3 \text{ times}}, 1; 2)$, and $(6, \underbrace{5,\ldots,5}_{n-2 \text{ times}}, 1; 2n+5)$ respectively. It can be easily verified that these path forests are $(n+2)$-burnable.

For the remaining cases:
$(2n+5, \underbrace{3,\ldots,3}_{n-1 \text{ times}}; 4)$, $(2n+5, 4, \underbrace{3,\ldots,3}_{n-2 \text{ times}}; 3)$, $(2n+5, 4, 4, \underbrace{3,\ldots,3}_{n-3 \text{ times}}; 2)$, and the last one $(2n+5, 5, \underbrace{3,\ldots,3}_{n-2 \text{ times}}; 2)$, we consider their corresponding T. In each T, the first burning source x_1 is placed at the vertex distance two away from the head on the longest arm. Now, our remaining path forests are $(2n+3, \underbrace{5,\ldots,5}_{n-1 \text{ times}}; 4)$, $(2n+3, 6, \underbrace{5,\ldots,5}_{n-2 \text{ times}}; 3)$, $(2n+3, 6, 6, \underbrace{5,\ldots,5}_{n-3 \text{ times}}; 2)$, and $(2n+3, 7, \underbrace{5,\ldots,5}_{n-2 \text{ times}}; 2)$ respectively which are clearly $(n+2)$-burnable.

Hence, this completes the proof. ■

Lemma 11. *Let $n \geq 3$. Suppose T is a disjoint union of an n-spider with arm lengths $5n+7, n+3, \ldots, n+3$ and a path of order two. Then T is not $(n+3)$-burnable.*

Proof. Assume there is a burning sequence of T in $n+3$ rounds. Let i be the smallest integer such that the head of the spider of T is within the associated neighbourhood of x_i. Note that if $i \neq 1$, then we may suppose x_1 is put on the longest arm of the spider. Consider the path forest T' formed by removing the vertices belonging to the associated neighbourhood of x_i. Clearly, we may assume $i \neq n+3$ since we can easily modify the burning sequence such that x_{n+3} is not placed at the head.

Suppose the shortest arm of the spider is not completely burned by x_i. Then T' is an $(n+1)$-path forest. Consider the case where the shortest path of T' has order more than one. Note that the order of the longest path of T' is at least $5n+7-(2n+6-2i) = 3n+1+2i$, as x_i burns at most $2n+6-2i$ vertices from the longest arm. As there are only $n+2$ remaining burning sources but $n+1$ paths, the longest path of T' can only be burned by a single burning source x_j for some $j \in \{1, 2, \dots, n+2\} \backslash \{i\}$ and possibly together with x_{n+3}, which gives a contradiction because $(2n+7-2j)+1 < 3n+1+2i$ regardless of i.

Next, we consider the case where the shortest path of T' is of order one. Hence, x_i is placed at the $(i-1)$-th vertex on the shortest arm of the spider. Suppose $i \neq 1$. Without loss of generality, we may assume that x_1 is the burning source closest to the head along the longest arm, since the burning sources can be easily rearranged along a burned path and the path remains burned. Therefore, by some rearrangement of x_1 and x_i, it is clear that we have a refined burning sequence where x_1 is put at the head of the spider (see Figure 2). Hence, we may suppose $i = 1$. However, when $i = 1$, $T' = (4n+5, \underbrace{1, \dots, 1}_{n-1 \text{ times}}; 2)$, but it is not $(n+2)$-burnable as $4n+5 > (2n+3) + (2n+1)$.

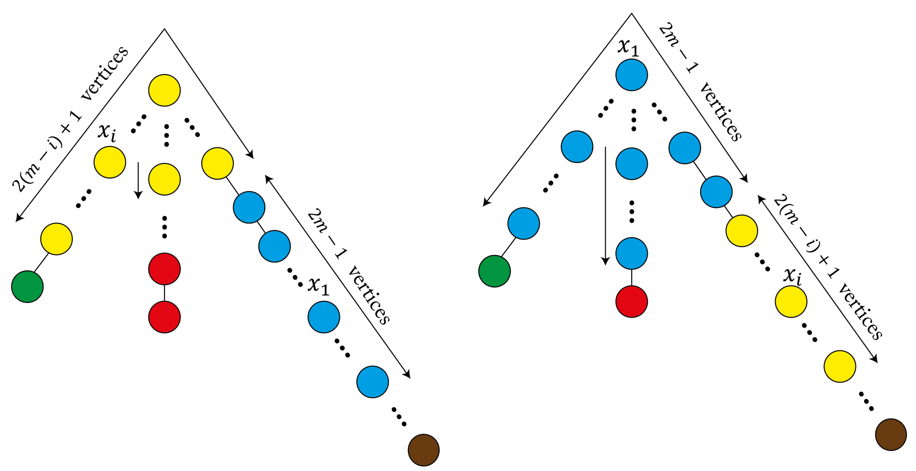

Figure 2. An illustration of rearrangement of x_1 and x_i.

Finally, suppose the shortest arm of the spider is completely burned by x_i. We may suppose x_i is placed at the i-th vertex on the shortest arm of the spider. If $i \neq 1$, by a similar rearrangement of x_1 and x_i as in Figure 2, it is clear that we have a refined burning sequence where x_1 is put at the first vertex on the shortest arm. Hence, we may suppose $i = 1$. However, when $i = 1$, $T' = (4n+6, \underbrace{2, \ldots, 2}_{n-2 \text{ times}}; 2)$, but it is not $(n+2)$-burnable as $4n+6 > (2n+3) + (2n+1) + 1$. ■

Lemma 12. *Let $n \geq 3$ and $m \geq n+3$. Suppose T is a disjoint union of an n-spider with arm lengths $l_1 \geq l_2 \geq \cdots \geq l_n$ and a path. If $|T| \leq m^2 + n - 2$ and $l_n \geq m$, then T is m-burnable in such a way that after the head is burned, there are at least $m-3$ rounds left unless $|T| = m^2 + n - 2$ with $m = n+3$, the independent path of T has order two, and the arm lengths of the spider are $5n+7, n+3, \ldots, n+3$.*

Proof. First, we suppose $l_n = m$. Then the first burning source is placed adjacent to the head on the shortest arm of the spider so that the shortest arm is completely burned in m rounds. The remaining unburned vertices by the first burning source in m rounds is an n-path forest, say T'. Therefore, T' has order at most $|T'| \leq m^2 + n - 2 - (1 + (n-1)(m-2) + m) = m^2 + 3n - mn - 5$. Note that

$$\begin{aligned}
&\big[m^2 + 3n - mn - 5\big] - \big[(m-1)^2 - (n-2)(n-1) + 1 - t_{T'}\big] \\
&= m(2-n) + n^2 - 5 + t_{T'} \\
&\leq (n+3)(2-n) + n^2 - 5 + t_{T'} \quad \text{(because } m \geq n+3 \text{ and } n \geq 3\text{)} \\
&= -n + 1 + t_{T'}.
\end{aligned}$$

If $t_{T'} = n$, then T' is clearly $(m-1)$-burnable since $m-1 > n+1$. Hence, we may assume $t_{T'} \leq n-1$. By Theorem 6, T' is $(m-1)$-burnable, unless $m = n+3$, $t_{T'} = n-1$, and T' is the path forest with path orders $4n+6, 2, 2, \ldots, 2$ (with $n-1$ paths of order 2). The possible T is either $(5n+7, n+3, \ldots, n+3 \mid 2)$ or $(n+3, n+3, \ldots, n+3 \mid 4n+6)$. The first case is the exceptional case given in this lemma, and it is clearly not m-burnable by Lemma 11. For the second case, x_1 is used to burn the independent path while x_2 is placed at the vertex adjacent to the head of the spider, resulting in $T'' = (\underbrace{3, \ldots, 3}_{n-1 \text{ times}}, 1; 2n+1)$. T'' can be verified that it is $(n+1)$-burnable.

Next, we suppose $l_n = m+1$. Then the first burning source x_1 is placed at the vertex at distance two away from the head on the shortest arm of the spider of T. Note that T' has n paths with $t_{T'} \leq 1$. Since

$$\begin{aligned}|T'| &\leq m^2 + n - 2 - (1 + (n-1)(m-3) + m + 1)\\ &= m^2 + 4n - mn - 7\\ &= m(m-n) + 3(n-2) + n - 1\\ &\leq m(m-n) + (m-n)(n-2) + n - 1\\ &= (m-n)(m+n-2) + n - 1\\ &= (m-1)^2 - (n-1)^2 + n - 1\\ &= (m-1)^2 - (n-1)(n-2).\end{aligned}$$

By Theorem 6, T' is $(m-1)$-burnable unless $t_{T'} = 1$ and $T' \in \mathcal{T}_{n,m-1}$. However, it can be verified that $T' \notin \mathcal{T}_{n,m-1}$ since $l_{n-1} - (m-3) \geq 4$.

Lastly, we consider $l_n \geq m+2$. If $m = n+3$, we are done by Lemma 10. Now, we consider putting the first burning source at the head of the spider. Hence, the remaining unburned vertices by the first burning source in m rounds form a path forest T'. Note that $|T'| \leq m^2 + n - 2 - (1 + n(m-1)) = m^2 + 2n - nm - 3$ where T' consist of $n+1$ paths and $t_{T'} \leq 1$ since $l_n - (m-1) \geq 3$.

Case 1. $m \geq n+5$. It can be verified that

$$\begin{aligned}&[m^2 + 2n - nm - 3] - [(m-1)^2 - n(n-1) + 1 - t_{T'}]\\ &= m(2-n) + n^2 + n - 5 + t_{T'}\\ &\leq (n+5)(2-n) + n^2 + n - 5 + t_{T'}\\ &= -2n + 5 + t_{T'}.\end{aligned}$$

Note that $-2n + 5 + t_{T'}$ is non-positive integer since $t_{T'} \leq 1$. By Theorem 6, T' is $(m-1)$-burnable, unless $n = 3$, $m = 8$, $t_{T'} = 1$, and T' is the path forest $(35, 3, 3, 2)$. This implies that the exceptional T is $(42, 10, 10 \mid 2)$. For this T, we put the first burning source at distance two away from the head on the shortest arm. The remaining unburned vertices by the first burning source in m rounds form the path forest $(37, 5, 1; 2)$, and this is 7-burnable.

Case 2. $m = n+4$. Note that

$$\begin{aligned}&[m^2 + 2n - nm - 3] - [(m-1)^2 - n(n-1) + 1 - t_{T'}]\\ &= (n+4)(2-n) + n^2 + n - 5 + t_{T'} = -n + 3 + t_{T'}.\end{aligned}$$

Then T' is $(m-1)$-burnable by Theorem 6 unless $t_{T'} = n-3$ and $T' \in \mathcal{T}_{n+1,m-1}$ or $-n + 3 + t_{T'} > 0$. Since $t_{T'} \in \{0, 1\}$, then the following are the only two possible cases when $t_{T'} = n-3$ and $T' \in \mathcal{T}_{n+1,m-1}$:

- $t_{T'} = 0$, $n = 3$, $m = 7$, and T' is either $(22, 3, 3, 3)$ or $(16, 5, 5, 5)$.
- $t_{T'} = 1$, $n = 4$, and $m = 8$ and T' is the path forest $(26, 3, 3, 3, 2)$.

The possible original T for the case above are $(28, 9, 9 \mid 3)$, $(9, 9, 9 \mid 22)$, $(22, 11, 11 \mid 5)$, $(11, 11, 11 \mid 16)$ and $(33, 10, 10, 10 \mid 2)$. In each of these cases, we put the first burning source at distance two from the head on the shortest arm of the spider. The remaining unburned vertices form the path forest $(24, 5, 1; 3)$, $(5, 5, 1; 22)$, $(18, 7, 3; 5)$ and $(7, 7, 3; 16)$, respectively for the first four T and they can be verify that it is 6-burnable while the last one gives rise to the path forest $(28, 5, 5, 1; 2)$ which is clearly 7-burnable.

Suppose $-n+3+t_{T'} > 0$. The only possible case occurs when $t_{T'} = 1$, $n = 3$, and $m = 7$. Note that $|T'| \leq 31$, $l_n \geq m + 2 = 9$ and the independent path is of order two as $t_{T'} = 1$. We may assume $|T'| = 31$. Then $10 \leq l'_1 \leq 23$. If any path of T' has order either 10 or 11, then deleting this path results in a T'' with three paths and $|T''| \leq 20$. By Theorem 6, T'' is 5-burnable. Hence, we now suppose T' has no path of order 10 or 11, in particular $l'_1 \geq 12$. Consider the 4-path forest T'' obtained from T' by removing 11 vertices from the longest path, so $|T''| = 20$.

Let l''_1 denote the order of the longest path of T''. Note that $l''_1 \geq 6$ and T'' consists of at least one length two path and $0 \leq t_{T''} \leq 2$. Now, if $6 \leq l''_1 \leq 9$, then we consider the 3-path forest T''' obtained from T'' by removing the longest path of T''. It is sufficient to show that T''' is 4-burnable. Note that $|T'''| \leq 14$ and it consists of at most two paths of order two. Consider the case where $t_{T'''} \leq 1$. By Theorem 6, T''' is 4-burnable unless $l''_1 = 6$ and $T''' = (9, 3, 2)$. However, this exceptional case contradicts $l''_1 = 6$. Now, we suppose that $t_{T'''} = 2$. By Theorem 6, T''' is 4-burnable unless T''' is $(10, 2, 2)$ or $(9, 2, 2)$. Note that $t_{T'''} = 2$ occurs only if $l'_1 = 13$. This implies that T' is either $(13, 10, 6; 2)$ or $(13, 9, 7; 2)$, which is clearly 6-burnable. Hence, if $6 \leq l''_1 \leq 9$, then T'' is 5-burnable.

Now, we assume $l''_1 \geq 10$. Since T' has no path of order 10 or 11, T' is one of the following

$$(12, 12, 5; 2), (13, 12, 4; 2), (13, 13, 3; 2), (14, 12, 3; 2),$$

$$(21, 5, 3; 2), (21, 4, 4; 2), (22, 4, 3; 2), (23, 3, 3; 2).$$

It can be verified those path forests are 6-burnable unless T' are $(13, 13, 3; 2)$, $(22, 4, 3; 2)$ and $(23, 3, 3; 2)$. The original T for these path forests are $(19, 19, 9 \mid 2)$, $(28, 10, 9 \mid 2)$ and $(29, 9, 9 \mid 2)$, respectively. In each of these cases, we put the first burning source at distance two from the head on the shortest arm. The remaining unburned vertices form the path forests $(15, 15, 1; 2)$, $(24, 6, 1; 3)$ and $(25, 5, 1; 2)$. These path forests are clearly 6-burnable. ■

Theorem 13. *Let $n \geq 3$ and $m \geq n+3$. Suppose T is a disjoint graph consisting of an n-spider with arm lengths $l_1 \geq l_2 \geq \cdots \geq l_n$ and a path of order l. If $|T| \leq m^2 + n - 2$, then T is m-burnable unless $l = 2$ and*

1. *$l_1 + l_2 = m^2 - 3$ and $l_3 = l_4 = l_5 = \cdots = l_n = 1$; or*

2. *$m = n + 3$, $l_1 = 5n + 7$ and $l_2 = l_3 = l_4 = \cdots = l_n = n + 3$.*

Furthermore, there is a burning sequence such that after the head is burned, there are at least $\min\{m - 3, l_n\}$ *rounds left.*

Proof. Let k be the number of arms of the spider in T with length at least m. Note that $k \leq n$. By Lemma 9, if $l_3 < m$, then T is m-burnable unless T is an exceptional case given in the first item in this theorem. Hence, we may suppose $l_3 \geq m$.

Suppose $k = n$, then T is m-burnable by Lemma 12 unless T is as given in the second item in this theorem. Now, we suppose $k \leq n - 1$. Note that $3 \leq k \leq n - 1$ since $l_3 \geq m$. Let T' be the disjoint graph obtained from T by deleting its $(k + 1)$-th to n-th arms of T.

Note that $|T'| \leq m^2 + k - 2$ with $l_k \geq m$ and $m \geq k + 4$. By Lemma 12, T' is m-burnable. Furthermore, there exist a burning sequence φ of T' such that after the head is burned, there are at least $m-3$ rounds left. For the case $l_{k+1} \leq m-3$, T is clearly m-burnable by using the burning sequence φ.

Next, we suppose $m-2 \leq l_{k+1} \leq m-1$. We consider the first burning source of T placed at the head of the spider. The remaining unburned vertices by the first burning source in m rounds form a path forest denoted as T''. Note that T'' is an $(k + 1)$-path forest. We have

$$
\begin{aligned}
|T''| &\leq m^2 + n - 2 - (1 + k(m - 1)) - (m - 2) - (n - k - 1) \\
&= (m - 1)^2 - m(k - 1) + 2k - 1 \\
&\leq (m - 1)^2 - (k + 4)(k - 1) + 2k - 1 \qquad \text{(because } m \geq n + 3 \geq k + 4) \\
&= (m - 1)^2 - k(k - 1) + 1 - (2k - 2).
\end{aligned}
$$

For the case $t_{T''} \leq k$, T'' is $(m - 1)$-burnable by Theorem 6 since $2k - 2 > k$ for all $k \geq 3$. When $t_{T''} = k + 1$, T'' is an $(k + 1)$-path forest where each path is of order 2. T'' is clearly $(m - 1)$-burnable since $m - 1 \geq k + 2$. ■

4. CONCLUSION

In this paper, we explored the burning behaviour of the disjoint union T of a spider and a path, addressing the gap in the study of graph burning for disconnected graphs. We demonstrated that T satisfies the bound suggested by Conjecture 1 for $m \geq n + 3$ with only some exceptional cases listed in Theorem 13. However, there are a number of uncharacterized exceptional cases which are not m-burnable when $|T| \leq m^2 + n - 2$ for $m = n + 2$. For clear visualisation, we list down those exceptional cases for $n = 3$ in Proposition 14.

Proposition 14. *Suppose T is a disjoint graph consisting of a 3-spider and a path. If $|T| \leq 26$, then T is 5-burnable unless T is*

$$(13,6,4 \mid 2),(6,6,4 \mid 10),(13,5,5 \mid 2),(12,6,5 \mid 2),(12,5,5 \mid 2),$$

$$(12,5,5 \mid 3),(5,5,5 \mid 10),(11,6,6 \mid 2),(9,8,6 \mid 2),(7,7,6 \mid 5),$$

$$(8,8,7 \mid 2),(7,7,7 \mid 4),(l_1,l_2,1 \mid 2) \text{ where } l_1+l_2=22.$$

To further understand the influence of increasing n on the occurrence of exceptional cases, we extended our investigation to the case of $n = 4$ and $m = 4+2 = 6$ (see Proposition 15). This analysis aims to determine whether the number of exceptional cases decreases or if identifiable structural characteristics emerge as n grows.

Proposition 15. *Suppose T is a disjoint graph consisting of a 4-spider and a path. If $|T| \leq 38$, then T is 6-burnable unless T is*

$$(22,6,6,1 \mid 2),(16,7,7,5 \mid 2),(15,7,7,6 \mid 2),(15,7,6,6 \mid 3),(16,6,6,6 \mid 3),$$

$$(15,6,6,6 \mid 3),(15,6,6,6 \mid 4),(7,6,6,6 \mid 12),(15,8,6,6 \mid 2),(16,7,6,6 \mid 2),$$

$$(6,6,6,6 \mid 13),(6,6,6,6 \mid 12),(17,6,6,6 \mid 2),(16,6,6,6 \mid 2),(15,6,6,6 \mid 2),$$

$$(15,7,6,6 \mid 2),(8,8,8,7 \mid 6),(14,7,7,7 \mid 2),(9,8,8,7 \mid 5),(9,9,9,8 \mid 2),$$

$$(9,8,8,8 \mid 4),(8,8,8,8 \mid 5),(l_1,l_2,1,1 \mid 2) \text{ where } l_1+l_2=33.$$

Propositions 14 and 15 imply that there are more non $(n+2)$-burnable T where $|T| \leq (n+2)^2+n-2$ for $n = 4$ compared to that of $n = 3$. Notice that some of the exceptional cases contain a balanced spider (a spider with arms of the same length) but there are also cases where an uncharacterised graph emerges. We believe that exceptional cases of T increase as n is larger. We leave the investigation of larger values of n for future study when $m = n+2$.

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