Discussiones Mathematicae
Graph Theory xx (xxxx) 1–11
https://doi.org/10.7151/dmgt.2599

STAR-CRITICAL RAMSEY NUMBERS OF CYCLES REVISITED

Yang Sun and Yanbo Zhang[1]

School of Mathematical Sciences
Hebei Normal University
Shijiazhuang 050024, P.R. China

Hebei Research Center of the Basic Discipline Pure Mathematics
Shijiazhuang 050024, P.R. China

e-mail: ysun.edu@outlook.com, ybzhang@hebtu.edu.cn

Abstract

For integers $n \geq m \geq 3$, let $r_*(C_n, C_m)$ denote the star-critical Ramsey number for a cycle of length n versus a cycle of length m. The exact value of $r_*(C_n, C_m)$ was determined for $m = 4$ by Wu, Sun, and Radziszowski (*Wheel and star-critical Ramsey numbers for quadrilateral*, Discrete Appl. Math. 186 (2015) 260–271). Subsequently, Zhang, Broersma, and Chen (*On star-critical and upper size Ramsey numbers*, Discrete Appl. Math. 202 (2016) 174–180) established the exact value for all odd integers $m \geq 3$. However, the case of even $m \geq 6$ has remained open. In this paper, we determine the exact value of $r_*(C_n, C_m)$ for all even integers $m \geq 6$ and $n \geq \max\{3m/2 + 1, m + 6\}$, showing that

$$r_*(C_n, C_m) = \frac{m}{2} + 3.$$

Keywords: Ramsey number, star-critical Ramsey number, cycle.

2020 Mathematics Subject Classification: 05C55, 05D10.

1. Introduction

A red-blue edge-coloring of a graph G refers to an assignment of each edge of G with one of two colors: red or blue. Given two graphs G_1 and G_2, we write $G \to (G_1, G_2)$ to indicate that for every red-blue edge-coloring of G, there exists

[1]Corresponding author.

either a red copy of G_1 or a blue copy of G_2 as a subgraph. The *Ramsey number* $r(G_1, G_2)$ is defined as

$$r(G_1, G_2) = \min\{r \mid K_r \to (G_1, G_2)\}.$$

When $r = r(G_1, G_2)$, it is clear that $K_r \to (G_1, G_2)$, while $K_{r-1} \not\to (G_1, G_2)$. A more refined question is: if a graph G is a proper subgraph of K_r and contains K_{r-1} as a proper subgraph, does $G \to (G_1, G_2)$ necessarily hold? To address this, Hook [10] introduced the notion of the star-critical Ramsey number. The graph $K_{r-1} \sqcup K_{1,k}$ consists of a complete graph K_{r-1} together with an additional vertex that is adjacent to exactly k vertices of K_{r-1}. The *star-critical Ramsey number* $r_*(G_1, G_2)$ is defined as

$$r_*(G_1, G_2) = \min\{k \mid K_{r-1} \sqcup K_{1,k} \to (G_1, G_2), \text{where } r = r(G_1, G_2)\}.$$

Cycles are among the most extensively studied graph classes in Ramsey theory. The Ramsey numbers of cycles were first investigated by Bondy and Erdős [1]. Shortly thereafter, Rosta [14, 15] and Faudree and Schelp [9] independently determined the Ramsey numbers for all cycles. Their result is as follows

$$r(C_n, C_m) = \begin{cases} 2n-1 & \text{for } 3 \le m \le n,\ m \text{ odd},\ (m,n) \neq (3,3), \\ n-1+m/2 & \text{for } 4 \le m \le n,\ m \text{ and } n \text{ even},\ (m,n) \neq (4,4), \\ \max\{n-1+m/2, 2m-1\} & \text{for } 4 \le m < n,\ m \text{ even and } n \text{ odd}. \end{cases}$$

In addition, $r(C_3, C_3) = r(C_4, C_4) = 6$. For other results concerning Ramsey numbers of cycles, we refer the reader to the dynamic survey [13].

We now turn our attention to star-critical Ramsey numbers. For recent developments in this area, we refer to the monograph [4] and the survey [11]. For certain pairs of graphs (G_1, G_2), their star-critical Ramsey number exhibits a particularly simple form

$$r_*(G_1, G_2) = r(G_1, G_2) - 1.$$

That is, when $r = r(G_1, G_2)$, even the removal of a single edge e from K_r yields $K_r - e \not\to (G_1, G_2)$. Such a pair (G_1, G_2) is said to be *Ramsey-full* [17].

The pair (C_3, C_3) serves as a classical example of a Ramsey-full pair. This fact was first pointed out by Erdős, Faudree, Rousseau, Schelp [7], who attributed the result to Chvátal. Consider the complete graph with vertex set $\{v_i \mid i \in [5]\}$, where $v_1v_2v_3v_4v_5v_1$ forms a red 5-cycle, and $v_1v_3v_5v_2v_4v_1$ forms a blue 5-cycle. Then, add a copy v_1' of the vertex v_1, preserving the same adjacency and edge-coloring pattern as v_1. This yields a red-blue edge-colored graph $K_6 - e$ that contains no monochromatic C_3. Therefore, (C_3, C_3) is Ramsey-full. Using a similar construction, for any $m \geq 3$ and $n \geq 3$, the graph pair (K_m, K_n) is Ramsey-full.

Erdős and Faudree [8] further proved that the pair (C_4, C_4) is also Ramsey-full. They also raised the question of whether it is possible to characterize all Ramsey-full pairs or to find infinite families of Ramsey-full pairs.

When m is odd, $n \geq m \geq 3$, and $(m, n) \neq (3, 3)$, the second author of this paper, together with Broersma and Chen [17], established that

$$r_*(C_n, C_m) = n + 1.$$

When $n \geq m = 4$, Wu, Sun, and Radziszowski [16] determined the corresponding star-critical Ramsey number, obtaining

$$r_*(C_n, C_4) = 5.$$

When m is even and $n \geq m \geq 6$, the determination of $r_*(C_n, C_m)$ has remained open for the past decade. In this paper, we make progress on this problem by completely determining the corresponding star-critical Ramsey numbers under a slightly enlarged range of n.

Theorem 1. *For even $m \geq 6$ and $n \geq \max\{3m/2 + 1, m + 6\}$, we have*

$$r_*(C_n, C_m) = m/2 + 3.$$

At this point, the only unresolved cases for the star-critical Ramsey numbers of cycles are

$$\text{even } m \geq 6 \text{ and } m \leq n \leq \max\{3m/2, m + 5\}.$$

In this range, based on the results of Section 3, we have $r_*(C_n, C_m) \geq m/2 + 3$. Furthermore, when n is odd and $n \leq 3m/2$, it holds that $r(C_n, C_m) = 2m - 1$. In this situation, a stronger lower bound $r_*(C_n, C_m) \geq m + 1$ can be established. However, the methods employed in this paper are not sufficient to handle the remaining cases within this interval.

We now introduce additional notation and terminology. For a positive integer k, we use $[k]$ to denote the set $\{1, \ldots, k\}$. For a graph G, we write $|G|$ and $e(G)$ to denote its number of vertices and edges, respectively, and $\overline{G}$ for its complement. Given a vertex set $V_1 \subseteq V(G)$, we denote by $G[V_1]$ the subgraph induced by V_1. For a vertex $v \in V(G)$, let $d(v)$ denote its degree. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The lengths of the longest and shortest cycles in G are denoted by $c(G)$ and $g(G)$, respectively. A graph G is said to be *pancyclic* if it contains cycles of every length from 3 up to the order of a Hamiltonian cycle. It is called *weakly pancyclic* if it contains cycles of every length from $g(G)$ to $c(G)$. A *matching* in a graph G is a set of edges no two of which share a common endpoint. For a red-blue edge-colored graph H, its subgraph H_R is defined as the graph with the same vertex set as H and whose edge set consists of all red edges of H; likewise, the subgraph H_B has the same vertex set as H, and its edge set comprises all the blue edges of H.

In Section 2, we present eight lemmas required for establishing the upper bound in the proof of the main theorem. The lower and upper bounds of Theorem 1 will be proved in Sections 3 and 4, respectively.

2. Useful Lemmas

Lemma 2 (Rosta [14, 15] and Faudree and Schelp [9] independently)**.** *$r(C_n, C_{2\ell}) = n + \ell - 1$ for $n \geq 3\ell$ and $\ell \geq 2$.*

Lemma 3 (Brandt [3])**.** *Every nonbipartite graph G with more than $(|G|-1)^2/4+1$ edges is weakly pancyclic with $g(G) = 3$.*

Lemma 4 (Erdős and Gallai [6])**.** *Let G be a graph of order n and $3 \leq c \leq n$. If $e(G) > (n-1)(c-1)/2 + 1$, then $c(G) \geq c$.*

Lemma 5 (Dirac [5])**.** *Let G be a graph with at least three vertices. If $\delta(G) \geq |G|/2$, then G is Hamiltonian.*

Bondy and Chvátal [2] provided a result indicating that the sufficient condition $\delta(G) \geq |G|/2$ for a graph G to be Hamiltonian can be relaxed. The *closure* of a graph $G = (V, E)$ is defined as the graph obtained by recursively adding edges between non-adjacent vertex pairs whose degree sum is at least $|V|$, until no such pairs remain.

Lemma 6 (Bondy and Chvátal [2])**.** *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Lemma 7 (Jackson [12])**.** *Let $G = (X, Y)$ be a bipartite graph with partite sets X and Y such that $d(x) \geq k$ for all $x \in X$, where $|X| \geq 2$ and $2 \leq k \leq |Y| \leq 2k-2$. Then G contains all cycles on $2m$ vertices for $2 \leq m \leq \min\{|X|, k\}$.*

Lemma 8. *For any graph G with at least six vertices, either $e(G) > (|G| - 1)^2/4 + 1$ or $e(\overline{G}) > (|G| - 1)^2/4 + 1$.*

Proof. Assume for contradiction that the statement does not hold. Then

$$\frac{|G|(|G|-1)}{2} = e(G) + e(\overline{G}) \leq 2\left(\frac{(|G|-1)^2}{4} + 1\right).$$

This implies $|G| \leq 5$, a contradiction. ■

Lemma 9. *Let G be a graph with a longest cycle C_p, and let X denote the set of vertices not on C_p. For a vertex $x \in X$, let W be the set of its neighbors on C_p. Fix an orientation of C_p and let W^+ denote the set of successors of the vertices in W along the orientation. Then $W^+ \cup \{x\}$ forms an independent set, and each vertex in $X \setminus \{x\}$ is adjacent to at most one vertex in W^+.*

Proof. Given an orientation $\overrightarrow{C_p}$ of the cycle C_p, let x_i^+ denote the successor of x_i along this orientation. Suppose that x is adjacent to some $x_i^+ \in W^+$. Since x is also adjacent to $x_i \in W$, it is adjacent to two consecutive vertices on C_p. Replacing the edge $x_i x_i^+$ on C_p with the path $x_i x x_i^+$ yields a cycle C_{p+1}, contradicting the maximality of C_p. Thus, x is not adjacent to any vertex in W^+.

Suppose two vertices x_i^+ and x_j^+ in W^+ are adjacent. Then $x_i x x_j \overleftarrow{C_p} x_i^+ x_j^+ \overrightarrow{C_p} x_i$ forms a cycle of length $p+1$, again a contradiction. Hence, W^+ is an independent set.

Furthermore, suppose there exists a vertex $y \in X \setminus \{x\}$ that is adjacent to at least two vertices in W^+, say x_i^+ and x_j^+. Then the path $x_i x x_j \overleftarrow{C_p} x_i^+ y x_j^+ \overrightarrow{C_p} x_i$ forms a cycle of length $p+2$, which is again a contradiction. Therefore, each vertex in $X \setminus \{x\}$ is adjacent to at most one vertex in W^+. ∎

3. PROOF OF THE LOWER BOUND

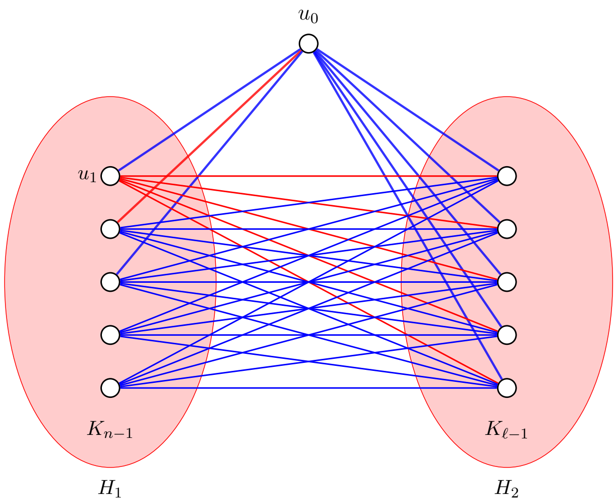

Figure 1. $K_{n+\ell-2} \sqcup K_{1,\ell+2}$ avoiding red C_n and blue $C_{2\ell}$.

In this section and the next, to avoid fractional expressions, we replace the even integer m in the theorem with 2ℓ. This substitution makes the arguments cleaner and more readable.

In this section, we establish the lower bound. We relax the conditions on n and ℓ, and prove that for $n \geq 4$, $\ell \geq 2$, and $n > \ell$, there exists a red-blue edge-coloring of the graph $K_{n+\ell-2} \sqcup K_{1,\ell+2}$ that contains neither a red copy of C_n nor a blue copy of $C_{2\ell}$. This coloring is illustrated in Figure 1 and described in detail below.

Let H_1 and H_2 be two disjoint red cliques on $n-1$ and $\ell-1$ vertices, respectively. Choose a vertex u_1 in H_1, and color all edges between u_1 and the vertices of H_2 red. For every vertex in $H_1 - u_1$, color all edges to the vertices in H_2 blue. It is straightforward to verify that the resulting coloring of the complete graph $K_{n+\ell-2}$ contains neither a red C_n nor a blue $C_{2\ell}$.

Next, we introduce an additional vertex u_0, and color all edges between u_0 and the vertices of H_2 blue. Also color the edge u_0u_1 blue, and between u_0 and the vertices of $H_1 - u_1$, include one blue edge and one red edge. This gives a red-blue edge-coloring of the graph $K_{n+\ell-2} \sqcup K_{1,\ell+2}$. Since u_0 is incident to only one red edge, it cannot lie on any red cycle C_n.

If u_0 is contained in a blue cycle $C_{2\ell}$, then such a cycle must include a blue edge from u_0 to $H_1 - u_1$ and another from u_0 to H_2. However, in the complete graph $K_{n+\ell-2}$, there is no blue path of even length connecting a vertex in H_1 to a vertex in H_2. Thus, u_0 cannot lie on any blue cycle $C_{2\ell}$.

Combining the above observations, we conclude that in this red-blue edge-coloring of $K_{n+\ell-2} \sqcup K_{1,\ell+2}$, there exists neither a red C_n nor a blue $C_{2\ell}$.

4. Proof of the Upper Bound

In this section, we establish the upper bound. Specifically, we prove that when $n \geq \max\{3\ell+1, 2\ell+6\}$ and $\ell \geq 3$, we have $K_{n+\ell-2} \sqcup K_{1,\ell+3} \to (C_n, C_{2\ell})$. Let G denote the graph $K_{n+\ell-2}$, and let u be the vertex joined to G by $\ell+3$ edges. We proceed by contradiction: suppose there exists a red-blue edge-coloring of $K_{n+\ell-2} \sqcup K_{1,\ell+3}$ that avoids both a red C_n and a blue $C_{2\ell}$.

Since $n-1 \geq 3\ell$, it follows from Lemma 2 that G contains a red cycle C_{n-1}. Denote by H the complete subgraph induced by $V(C_{n-1})$, and let $X = V(G) \setminus V(C_{n-1})$. Then $|X| = \ell - 1$. We now present a series of claims.

Claim 10. *G_R is not bipartite.*

Proof. Since $n \geq 3\ell+1$, we have $|G| = n+\ell-2 \geq 4\ell-1$. If G_R is bipartite, then by the pigeonhole principle, one of its partite sets would contain at least 2ℓ vertices. Thus, G would contain a blue $K_{2\ell}$ as a subgraph, which in turn contains a blue $C_{2\ell}$ as a subgraph, a contradiction. ∎

Claim 11. *Suppose $V(G)$ is partitioned into two sets A and B with $|A| = n-1$ and $|B| = \ell - 1$. If each vertex in B is incident to at most one red edge to A, then u is incident to at most two blue edges to A.*

Proof. Suppose not. Then there exist $a_1, a_2, a_3 \in A$ such that the edges ua_i are blue for each $i \in [3]$.

For any $b_1, b_2 \in B$, each of them is connected to at least two of $\{a_1, a_2, a_3\}$ by blue edges. Hence, there exists a matching of two blue edges between $\{b_1, b_2\}$ and $\{a_1, a_2, a_3\}$. Without loss of generality, assume a_1b_1 and a_2b_2 are blue edges.

In $A \setminus \{a_1, a_2, a_3\}$, there are at most $|B|$ vertices adjacent to B via red edges. The remaining vertices in $A \setminus \{a_1, a_2, a_3\}$ are at least

$$|A \setminus \{a_1, a_2, a_3\}| - |B| = (n-1-3) - (\ell - 1) \geq \ell + 3.$$

All edges between these vertices and B are blue. In other words, the induced bipartite subgraph between $A \setminus \{a_1, a_2, a_3\}$ and B contains a blue $K_{\ell+3,\ell-1}$ as a subgraph. It is easy to find within this subgraph a blue path of length $2\ell - 4$ with endpoints b_1 and b_2. Combined with the path $b_1a_1ua_2b_2$, this forms a blue cycle $C_{2\ell}$, leading to a contradiction. ■

Claim 12. *G_B is not a bipartite graph.*

Proof. We proceed by contradiction. Suppose that G_B is bipartite, and let its two partite sets be denoted by V_1 and V_2. Without loss of generality, assume $|V_1| \geq |V_2|$. Clearly, $|V_1| \leq n-1$, and hence $|V_2| \geq |G| - |V_1| \geq \ell - 1$.

If $|V_1| = n-1$, then $|V_2| = \ell - 1$. Since G_B is bipartite, $G[V_1]$ induces a red complete subgraph K_{n-1}. As G does not contain a red C_n, each vertex in $V_2 \cup \{u\}$ has at most one red edge to V_1. By Claim 11, the vertex u has at most two blue edges to V_1. Therefore, the total degree of u is at most $3 + |V_2| \leq \ell + 2$, which leads to a contradiction. This contradiction implies that $|V_1| \leq n-2$, and hence $|V_2| \geq \ell$.

We claim that there does not exist a matching of two red edges between V_1 and V_2. Otherwise, suppose that v_1v_2 and v_3v_4 are red edges with $v_1, v_3 \in V_1$ and $v_2, v_4 \in V_2$. In $G[V_1]$, there exists a red path of order $|V_1|$ with endpoints v_1 and v_3; similarly, in $G[V_2]$, there exists a red path of order $n - |V_1|$ with endpoints v_2 and v_4. The latter path exists since $|V_2| \geq n - |V_1| \geq 2$. These two red paths, together with the red edges v_1v_2 and v_3v_4, form a red cycle C_n, which is again a contradiction. Therefore, there is no matching of two red edges between V_1 and V_2. It follows that there exists a vertex v_0 in G_B such that all edges between $V_1 \setminus \{v_0\}$ and $V_2 \setminus \{v_0\}$ are blue. Note that for $i \in [2]$, if $v_0 \notin V_i$, then $V_i \setminus \{v_0\} = V_i$.

Since G has at least $4\ell - 1$ vertices and $|V_1| \geq |V_2|$, it follows that $V_1 \setminus \{v_0\}$ contains at least $2\ell - 1$ vertices. If $V_2 \setminus \{v_0\}$ contains at least ℓ vertices, then the graph contains a blue complete bipartite subgraph $K_{\ell,\ell}$, contradicting the assumption that G does not contain a blue $C_{2\ell}$. Therefore, $V_2 \setminus \{v_0\}$ has at most $\ell - 1$ vertices. Since $|V_2| \geq \ell$, we must have $|V_2| = \ell$ and $v_0 \in V_2$. In this case, $|V_1| = n - 2$.

If v_0 has at least two blue edges to V_1, let v_5 and v_6 be two such neighbors in V_1. Because every vertex in V_1 is connected to every vertex in $V_2 \setminus \{v_0\}$ via blue edges, one can easily find a blue path of length $2\ell - 2$ with endpoints v_5 and v_6. This blue path, together with $v_5 v_0 v_6$, forms a blue cycle $C_{2\ell}$, which yields a contradiction.

Therefore, we only need to consider the remaining case: v_0 has at most one blue edge to V_1, i.e., v_0 has at least $n - 3$ red edges to V_1. In this case, if u has at least two red edges to $V_1 \cup \{v_0\}$, then by Lemma 6, the subgraph $G[V_1 \cup \{v_0, u\}]$ contains a red Hamiltonian cycle C_n, which is a contradiction. Hence, the vertex u has at most one red edge to $V_1 \cup \{v_0\}$.

The sets $V_1 \cup \{v_0\}$ and $V_2 \setminus \{v_0\}$ correspond to the sets A and B, respectively, in Claim 11. By that claim, u has at most two blue edges to $V_1 \cup \{v_0\}$.

Consequently, the total degree of u is at most $1 + 2 + (|V_2| - 1) = \ell + 2$, again a contradiction. Therefore, G_B is not bipartite. ■

Claim 13. *The number of edges in G_B is at most $(|G| - 1)^2/4 + 1$.*

Proof. Suppose the claim is false. By Lemma 3 and Claim 12, the graph G_B is weakly pancyclic with girth $g(G_B) = 3$.

Since $|G| \geq n + \ell - 2 \geq 4\ell - 1$, it follows that

$$(|G| - 1)^2/4 + 1 > (|G| - 1)(2\ell - 1)/2.$$

By Lemma 4, we then have $c(G) \geq 2\ell$.

This implies that G_B contains a cycle $C_{2\ell}$ as a subgraph, a contradiction. ■

Claim 14. *The graph G_R contains no cycle of length at least n.*

Proof. By Lemmas 3 and 8, together with Claims 10 and 13, the graph G_R is weakly pancyclic with girth $g(G_R) = 3$.

If G_R contains a cycle of length at least n, then it contains a copy of C_n as a subgraph, which gives a contradiction. ■

Claim 15. *Each vertex in X has at most ℓ red edges to $V(H)$.*

Proof. Suppose not. Assume there exists a vertex $x \in X$ that has at least $\ell + 1$ neighbors in $V(H)$ via red edges. Let W be a set of $\ell + 1$ neighbors of x in $V(H)$ joined by red edges. For a orientation of the cycle C_{n-1}, let W^+ denote the set of successors of the vertices in W. By Lemma 9, the subgraph $G[W^+ \cup \{x\}]$ is a blue complete graph, and each vertex in $X \setminus \{x\}$ has at most one red edge to W^+.

Consider the graph $G[W^+ \cup X]$, which has 2ℓ vertices and each vertex is incident to at least ℓ blue edges. By Lemma 5, the subgraph $G[W^+ \cup X]$ contains a blue cycle $C_{2\ell}$. This contradiction completes the proof of the claim. ■

Claim 16. $\Delta(H_B) \leq \ell + 1$.

Proof. Suppose to the contrary that $\Delta(H_B) \geq \ell + 2$. That is, there exists a vertex v in H that is incident to at least $\ell + 2$ blue edges in H. We choose a set of $2\ell + 2$ vertices from $H - v$, among which at least $\ell + 2$ are neighbors of v via blue edges. Denote this set of $2\ell + 2$ vertices by Y.

Since $|Y| = 2\ell + 2$ and by Claim 15, each vertex in X has at least $\ell + 2$ blue edges to Y. Viewing $X \cup \{v\}$ as the set X in Lemma 7 and taking $k = \ell + 2$ in the lemma, it follows from Lemma 7 that there exists a blue cycle of length 2ℓ. This contradiction proves that $\Delta(H_B) \leq \ell + 1$. ■

Claim 17. *Each vertex in $X \cup \{u\}$ has at most one red edge to $V(H)$.*

Proof. Suppose not. Then there exists a vertex $x \in X \cup \{u\}$ with at least two red edges to $V(H)$.

By Claim 16, we have $\delta(H_R) \geq |H| - 1 - (\ell + 1) = n - \ell - 3$. Since $n \geq 2\ell + 6$, it follows that $\delta(H_R) \geq n/2$. By Lemma 6, the subgraph induced by $V(H) \cup \{x\}$ contains a red Hamiltonian cycle, contradicting the assumption that the graph contains no red C_n. ■

By Claims 11 and 17, the vertex u has at most two blue edges to $V(H)$. Together with Claim 17, it follows that u has at most three edges to $V(H)$ in total. Since u has at most $|X| = \ell - 1$ edges to X, the total degree of u is at most $\ell + 2$, yielding a final contradiction.

Acknowledgements

We appreciate the referees' careful reading and valuable suggestions. Yanbo Zhang (the corresponding author) was partially supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11601527 and by the Natural Science Foundation of Hebei Province under Grant No. A2023205045.

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Received 19 April 2025
Revised 30 July 2025
Accepted 31 July 2025
Available online 29 August 2025