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MONOCHROMATIC STARS AND MATCHINGS IN COMPLETE MULTIPARTITE GRAPHS

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Abstract

For graphs $G_1, \dots, G_l$ and G, let $G \to (G_1, \dots, G_l)$ denote that any l-coloring of $E(G)$ yields a monochromatic G_i in color i for some $i \in [l]$. Let $K_{1,n}$ be the star of order $n+1$, mK_2 be the matching of size m, and $K_{N_1,\dots,N_k}$ be the complete k-partite graph whose partite sets have sizes $N_1, \dots, N_{k-1}$ and N_k, respectively. In this paper, we prove that if $\sum_{l=1}^{k} N_l \geq \max\{2n+m-2, 2m\}$ and $\sum_{l=1}^{k} N_l - N_c \geq m$ for each $c \in [k]$, then $K_{N_1,\dots,N_k} \to (K_{1,n}, mK_2)$. Furthermore, we extend it to multicolors.

Keywords: bipartite Ramsey numbers, set and size multipartite Ramsey numbers, multipartite graphs.

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1. Introduction

In this paper, all graphs are simple. For a graph G, let $V(G)$ and $E(G)$ denote the vertex and the edge sets of G, respectively. The order of G is $|V(G)|$, and the size of G is $|E(G)|$. For a positive integer $l \geq 2$, let $G_1, \dots, G_l$ and G be graphs, and let $G \to (G_1, \dots, G_l)$ denote that any l-coloring of $E(G)$ yields a monochromatic copy of G_i in color i for some $i \in [l]$. The Ramsey number for $G_1, \dots, G_l$, $r(G_1, \dots, G_l)$, is the minimum integer N such that $K_N \to (G_1, \dots, G_l)$. For positive integers n and m, let $K_{1,n}$ be the star of order $n+1$, and let mK_2 be the matching of size m. For given positive integers $n_1, \dots, n_s, m_1, \dots, m_{t-1}$ and m_t, let $\sum = \sum_{i=1}^{s}(n_i - 1)$ and $\Lambda = \sum_{j=1}^{t}(m_j - 1)$. In 1972, Harary [8] determined the value of $r(K_{1,n_1}, K_{1,n_2})$. Burr and Roberts [3] extended it to multicolors by showing that

$$r(K_{1,n_1},\dots,K_{1,n_s}) = \begin{cases} \sum +1, & \text{if } \sum \text{ and some } n_i \text{ are even},\\ \sum +2, & \text{otherwise.} \end{cases}$$

In 1975, Cockayne and Lorimer [5] determined the Ramsey numbers for matchings by showing that $r(m_1K_2,\dots,m_tK_2) = \max_{1\le j\le t}\{m_j\} + \Lambda + 1$. Furthermore, they [4] determined the Ramsey numbers for stars versus a matching. In 2018, Omidi, Raeisi and Rahimi [10] determined the Ramsey numbers for stars versus matchings by showing that if $m_1 \ge \cdots \ge m_t$, then

$$\begin{aligned} &r(K_{1,n_1},\dots,K_{1,n_s}, m_1K_2,\dots,m_tK_2)\\ &= \begin{cases} r+\Lambda+1, & \text{if } m_1 < r \le 2m_2,\ \sum \text{ and some } n_i \text{ are even},\\ \max\{r-1, m_1\} + \Lambda + 1, & \text{otherwise,} \end{cases} \end{aligned}$$

where $r = r(K_{1,n_1},\dots,K_{1,n_s})$.

For a positive integer $k \ge 2$, let $N_1,\dots,N_k$ be positive integers and let $K_{N_1,\dots,N_k}$ be the complete k-partite graph whose partite sets have sizes $N_1,\dots,$ N_{k-1} and N_k, respectively. If $|N_1| = \cdots = |N_k| = t$, then simplify it as $K_{k\times t}$. In 1973, Gyárfás and Lehel [7], and Faudree and Schelp [6] introduced the bipartite Ramsey number for bipartite graphs $G_1,\dots,G_l$, $br(G_1,\dots,G_l)$, which is the minimum integer N such that $K_{2\times N} \to (G_1,\dots,G_l)$. In 2015, Raeisi [12] determined the bipartite Ramsey numbers for stars versus matchings by showing that

$$br(K_{1,n_1},\dots,K_{1,n_s}, m_1K_2,\dots,m_tK_2) = \begin{cases} \Lambda+1, & \text{if } \sum < \left\lfloor \frac{\Lambda+1}{2} \right\rfloor,\\ \sum + \left\lfloor \frac{\Lambda}{2} \right\rfloor + 1, & \text{otherwise.} \end{cases}$$

Let $a \ge 1$ and $b \ge 2$ be positive integers. In 2004, Burger and van Vuuren [1] introduced the set multipartite Ramsey number for $G_1,\dots,G_l$, $M_a(G_1,\dots,G_l)$, which is the minimum integer k such that $K_{k\times a} \to (G_1,\dots,G_l)$. They [2] also introduced the size multipartite Ramsey number for $G_1,\dots,G_l$, $m_b(G_1,\dots,G_l)$, which is the minimum integer N such that $K_{b\times N} \to (G_1,\dots,G_l)$. In 2016, Jayawardene and Samarasekara [9] determined the size multipartite Ramsey numbers for matching versus matching by showing that if $m_1 \ge m_2$, then

$$m_b(m_1K_2, m_2K_2) = \begin{cases} m_1+m_2-1, & \text{if } b=2,\\ \left\lceil \frac{2m_1+m_2-1}{b} \right\rceil, & \text{otherwise.} \end{cases}$$

In 2018, Perondi and Monte Carmelo [11] determined the set and the size multipartite Ramsey numbers for stars by showing that

$$M_a(K_{1,n_1},\dots,K_{1,n_s}) = \begin{cases} \frac{\sum}{a}+1, & \text{if } a \text{ is odd, } \frac{\sum}{a} \text{ and some } n_i \text{ are even,}\\ \left\lfloor \frac{\sum}{a} \right\rfloor + 2, & \text{otherwise.} \end{cases}$$

and

$$m_b(K_{1,n_1},\dots,K_{1,n_s}) = \begin{cases} \frac{\sum}{b-1}, & \text{if } b \text{ and } \frac{\sum}{b-1} \text{ are odd, and some } n_i \text{ is even,}\\ \left\lceil \frac{\sum+1}{b-1} \right\rceil, & \text{otherwise.} \end{cases}$$

In this paper, we prove the following theorems.

Theorem 1. *If* $\sum_{l=1}^{k} N_l \geq \max\{2n+m-2, 2m\}$ *and* $\sum_{l=1}^{k} N_l - N_c \geq m$ *for each* $c \in [k]$*, then* $K_{N_1,\dots,N_k} \to (K_{1,n}, mK_2)$.

Furthermore, we extend it to multicolors.

Theorem 2. *If* $\sum_{l=1}^{k} N_l \geq \max\{2\sum+\Lambda+1, 2\Lambda+2\}$ *and* $\sum_{l=1}^{k} N_l - N_c \geq \Lambda+1$ *for each* $c \in [k]$*, then* $K_{N_1,\dots,N_k} \to (K_{1,n_1},\dots,K_{1,n_s}, m_1K_2,\dots,m_tK_2)$.

Notations. For sets of vertices U and V satisfying $U \cap V = \emptyset$, let $E(U,V)$ be the set of all edges with one end vertex in U and the other in V. Denote the subgraph H of G by $H \subset G$. For a red-blue edge-colored graph G, let R and B be the induced subgraphs of G induced by the red and blue edges, respectively. Let $N_R(v)$ and $N_B(v)$ be the set of all vertices adjacent to v in R and B, respectively. Let $d_R(v) = |N_R(v)|$ and $d_B(v) = |N_B(v)|$.

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Lemma 3. *If* $N_1 + N_2 \geq \max\{2n+m-2, 2m\}$ *and* $N_i \geq m$ *for each* $i \in [2]$*, then* $K_{N_1,N_2} \to (K_{1,n}, mK_2)$.

Proof. Let $V(K_{N_1,N_2}) = V_1 \cup V_2$ with $|V_i| = N_i$ for each $i \in [2]$. Color $E(K_{N_1,N_2})$ arbitrarily by red and blue and let K be the resulting graph. Let $H = sK_2 = \{u_jv_j | u_j \in V_1, v_j \in V_2, j \in [s]\}$ be a maximum blue matching of K, and assume that $s \leq m-1$ (otherwise, there is a blue copy of mK_2 in K, and we are done). Moreover, let $U_i = V_i \setminus V(H)$ for each $i \in [2]$. Note that $U_i \neq \emptyset$ since $N_i \geq m$ and $s \leq m-1$. Let U_1' be the set of all u_j such that $E(\{u_j\}, U_2)$ is red, and let U_2' be the set of all v_j such that $E(\{v_j\}, U_1)$ is red. For each $j \in [s]$, if there are vertices $w \in U_1$ and $w' \in U_2$ such that v_jw and u_jw' are blue, then there is a blue copy of $(s+1)K_2$ (replace u_jv_j with v_jw and u_jw'). It is a contradiction since H is a maximum blue matching. Consequently, $E(\{u_j\}, U_2)$ or $E(\{v_j\}, U_1)$ is red, and thus $|U_1'| + |U_2'| \geq s$. Note that $E(U_1, U_2)$ is red since H is a maximum blue matching. Now, there are vertices $u \in U_1$ and $v \in U_2$ such that $d_R(u) \geq |U_2| + |U_2'|$ and $d_R(v) \geq |U_1| + |U_1'|$. Note that

$$\begin{aligned} d_R(u) + d_R(v) &\geq |U_2| + |U_2'| + |U_1| + |U_1'| \\ &\geq N_1 + N_2 - 2s + s \\ &\geq 2n + m - 2 - s \geq 2n - 1. \end{aligned}$$

By the pigeonhole principle, $d_R(u) \geq n$ or $d_R(v) \geq n$ and so, K contains a red copy of $K_{1,n}$, completing the proof. ■

Remark 4. The lower bounds presented in Lemma 3 are the best. Indeed, if $N_1 = m - 1$, or $N_2 = m - 1$, or $N_1 + N_2 = 2m - 1$, then a blue copy of K_{N_1,N_2} contains neither a red copy of $K_{1,n}$ nor a blue copy of mK_2. If $N_1 \geq m, N_2 \geq m$ and $N_1 + N_2 = 2n + m - 3$, then let $N_1 \geq n - 1$, and let $N_2 \geq n - 1$. In K_{N_1,N_2}, color a copy of $K_{n-1,n-1}$ in red and all other edges in blue. The resulting graph contains neither a red copy of $K_{1,n}$ nor a blue copy of mK_2.

Theorem 5. *If* $\sum_{l=1}^{3} N_l \geq \max\{2n + m - 2, 2m\}$ *and* $\sum_{l=1}^{3} N_l - N_c \geq m$ *for each* $c \in [3]$*, then* $K_{N_1,N_2,N_3} \to (K_{1,n}, mK_2)$.

Proof. Let $V(K_{N_1,N_2,N_3}) = V_1 \cup V_2 \cup V_3$. Color $E(K_{N_1,N_2,N_3})$ arbitrarily by red and blue and let K be the resulting graph. Let sK_2 be a maximum blue matching of K and denote it by S. Furthermore, we may assume that $s \leq m - 1$. In the sequel, the index of sets is considered in modulo 3, and we will consider four cases depending on the coverage relationships between S and all V_i.

Case 1. S covers V_1, V_2 and V_3. Then $2m \leq \sum_{l=1}^{3} N_l = |V(S)| = 2s \leq 2(m - 1)$. This is a contradiction.

Case 2. S covers precisely one of V_1, V_2 and V_3. Without loss of generality, assume that $V_3 \subset V(S)$. For each $k \in [3]$, let $S_k = \{ab \in E(S) | a \in V_{k+1}, b \in V_{k+2}\}$. Let $A_1 = V_2 \cap V(S_1), A_2 = V_3 \cap V(S_2), A_3 = V_2 \cap V(S_3), C_1 = V_3 \cap V(S_1), C_2 = V_1 \cap V(S_2), C_3 = V_1 \cap V(S_3), A = V_1 \setminus V(S)$, and $C = V_2 \setminus V(S)$. Note that $A \neq \emptyset$ and $C \neq \emptyset$.

For each $i \in [3]$, let $S_{i,A} = \{x \in A_i | E(\{x\}, A)$ is red$\}$ and $S_{i,C} = \{x \in C_i | E(\{x\}, C)$ is red$\}$. For each edge $u_i v_i \in S_i$, let $u_i \in A_i$ and $v_i \in C_i$. If there are vertices $w \in A$ and $w' \in C$ such that $u_i w$ and $v_i w'$ are blue, then there is a blue copy of $(s+1)K_2$ (replace $u_i v_i$ with $u_i w$ and $v_i w'$). It is a contradiction since S is a maximum blue matching. Consequently, $E(\{u_i\}, A)$ or $E(\{v_i\}, C)$ is red, and thus $|S_{i,A}| + |S_{i,C}| \geq |S_i|$. Note that $E(A, C)$ is red since S is a maximum blue matching.

Now, there are vertices $u \in A$ and $v \in C$ such that $d_R(u) \geq \sum_{j=1}^{3} |S_{j,A}| + |C|$ and $d_R(v) \geq \sum_{j'=1}^{3} |S_{j',C}| + |A|$. Note that

$$\begin{aligned}
d_R(u) + d_R(v) &\geq \sum_{j=1}^{3} |S_{j,A}| + |C| + \sum_{j'=1}^{3} |S_{j',C}| + |A| \\
&\geq |A| + |C| + \sum_{l=1}^{3} |S_l| = \sum_{l=1}^{3} N_l - 2s + s \\
&\geq 2n + m - 2 - s \geq 2n - 1.
\end{aligned}$$

By the pigeonhole principle, $d_R(u) \geq n$ or $d_R(v) \geq n$ and so, K contains a red copy of $K_{1,n}$, and we are done.

Case 3. S covers precisely two of V_1, V_2 and V_3. Without loss of generality, we may assume that $V_2 \cup V_3 \subset V(S)$. For each $k \in [3]$, let $S_k = \{ab \in E(S) | a \in V_{3-k}, b \in V_{2-k}\}$. For each $i \in [2]$, let $A_i = V_1 \cap V(S_i)$ and $C_i = V_{i+1} \cap V(S_i)$. Let $C = V_1 \setminus V(S)$. Note that $|C| \geq 2$ since $\sum_{l=1}^{3} N_l \geq 2m$ and $s \leq m - 1$.

For each $i \in [2]$, let $S_{i,S_3} = \{x \in A_i | E(\{x\}, V(S_3))$ is red$\}$ and $S_{i,C} = \{x \in C_i | E(\{x\}, C)$ is red$\}$. For each edge $u_i v_i \in S_i$, let $u_i \in C_i$ and $v_i \in A_i$. If there is a vertex $w \in C$ and an edge $w'w'' \in S_3$ such that $u_i w$ and $v_i w'$ are blue, then there is a blue copy of sK_2 (replace $u_i v_i$ and $w'w''$ with $u_i w$ and $v_i w'$), which covers only one of V_2 and V_3 since $|C| \geq 2$. Then K contains a red copy of $K_{1,n}$ by Case 2, and we are done. Consequently, we may assume that $E(\{u_i\}, C)$ or $E(\{v_i\}, V(S_3))$ is red, and thus $|S_{i,S_3}| + |S_{i,C}| \geq |S_i|$.

If there is a vertex $u' \in C$ and an edge $v'v'' \in S_3$ such that $u'v'$ is blue, then there is a blue copy of sK_2 (replace $v'v''$ with $u'v'$), which covers only one of V_2 and V_3 since $|C| \geq 2$. Then K contains a red copy of $K_{1,n}$ by Case 2, and we are done. Consequently, we may assume that $E(C, V(S_3))$ is red.

Now, there are vertices $u \in C$ and $v \in V(S_3)$ such that $d_R(u) \geq |V(S_3)| + \sum_{j=1}^{2} |S_{j,C}|$ and $d_R(v) \geq |C| + \sum_{j'=1}^{2} |S_{j',S_3}|$. Note that

$$\begin{aligned}
d_R(u) + d_R(v) &\geq |V(S_3)| + \sum_{j=1}^{2} |S_{j,C}| + |C| + \sum_{j'=1}^{2} |S_{j',S_3}| \\
&\geq |C| + |S_1| + |S_2| + 2|S_3| \\
&= \sum_{l=1}^{3} N_l - 2s + s + |S_3| \\
&\geq 2n + m - 2 - s \geq 2n - 1.
\end{aligned}$$

By the pigeonhole principle, $d_R(u) \geq n$ or $d_R(v) \geq n$ and so, K contains a red copy of $K_{1,n}$, and we are done.

Case 4. S covers none of the V_1, V_2 and V_3. For each $k \in [3]$, let $S_k = \{ab \in E(S) | a \in V_{k+1}, b \in V_{k+2}\}$. For each $k \in [3]$, let $A_k = V_{k+2} \cap V(S_k)$, $C_k = V_{k+1} \cap V(S_k)$, and $D_k = V_k - V(S)$. Note that for each $k \in [3]$, $D_k \neq \emptyset$.

For each $i \in [3]$, let $S_{i+1,A_i} = \{x \in A_i | E(\{x\}, D_{i+1})$ is red$\}$ and $S_{i+2,C_i} = \{x \in C_i | E(\{x\}, D_{i+2})$ is red$\}$. For each edge $u_iv_i \in S_i$, let $u_i \in A_i$ and $v_i \in C_i$. If there are vertices $w \in D_{i+1}$ and $w' \in D_{i+2}$ such that u_iw and v_iw' are blue, then there is a blue copy of $(s+1)K_2$ (replace u_iv_i with u_iw and v_iw'). It is a contradiction since S is a maximum blue matching. Consequently, $E(\{u_i\}, D_{i+1})$ or $E(\{v_i\}, D_{i+2})$ is red, and thus $|S_{i+1,A_i}| + |S_{i+2,C_i}| \geq |S_i|$.

For each $i \in [3]$, let $S'_i = \{uv \in S_i | N_B(u) \cap D_i \neq \emptyset, N_B(v) \cap D_i \neq \emptyset\}$. Assume that there is an edge $uv \in S'_i$ such that $|N_B(u) \cap D_i| \geq 2$ and $|N_B(v) \cap D_i| \geq 2$, or $(N_B(u) \cap D_i) \setminus (N_B(v) \cap D_i) \neq \emptyset$. Let $w \in N_B(u) \cap D_i$ and let $w' \in N_B(v) \cap D_i$ satisfy $w \neq w'$. Then there is a blue copy of $(s+1)K_2$ (replace uv with uw and vw'). It is a contradiction since S is a maximum blue matching. Consequently, for each edge $u_iv_i \in S'_i$, $|N_B(u_i) \cap D_i| = |N_B(v_i) \cap D_i| = 1$ and $N_B(u_i) \cap D_i = N_B(v_i) \cap D_i$. If $|N_B(x_i) \cap V(S'_i)| \geq 1$ for each vertex $x_i \in D_i$, then there is a blue copy of sK_2 (replace the blue edges in S'_i with the blue edges in $E(D_i, V(S'_i))$), which covers only V_i. Then K contains a red copy of $K_{1,n}$ by Case 2, and we are done. Consequently, we may assume that there is a vertex $u'_i \in D_i$ such that $E(\{u'_i\}, V(S'_i))$ is red. Note that for any $uv \in S_i \setminus S'_i$, $E(\{u\}, D_i)$ or $E(\{v\}, D_i)$ is red by the definition of $S_i \setminus S'_i$, and thus $|N_R(u'_i) \cap V(S_i)| \geq 2|S'_i| + |S_i| - |S'_i| = |S'_i| + |S_i|$. Furthermore, for each $i \in [3]$, $E(D_i, D_{i+1})$ is red since S is a maximum blue matching. Note that

$$
\begin{aligned}
\sum_{i=1}^{3} d_R(u'_i) &= \sum_{i=1}^{3} \left(|S_{i,A_{i-1}}| + |S_{i,C_{i-2}}| + |N_R(u'_i) \cap V(S_i)| + |D_{i+1}| + |D_{i+2}|\right) \\
&\geq \sum_{i=1}^{3} \left(|S_{i,A_{i-1}}| + |S_{i,C_{i-2}}| + |S'_i| + |S_i| + |D_{i+1}| + |D_{i+2}|\right) \\
&\geq 2\left(\sum_{i=1}^{3} |D_i| + \sum_{j=1}^{3} |S_j|\right) + \sum_{k=1}^{3} |S'_k| = 2\left(\sum_{i=1}^{3} N_i - 2s + s\right) + \sum_{k=1}^{3} |S'_k| \\
&\geq 2(2n + m - 2 - s) \geq 4n - 2 \geq 3n - 2.
\end{aligned}
$$

By the pigeonhole principle, $d_R(u'_1) \geq n, d_R(u'_2) \geq n$ or $d_R(u'_3) \geq n$ and so, K contains a red copy of $K_{1,n}$, and we are done.

All cases have been discussed, and we have finished the proof. ■

Now, we are ready to prove our main theorems.

Proof of Theorem 1. We may assume that $N_1 \geq \cdots \geq N_k$. Then the condition can be simplified as $\sum_{l=1}^{k} N_l \geq \max\{2n+m-2, 2m\}$ and $\sum_{l=2}^{k} N_l \geq m$. Now, we use induction on k to prove the theorem. Note that the assertion holds for $k = 2, 3$ by Lemma 3 and Theorem 5. Assume that the assertion holds for $k-1$ and $k \geq 4$. If $N_1 \geq m$, then we are done since $K_{N_1, \sum_{l=2}^{k} N_l} \to (K_{1,n}, mK_2)$ by Lemma 3 and $K_{N_1, \sum_{l=2}^{k} N_l} \subset K_{N_1,\ldots,N_k}$. Thus, we may assume that $N_1 \leq m-1$, and let $M = N_{k-1} + N_k$. If $M \leq N_1$, then we are done since $K_{N_1,\ldots,N_{k-2},M} \to (K_{1,n}, mK_2)$ by the induction hypothesis and $K_{N_1,\ldots,N_{k-2},M} \subset K_{N_1,\ldots,N_k}$. Consequently, we may assume that $M > N_1$. If $\sum_{l=1}^{k-2} N_l \geq m$, then we are done since $K_{M,N_1,\ldots,N_{k-2}} \to (K_{1,n}, mK_2)$ by the induction hypothesis and $K_{M,N_1,\ldots,N_{k-2}} \subset K_{N_1,\ldots,N_k}$. Thus, we may assume that $\sum_{l=1}^{k-2} N_l \leq m-1$. Note that $M = N_{k-1} + N_k \leq N_1 + N_2 \leq \sum_{l=1}^{k-2} N_l \leq m-1$. Then $2m \leq \sum_{l=1}^{k} N_l = \sum_{l=1}^{k-2} N_l + M \leq 2m-2$. This is a contradiction. ■

Remark 6. The lower bounds presented in Theorem 1 are the best for $2 \leq k \leq m+1$. The assertion holds for $k = 2$ by Remark 4. Assume that $k \geq 3$. Let $V(K_{N_1,\ldots,N_k}) = \bigcup_{l=1}^{k} V_l$ with $|V_l| = N_l$ for each $l \in [k]$. If there is $c_0 \in [k]$ such that $\sum_{l=1}^{k} N_l - N_{c_0} = m-1$, or $\sum_{l=1}^{k} N_l = 2m-1$, then a blue copy of $K_{N_1,\ldots,N_k}$ contains neither a red copy of $K_{1,n}$ nor a blue copy of mK_2. If $\sum_{l=1}^{k} N_l = 2n+m-3$ and $\sum_{l=1}^{k} N_l - N_c \geq m$ for each $c \in [k]$, then let $N_1 = N_2 = n-1$ and $\sum_{l=3}^{k} N_l = m-1$. Color $E(V_1, V_2)$ in red and all other edges in blue. The resulting graph contains neither a red copy of $K_{1,n}$ nor a blue copy of mK_2.

Proof of Theorem 2. Color $E(K_{N_1,\ldots,N_k})$ arbitrarily by $s+t$ colors. If the color of an edge is the first s colors, then recolor it with red. Otherwise, recolor it with blue. Note that $K_{N_1,\ldots,N_k} \to \big(K_{1,\sum+1}, (\Lambda+1)K_2\big)$ by Theorem 1. If there is a red copy of $K_{1,\sum+1}$, then there is a monochromatic copy of K_{1,n_i} in color i for some $i \in [s]$ by the pigeonhole principle. If there is a blue copy of $(\Lambda+1)K_2$, then there is a monochromatic copy of $m_j K_2$ in color $s+j$ for some $j \in [t]$ by the pigeonhole principle. Consequently, $K_{N_1,\ldots,N_k} \to (K_{1,n_1}, \ldots, K_{1,n_s}, m_1K_2, \ldots, m_tK_2)$. ■

The following is from Theorem 2 directly by the definition of the set and the size multipartite Ramsey numbers.

Corollary 7.

$$\begin{aligned} &M_a(K_{1,n_1}, \ldots, K_{1,n_s}, m_1K_2, \ldots, m_tK_2) \\ &\leq \max\left\{ \left\lceil \frac{2\sum+\Lambda+1}{a} \right\rceil, \left\lceil \frac{2\Lambda+2}{a} \right\rceil, \left\lceil \frac{\Lambda+1}{a} \right\rceil + 1 \right\}. \end{aligned}$$

Corollary 8.

$$\begin{aligned} &m_b(K_{1,n_1}, \ldots, K_{1,n_s}, m_1K_2, \ldots, m_tK_2) \\ &\leq \max\left\{ \left\lceil \frac{2\sum+\Lambda+1}{b} \right\rceil, \left\lceil \frac{2\Lambda+2}{b} \right\rceil, \left\lceil \frac{\Lambda+1}{b-1} \right\rceil \right\}. \end{aligned}$$

We can also resolve the bipartite Ramsey numbers for stars versus matchings.

Theorem 9.
$$\begin{aligned}&br(K_{1,n_1},\dots,K_{1,n_s},m_1K_2,\dots,m_tK_2)\\&=m_2(K_{1,n_1},\dots,K_{1,n_s},m_1K_2,\dots,m_tK_2)\\&\le\max\left\{\sum+\left\lceil\frac{\Lambda+1}{2}\right\rceil,\Lambda+1\right\}.\end{aligned}$$

The upper bound in Theorem 9 is tight. The lower bound is omitted here and one can find it in [12].

3. Remark

We have proven that the lower bounds of Theorem 1 are the best for $2\le k\le m+1$. If $k\ge m+2$, then in Remark 6, we have to partition V_1 or V_2 into more sets. In this case, the lower bound of $\sum_{l=1}^{k}N_l$ can be improved.

Conjecture 10. *Let $k\ge m+2$ be an integer. If $\sum_{l=1}^{k}N_l\ge\max\left\{\frac{k-m+1}{k-m}(n-1)+m,2m\right\}$ and $\sum_{l=1}^{k}N_l-N_c\ge m$ for each $c\in[k]$, then $K_{N_1,\dots,N_k}\to(K_{1,n},mK_2)$.*

We believe that the lower bounds of Theorem 2 can be improved since $r(m_1K_2,\dots,m_tK_2)=\max\limits_{1\le j\le t}\{m_j\}+\Lambda+1$ from [5].

Conjecture 11. *If $m_1\ge\cdots\ge m_t$, $\sum_{l=1}^{k}N_l\ge\max\{2\sum+\Lambda+1,\Lambda+m_1+1\}$ and $\sum_{l=1}^{k}N_l-N_c\ge m_1$ for each $c\in[k]$, then $K_{N_1,\dots,N_k}\to(K_{1,n_1},\dots,K_{1,n_s},m_1K_2,\dots,m_tK_2)$.*

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