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BOUNDS FOR PACKING CHROMATIC NUMBER OF SOME SUBCLASSES OF TREES

K. Mohamed Harith, A.V. Jayanthan

Department of Mathematics
IIT Madras
Chennai, INDIA – 600036

e-mail: harry1496@gmail.com,
jayanav@iitm.ac.in

and

Raghavendra Rao B.V.

Department of Computer Science and Engineering
IIT Madras
Chennai, INDIA – 600036

e-mail: bvrr@iitm.ac.in

Abstract

It is known that computing packing chromatic number is NP-hard even for trees. In this article, we derive an exact formula for the packing chromatic number for trees of diameter five in terms of number of vertices of degree at least four. Additionally, we improve the upper bound for *shifted* packing chromatic number of an infinite path. We also establish a new bound for the packing chromatic number of any tree, related to the number of vertices of degree at least four. Finally, we identify an infinite class of trees containing caterpillars, which has a bounded packing chromatic number.

Keywords: tree, vertex degree, graph diameter, graph coloring.

2020 Mathematics Subject Classification: 05C05, 05C07, 05C15.

1. Introduction

Let $G = (V, E)$ be a finite simple undirected graph. For $k \in \mathbb{N}$, a packing k-coloring is a partition of the vertex set V into k subsets $V_1, \ldots, V_k$ such that if $u, v \in V_i$, $u \neq v$, then the length of a shortest path between the vertices u and

v in G is greater than i, for all $i \in [k] := \{1, \ldots, k\}$. The smallest integer k for which there exists a packing k-coloring of G is called the packing chromatic number of G, denoted by $\chi_\rho(G)$.

The term "packing chromatic number" was introduced by Brešar *et al.* [6], as it combines aspects of both packing (densely collecting vertices) and coloring (partitioning the vertex set). However, the concept was initially introduced by Goddard *et al.* in [11] under the name "broadcast chromatic number" with applications to frequency assignment problems. Prior to the formal introduction of packing coloring, Sloper studied a related problem called the eccentric coloring of trees in [13]. He established some upper bounds and calculated the eccentric chromatic numbers for certain classes of trees, including caterpillars and binary trees. Since eccentric colorings are a specific type of packing coloring, the same upper bounds apply to packing chromatic numbers as well.

Many authors have explored various structural and algorithmic aspects of packing coloring for special classes of graphs, such as planar graphs, subcubic graphs, and certain types of trees, leading to numerous open questions and conjectures. In [11], Goddard *et al.* showed that the decision version of packing chromatic number is NP-complete and later, Fiala and Golovach [8] showed that it remains NP-complete even for trees. Obtaining bounds on the packing chromatic number of various classes of graphs and obtaining characterization of graphs with a given packing chromatic number remain key areas of research. For example, bounding packing chromatic number of subcubic graphs received wide attention in the literature. Initially, it was questioned in [7] whether the class of subcubic graphs have bounded packing chromatic number. However, this was disproved by Balogh *et al.* in [2], and later improved by several authors [3, 10, 12] (see also the survey [5]).

Since determining packing chromatic number is NP-hard even for trees [8], bounding packing chromatic number remains interesting even for trees. In a graph G, a vertex is called a *large vertex* if its degree is greater than or equal to 4. In [11], the authors present an exact formula for the packing chromatic number of trees of diameter four, based on the number of large-degree vertices. Utilizing this, they showed that packing chromatic number of trees is bounded by $\frac{n+7}{4}$ which is sharp even for lobsters of small diameters.

Using standard apparatus from model checking for monadic second order logic, Fiala and Golovach [8] observed that if the packing chromatic number of a tree is bounded by a constant, then determining the exact value can be done in polynomial time. Thus it is interesting to identify classes of graphs for which the packing chromatic number is bounded. Argiroffo *et al.* [1], obtained bounds on the packing chromatic number of lobsters in terms of the maximum number of large vertices adjacent to a vertex in its spine. In this article, we first determine the exact values of the packing chromatic number for trees with diameter five, based on the number of large vertices, by identifying certain packing chromatic critical subtrees.

When a graph has many pendant vertices, color 1 can be assigned to all the pendant vertices. It is therefore reasonable to inquire about the minimum number of colors required to packing color a graph when the use of smaller colors is restricted. In other words, for $1 \leq s \leq k$, the *s-shifted packing k-coloring* is the partition of the vertex set V of a graph into k subsets $V_1, \ldots, V_k$ such that for any $u, v \in V_i$, $u \neq v$, the distance between u and v in G is at least $s + i$. In [9], an upper bound for the shifted packing chromatic number of path graphs was determined. Building on this, Argiroffo *et al.* in [1] offer an upper bound for the packing chromatic number of lobsters and show that it can be computed in polynomial time for an infinite subclass of them, including caterpillars. It may be noted that they studied s-shifted packing k-coloring of graphs under the name (k, s)-packing coloring.

We enhance this upper bound for the shifted packing chromatic number of the infinite path. The main idea is to construct a packing coloring for the infinite path using the packing coloring of a finite path. By utilizing the packing coloring for binary trees and realising the role played by large vertices, we establish a new upper bound for the packing chromatic number of any tree. Finally, we identify an infinite class of trees that includes caterpillars but is not contained within lobsters, where the packing chromatic number is bounded by leveraging the bound on the shifted packing chromatic number of paths.

The paper is organized as follows. In the next section, we review some graph theory terminologies that will be used throughout the article. In Section 3, we discuss the packing chromatic number of trees with diameter five. In Section 4, we derive an upper bound for the shifted packing coloring of an infinite path. Section 5 covers new upper bounds for the packing chromatic number of lobsters and trees in general. Finally, in Section 6, we examine the packing chromatic number for the class of bounded width trees, which includes caterpillars.

2. Preliminaries

In this section, we recall some definitions and notations in graph theory. More details can be found in the textbook by West, [14]. Let G be a graph. We denote its vertex set by $V(G)$ (or simply V) and its edge set by $E(G)$ (or simply E). All graphs considered here are undirected, unweighted, finite, and simple. Thus E can be viewed as a set of unordered pairs of vertices. For a vertex $v \in V$, the total number of vertices in G that are adjacent to v is called the *degree* of v. If a vertex v has degree one in G, it is referred to as a *leaf* or a *pendant vertex.* We shall call a vertex a *large vertex* if it has degree greater than or equal to four, [11].

Let $u, v \in V$ be any two distinct vertices. The *distance* between vertices u and v is the length of the shortest path connecting the two vertices, denoted by $\mathrm{d}(u, v)$. This is well defined for any pair of vertices when the graph considered is connected. The *eccentricity* of a vertex $v \in V$, denoted by $\epsilon_G(v)$, is defined as the maximum distance between v and any other vertex in G: $\epsilon_G(v) = \max_{u \in V}\{\mathrm{d}(u, v)\}$. The *diameter* of a graph is the greatest distance between any pair of vertices, that is, $\mathrm{diam}(G) = \max_{v \in V}\{\epsilon_G(v)\}$. The *center* of a graph G, denoted by $\mathcal{C}(G)$, is the set of vertices with minimum eccentricity, that is, $\mathcal{C}(G) = \{v \in G \mid \epsilon_G(v) \text{ is minimum}\}$.

A subset $S \subseteq V$ is said to be an *independent set* if G does not contain an edge between any pair of vertices in S. For a positive integer k, a *proper k-vertex coloring* of graph G is a function $f : V \to [k]$ which partitions V into k independent sets

$$V = V_1 \cup \cdots \cup V_k,$$

where each V_i is called a *color class*. The least positive integer k for which such partition of vertex set exists is called the *chromatic number* of the graph, denoted by $\chi(G)$. For $s \in \mathbb{N}$, $U \subseteq V$ is said to be an *s-packing independent set*, if for any two distinct vertices $u, v \in U$, $\mathrm{d}(u, v) > s$. A *packing k-coloring* of graph G is a partition of the vertex set V into k independent sets

$$V = V_1 \cup \cdots \cup V_k,$$

where each V_i is an i-packing independent set. If such a partition of V exists, then the graph G is said to be *packing k-colorable*. The least positive integer k for which G is packing k-colorable is called the *packing chromatic number*, denoted by $\chi_\rho(G)$.

In this article, we limit our focus on trees. Recall that an acyclic graph is called a forest and a connected forest is a tree. Very often, a graph that is a tree is denoted by T instead of G. A subtree T' of a tree T is an induced subgraph which has $V(T') \subseteq V(T)$, and it is also denoted by $T' \subset T$.

A tree T is a *packing chromatic critical graph*, or *χ_ρ-critical graph*, if for every proper subtree T' of T, $\chi_\rho(T') < \chi_\rho(T)$. If T is χ_ρ-critical and $\chi_\rho(T) = k$, then T is referred to as a *k-χ_ρ-critical tree.* Furthermore, if for all $v \in V$, $\chi_\rho(T - v) < \chi_\rho(T)$, then T is called *χ_ρ-vertex-critical tree*. It has been shown by Brešar and Ferme [4] that these two notions coincide for trees. More generally, let $\mathcal{G}$ be a class of graphs. For a positive integer k, we say a graph $G \in \mathcal{G}$ is *k-χ_ρ-critical in $\mathcal{G}$*, if $\chi_\rho(G) = k$ and for every proper subgraph $H \in \mathcal{G}$ of G, $\chi_\rho(H) < \chi_\rho(G)$. Observe that, if $G \in \mathcal{G}$ is k-χ_ρ-critical and H is a subgraph of G such that $\chi_\rho(H) = \chi_\rho(G) = k$, then $H \notin \mathcal{G}$.

For any two graphs G_1, G_2, define their *intersection* $G_1 \cap G_2$ as the graph with the vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. Note that the diameter of $G_1 \cap G_2$ is at most the minimum of the diameters of G_1 and G_2.

For a positive integer n, let the path P_n be the tree with vertex set $V(P_n) = \{1, \ldots, n\}$ and edge set $E(P_n) = \{\{i, i+1\} \mid 1 \leq i \leq n-1\}$. For $n \geq 3$, let the cycle C_n be the graph with vertex set $V(C_n) = \{1, \ldots, n\}$ and edge set $E(C_n) = \{\{i, i+1\} \mid 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$.

Let G be a finite graph. For $v \in V(G)$, let $G(v)$ denote the BFS Tree which is obtained by running the Breadth First Search algorithm with initial vertex v, [14]. Note that, for a tree T, its BFS tree $T(v)$ is isomorphic to T, while there is an ordering of the vertices in $T(v)$. In fact, the vertices in $T(v)$ are totally ordered and the order can be obtained by the breadth first search traversal.

A *caterpillar* is a tree T such that the induced subtree obtained by removing all the pendant vertices of T is a path. The central path is called the *spine* of the caterpillar. A *lobster* is a tree T such that the induced subtree obtained by removing all the pendant vertices of T is a caterpillar. The spine of the induced caterpillar is called the *spine* of the lobster.

3. Packing Coloring for Trees of Diameter Five

In this section, we determine the packing chromatic number for all trees of diameter five. If a tree T has diameter two or three, then it can be easily verified that $\chi_\rho(T) = 2$ or 3, respectively. Goddard *et al.* have derived an explicit formula for the packing chromatic number of trees of diameter four.

Theorem 3.1 [11]. *Let T be a tree of diameter 4 with center v. For $i = 1, 2, 3$, let n_i denote the number of neighbours of v of degree i, and let L denote the number of large neighbours of v. If $L = 0$ then*

$$\chi_\rho(T) = \begin{cases} 4, & \text{if } n_3 \geq 2 \text{ and } n_1 + n_2 + n_3 \geq 3, \\ 3, & \text{otherwise} \end{cases}$$

and if $L > 0$ then

$$\chi_\rho(T) = \begin{cases} L+3, & \text{if } n_3 \geq 1 \text{ and } n_1 + n_2 + n_3 \geq 2, \\ L+1, & \text{if } n_1 = n_2 = n_3 = 0, \\ L+2, & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}_5$ denote the class of all trees of diameter five. For any $T \in \mathcal{D}_5$, the set of centers is $\mathcal{C}(T) = \{v_1, v_2\}$. Define L_j, for $j = 1, 2$, as the number of large neighbors of v_j, excluding the other center. For positive integers m_1, m_2, define a subclass $\mathcal{D}_5(m_1, m_2)$ of $\mathcal{D}_5$ consisting of trees with diameter five, where $L_1 = m_1$ and $L_2 = m_2$.

Every tree $T \in \mathcal{D}_5$ contains P_6 as an induced subtree. P_6 requires a minimum of three colors to be packing colored. Therefore, for every $T \in \mathcal{D}_5$, we have $\chi_\rho(T) \geq 3$.

In the following, an exact formula for packing chromatic number is obtained in terms of L_1, L_2 of the tree. The strategy involves identifying the χ_ρ-critical tree within some subclass of $\mathcal{D}_5$ and then developing a formula for the packing chromatic number. In the figures throughout the rest of this section, each tree is depicted with a valid packing coloring indicated within its vertices, illustrating the upper bounds on their packing chromatic numbers.

3.1. Packing coloring of $\mathcal{D}_5(0,0)$

Let us consider the case when a diameter five tree does not have any large-degree vertices adjacent to a center. There are exactly three χ_ρ-critical trees in $\mathcal{D}_5(0,0)$. We name them as T_1, T_2, T_3, and describe them below.

Consider the tree T_1 in Figure 1. If any of the centers is colored 1, then three new colors are required to color its neighbors. Assume that none of the centers is colored 1. Then the induced subtree P_6 requires minimum four colors to be packing colored. Thus $\chi_\rho(T_1) = 4$. It is not very difficult to see that T_1 is 4-χ_ρ-critical in $\mathcal{D}_5$.

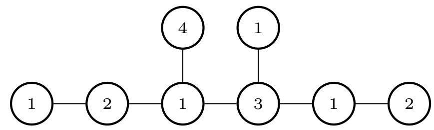


Figure 1. Tree T_1.

Consider the tree T_2 in Figure 2. If any degree three vertex is colored 1, then three more new colors are required to color its neighbors. If none of them are colored 1, then we are done. Thus, $\chi_\rho(T_2) = 4$. For criticality, we consider subtrees of T_2 which lie in $\mathcal{D}_5$. These are obtained by removing one or more leaves of a degree three neighbor of a center. In any such case, the tree can be packing colored using 3 colors. Therefore, T_2 is 4-χ_ρ-critical in $\mathcal{D}_5$.

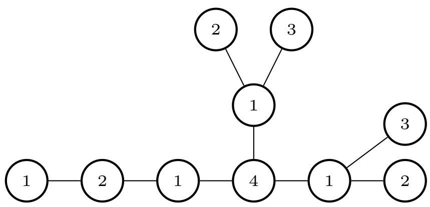


Figure 2. Tree T_2.

Consider the tree T_3 in Figure 3. We want to show that $\chi_\rho(T_3) \geq 5$. In any packing coloring of T_3 where one of the centers is assigned the color 1, we need at least 5 colors. Assume that both centers are assigned colors other than 1. Consider the degree three vertices in T_3. Suppose that all three vertices do not receive color 1, then we are done, as at least two more unused colors are required to color them. Suppose that only one of the degree three vertices is colored with 1, then either one of its leaves must receive a color different from that of the centers and the remaining degree three vertices, or the other two degree three vertices must receive different colors. In both cases, we are done.

Suppose that two of the degree three vertices receive color 1, then the third degree three vertex must receive a color different from that of the centers. Moreover, in this case, either a leaf of a degree three vertex colored with 1 will receive an unused color or a leaf of the degree two vertex will receive an unused color. If all three degree vertices receive color 1, then also we are done. Thus, $\chi_\rho(T_3) = 5$.

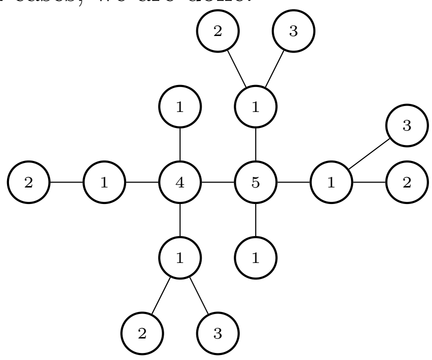


Figure 3. Tree T_3.

Lemma 3.2. *The tree T_3 in Figure 3 is 5-χ_ρ-critical in $\mathcal{D}_5$.*

Proof. We need to consider only subtrees of T_3 that lie in $\mathcal{D}_5$. The maximal such subtrees of T_3 can be obtained by removing exactly one leaf from {a,b,c,d,e} in Figure 4. It is sufficient to provide a valid packing coloring using four or fewer colors for these maximal subtrees. To obtain a valid packing coloring for a subtree of T_3, we will follow the same packing coloring given in Figure 3, with some minor changes listed in Table 1. Thus, T_3 is 5-χ_ρ-critical in $\mathcal{D}_5$. ∎

It remains to prove that T_1, T_2, T_3 are the only χ_ρ-critical trees in $\mathcal{D}_5(0,0)$. We do this in the following proposition.

Proposition 3.3. *If $T \in \mathcal{D}_5(0,0)$ such that*

(1) *T is 4-χ_ρ-critical in $\mathcal{D}_5(0,0)$, then T is isomorphic to T_1 or T_2.*

(2) *T is 5-χ_ρ-critical in $\mathcal{D}_5(0,0)$, then T is isomorphic to T_3.*

Proof. (1) The degree of both centers in T_1 is three. T_2 has only one center of degree greater than two and, furthermore, that center has two neighbors of degree three. Let T be a 4-χ_ρ-critical tree in $\mathcal{D}_5(0,0)$. If T contains T_1 or T_2, then T is isomorphic to T_1 or T_2 respectively, as T_1, T_2 are 4-χ_ρ-critical trees in $\mathcal{D}_5$. Let us suppose T does not contain T_1 and T_2, then T has the property that the degree of at least one of the centers is two. Also, there can be at most one neighbor of degree three adjacent to any center. Therefore, the packing coloring in Figure 5 can be extended to any tree in $\mathcal{D}_5(0,0)$ which does not contain T_1, T_2 as a subtree. Thus $\chi_\rho(T) = 3$, which is a contradiction.

(2) The tree T_3 has the following structural properties.

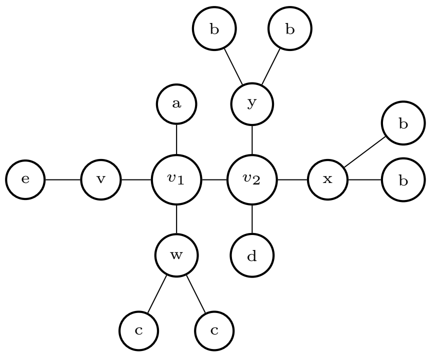


Figure 4. 5-χ_ρ-critical.

Pendant vertex removed	Change in packing coloring
a	$v_1 \to 1$, $v_2 \to 4$, w $\to 2$, v $\to 3$, c $\to 1$, e $\to 1$.
b	$v_2 \to 3$, w$\to 2$, c $\to 1$, y $\to 2$, leaves of y $\to 1$, x $\to 1$, leaf of x $\to 2$ **or** $v_2 \to 3$, w $\to 2$, c $\to 1$, x $\to 2$, leaves of x $\to 1$, y $\to 1$, leaf of y$\to 2$.
c	$v_1 \to 2$, $v_2 \to 4$, c $\to 3$, e$\to 3$.
d	$v_2 \to 1$, x $\to 2$, y $\to 3$, b $\to 1$.
e	$v_1 \to 2$, $v_2 \to 4$, w $\to 3$, c $\to 1$.

Table 1. Packing coloring using four or less number of colors.

1. Both centers have degree 4.
2. One of the centers has exactly one neighbor of degree three while the other center has exactly two neighbors of degree three.

Let T be a 5-χ_ρ-critical tree in $\mathcal{D}_5(0,0)$. If T contains T_3, then T is isomorphic to T_3 by Lemma 3.2. Now let us suppose that T does not contain T_3. We consider two cases.

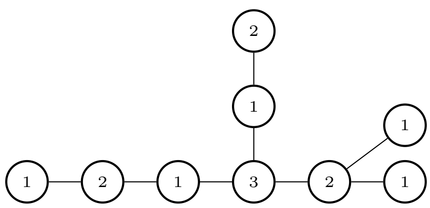


Figure 5. A 3-packing coloring.

Case 1. T has a center whose degree is strictly less than 4. Suppose v_1 has degree strictly less than 4, then use the following packing coloring. Assign $v_1 \to 1$, v_1 can have at most two neighbors other than v_2, assign them $2, 3$ and their leaves 1. Assign $v_2 \to 4$, the neighbors of v_2 (other than v_1) $\to 1$ and their leaves $2, 3$.

Case 2. T does not satisfy property (ii) of T_3. In this case, the distribution of degree three vertices among the centers differs from that of T_3, and a maximal common subtree $T \cap T_3$ that can be obtained is by removing either vertex b or c from T_3. Again, Table 1 provides valid packing colorings using at most 4 colors for such subtrees, which can be extended to T itself. Hence, $\chi_\rho(T) \leq 4$, again contradicting the assumption.

Let us suppose that one of the centers does not have any neighbors of degree three (excluding the other center). Then the packing coloring used when pendant vertex c was removed can be used to extend the coloring of this tree, refer Table 1.

In both cases, we arrive at a contradiction to the criticality of T. Therefore, if T is 5-χ_ρ-critical in $\mathcal{D}_5(0,0)$, then T must be isomorphic to T_3. ∎

Theorem 3.4. *Let T be a tree of diameter 5 with the centers v_1 and v_2. Let L_1, L_2 be the number of large-degree vertices adjacent to v_1 and v_2, respectively (excluding v_1 and v_2 themselves). Suppose that $L_1 = 0$ and $L_2 = 0$, then*

$$\chi_\rho(T) = \begin{cases} 5, & \text{if } T_3 \subset T, \\ 4, & \text{if } T_3 \not\subset T \text{ and } (T_1 \subset T \text{ or } T_2 \subset T), \\ 3, & \text{otherwise.} \end{cases}$$

Proof. The lower bounds for each of the cases have been established by the discussion on T_1, T_2, and T_3. One good choice for packing coloring is to assign the two centers $4, 5$ and color every neighbor of centers with 1 and finally assign the vertices that are distance two from centers colors 2 and 3. Thus for any tree $T \in \mathcal{D}_5(0,0)$, we have $\chi_\rho(T) \leq 5$. By Proposition 3.3, the upper bound for the packing chromatic number of any tree in $\mathcal{D}_5(0,0)$ that does not contain T_3 is 4, while for any tree that does not contain T_1 or T_2 is 3. ∎

3.2. Packing coloring of $\mathcal{D}_5(>0,0)$.

Next, let us consider the trees that have large-degree vertices adjacent to exactly one of the centers. For any tree $T \in \mathcal{D}_5(0,>0)$, we can interchange the labeling of the two centers and end up in the case of $\mathcal{D}_5(>0,0)$. Therefore, it suffices to study the case $L_1 > 0$ and $L_2 = 0$.

The next lemma is a structural result for existence of optimal packing coloring which assigns unique colors to the large vertices of $T \in \mathcal{D}_5(>0,0)$.

Lemma 3.5. *Let $T \in \mathcal{D}_5(L_1, 0)$, where $L_1 \geq 3$. Then there exists an optimal packing coloring of T with a large vertex receiving a unique color.*

Proof. Let f be any optimal packing coloring of $T \in \mathcal{D}_5(\geq 3, 0)$. Suppose there is a large vertex in T which receives a color greater than or equal to 4 under f, then by diameter condition, it will be a unique color. Therefore, we can assume that under the packing coloring f, every large vertex receives the color $1, 2$ or 3.

There can be at most one large vertex having color 2 or 3. Suppose there are two large vertices that are colored 1 under f. Then at least one of the two large vertices has a leaf that will receive a unique color in the tree. We can recolor a large vertex with the unique color and its leaves with 1, to obtain another packing coloring, which is again optimal, as it uses same colors as f. Suppose there are two large vertices which are colored 1 and 3, then there is a unique color assigned to one of the leaves of the large vertex colored 1. Again, we can recolor a large vertex with the unique color and obtain an optimal packing coloring.

Therefore, in all possible optimal packing colorings f, either a large vertex already receives a unique color (when colored with a value ≥ 4), or we can find a large vertex that can be recolored with a unique color originally assigned to one of its leaves, while preserving the packing constraints and the number of colors used. In either case, we obtain a new optimal packing coloring in which at least one large vertex receives a unique color. ■

Note that any large vertex which is colored 1 in any $T \in \mathcal{D}_5$ must have degree less than or equal to four. Otherwise, one of its leaves will necessarily be assigned a unique color. Moreover, the following observation is an immediate deduction from the proof of Lemma 3.5.

Observation 3.6. *Let $T \in \mathcal{D}_5$ be any tree.*

1. *If T has a large vertex adjacent to a center, then there exists an optimal packing coloring that assigns the colors $1, 2$ or 3 to the large vertex.*
2. *If T has two large vertices adjacent to a center, then there exists an optimal packing coloring that assigns either the colors $1, 2$ or $2, 3$ to the large vertices.*

For a tree T and $w \in V(T)$, for convenience, we let $T \setminus \{w\}$ denote the subtree of T that is obtained by removing w and its pendant vertices from T.

Lemma 3.7. *For $L_1 > 0$, let $T \in \mathcal{D}_5(L_1, 0)$ be such that there exists an optimal packing coloring of T with a large neighbor w of a center receiving a unique color. Then $\chi_\rho(T) = \chi_\rho(T \setminus \{w\}) + 1$.*

Proof. Let $T \in \mathcal{D}_5(L_1, 0)$ be any tree with an optimal packing coloring f. Suppose that f assigns a unique color to a large vertex w of T, then the induced subtree $T \setminus \{w\}$ can be packing colored using the restriction of f. Thus, $\chi_\rho(T \setminus \{w\}) \leq \chi_\rho(T) - 1$. On the other hand, let f' be an optimal packing coloring of $T \setminus \{w\}$. Then f' can be extended to a packing coloring of T, by setting $f'(w)$ to be a unique color and the pendant vertices at w in T can be set to 1. Thereby, we obtain a packing coloring of T from an optimal packing coloring of $T \setminus \{w\}$ and thus, $\chi_\rho(T) \leq \chi_\rho(T \setminus \{w\}) + 1$. ■

The results of Lemmas 3.5 and 3.7 facilitate the reduction of trees in $\mathcal{D}_5(\geq 3, 0)$. This enables us to concentrate on χ_ρ-critical trees within the subclasses $\mathcal{D}_5(1, 0)$ and $\mathcal{D}_5(2, 0)$.

Case (2a). Packing coloring of $T \in \mathcal{D}_5(1, 0)$. Let $T \in \mathcal{D}_5(1, 0)$ be any tree. By Observation 3.6, there is an optimal packing coloring which assigns colors $1, 2$ or 3 to the large vertex in T. Furthermore, we have a packing coloring using the following schema: center $v_1, v_2 \to 4, 5$, large vertex $\to 2$, every pendant vertex at distance two away from the center (other than the leaves of the large vertex) $\to 2, 3$ and the remaining vertices $\to 1$. Therefore, $\chi_\rho(T) \leq 5$.

Remark 3.8. For any $T \in \mathcal{D}_5(1, 0)$, we may assume that there is an optimal packing coloring which assigns the colors 2 or 3 to the large vertex. This is because utilizing the color 1 for the large vertex uses at least 5 colors, which is an upper bound of packing chromatic number for this class.

Consider the tree T_4 in Figure 6. It contains T_1 as a subtree. Therefore, $\chi_\rho(T_4) \geq 4$. The packing coloring given in Figure 6 shows that $\chi_\rho(T_4) = 4$. It can be easily verified that T_4 is 4-χ_ρ-critical in $\mathcal{D}_5(1, 0)$.

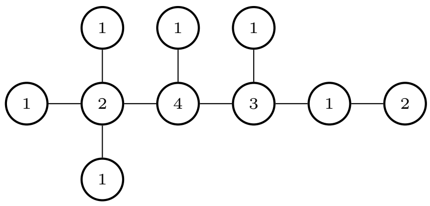


Figure 6. Tree T_4.

Consider the tree T_5 in Figure 7. It contains T_2 as a subtree. Therefore, we can conclude that $\chi_\rho(T_5) = 4$. For criticality, we consider subtrees of T_5 which lie in $\mathcal{D}_5(1, 0)$. These are obtained by removing one or more leaves of a degree three neighbor of a center. In any such case, the obtained subtree can be packing colored using 3 colors. Therefore, T_5 is 4-χ_ρ-critical in $\mathcal{D}_5(1, 0)$.

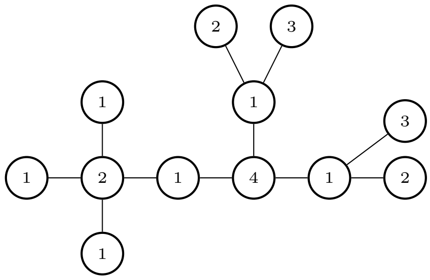


Figure 7. Tree T_5.

Consider the tree T_6 in Figure 8. It contains T_2 as a subtree. As before, we conclude that $\chi_\rho(T_6) = 4$. For criticality, we consider subtrees of T_6 which lie in $\mathcal{D}_5(1, 0)$. These are obtained by removing one or more leaves of the degree three neighbor of the center. In any such case, the obtained subtree can be packing colored using 3 colors. Therefore, T_6 is 4-χ_ρ-critical in $\mathcal{D}_5(1, 0)$.

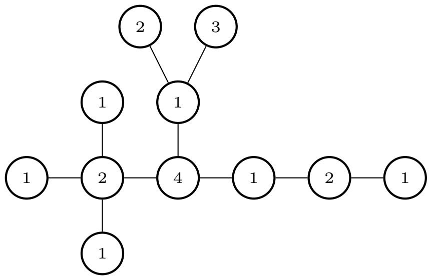


Figure 8. Tree T_6.

Consider the tree T_7 in Figure 9. We note that T_7 contains T_3 as a subtree. To see this, let v be a leaf adjacent to the large neighbor of v_1 (as illustrated in Figure 9) and let T' be the subtree $T_7 \setminus \{v\}$. It is not hard to see that T_3 is isomorphic to T'. Therefore, $\chi_\rho(T_7) = 5$.

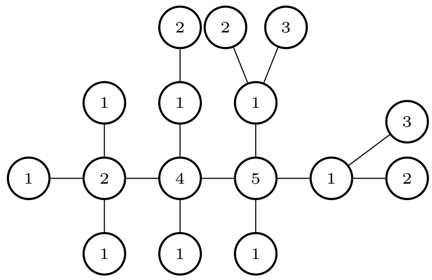


Figure 9. Tree T_7.

Consider the tree T_8 in Figure 10. We want to show that $\chi_\rho(T_8) \geq 5$. If a center or the large-degree vertex is colored 1, then four unused colors are required to color its neighbors. Therefore, assume that none of the centers or large vertex is colored 1, then three colors which are not 1 are required to color them. Now consider the degree three vertex. If it is not colored 1, then it must be an unused color, thus attaining a minimum bound of 5 colors. If it is colored 1, then one of its leaves must be assigned an unused color. Thus, $\chi_\rho(T_8) = 5$. The criticality of T_7 and T_8 is proved in Lemma 3.9.

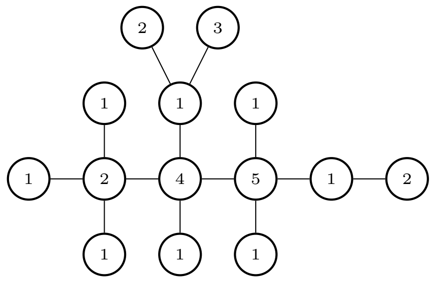


Figure 10. Tree T_8.

There are exactly 5 χ_ρ-critical trees in $\mathcal{D}_5(1,0)$. We name them as $T_4, \ldots, T_8$, and describe them below.

Lemma 3.9. *The trees T_7, T_8 are 5-χ_ρ-critical trees in the subclass of $\mathcal{D}_5(1,0)$.*

Proof. First we prove the criticality of T_7. We need to consider only subtrees of T_7 that lie in $\mathcal{D}_5(1,0)$. Note that a subtree of T_7 will still be in $\mathcal{D}_5(1,0)$ only if it is obtained by removing anyone or more of the leaves {b,c,d,e} in Figure 11. It is sufficient to provide a valid packing coloring using four colors or less for the subtrees obtained by removing one of those leaves. To obtain a valid packing coloring for a subtree of T_7, we will follow the same packing coloring given in Figure 9, with some minor changes listed in Table 2. Thus, T_7 is 5-χ_ρ-critical in $\mathcal{D}_5(1,0)$.

For the criticality of T_8, we need to consider only subtrees of T_8 that lie in $\mathcal{D}_5(1,0)$. Note that a subtree of T_8 will still be in $\mathcal{D}_5(1,0)$ only if it is obtained by removing anyone or more of the leaves {b,c,e} in Figure 12. It is sufficient to provide a valid packing coloring using four colors or less for the subtrees obtained

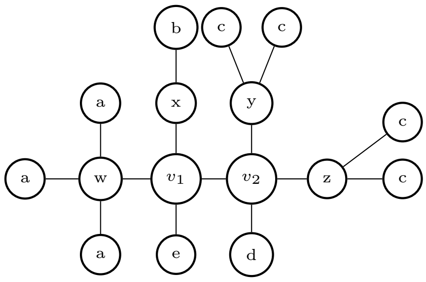


Figure 11. T_7 is 5-χ_ρ-critical in $\mathcal{D}_5(1,0)$.

pendant vertex removed	Change in packing coloring
b	$v_1 \to 2$, $v_2 \to 4$, w $\to 3$.
c	$v_2 \to 3$, y $\to 2$, leaves of y$\to 1$, z $\to 1$, leaf of z $\to 2$ **or** $v_2 \to 3$, z $\to 2$, leaves of z $\to 1$, y $\to 1$, leaf of y $\to 2$.
d	$v_2 \to 1$, y $\to 2$, z $\to 3$, c $\to 1$.
e	$v_1 \to 1$, $v_2 \to 4$, x $\to 3$, b $\to 1$.

Table 2. Packing coloring using four or less number of colors.

by removing one of those leaves. To obtain a valid packing coloring for a subtree of T_8, we will follow the same packing coloring given in Figure 10, with some minor changes listed in Table 3. Thus, T_8 is 5-χ_ρ-critical in $\mathcal{D}_5(1,0)$. ■

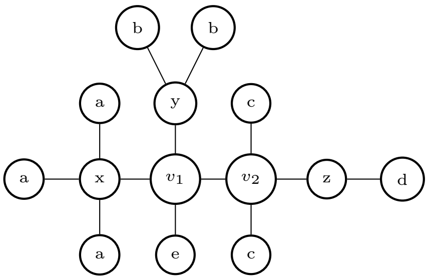


Figure 12. T_8 is 5-χ_ρ-critical in $\mathcal{D}_5(1,0)$.

pendant vertex removed	Change in packing coloring
b	$v_2 \to 3$, b $\to 2$.
c	$v_2 \to 1$, c $\to 2$, z $\to 3$, d $\to 1$.
e	$v_1 \to 1$, $v_2 \to 4$, y $\to 3$, b $\to 1$.

Table 3. Packing coloring using four or less number of colors.

It remains to prove that $T_4, \ldots, T_8$ are the only χ_ρ-critical trees in $\mathcal{D}_5(1,0)$. We do this in the following proposition.

Proposition 3.10. *If* $T \in \mathcal{D}_5(1,0)$ *such that*

(1) T *is* 4-χ_ρ*-critical, then* T *is isomorphic to* T_4 *or* T_5 *or* T_6*.*

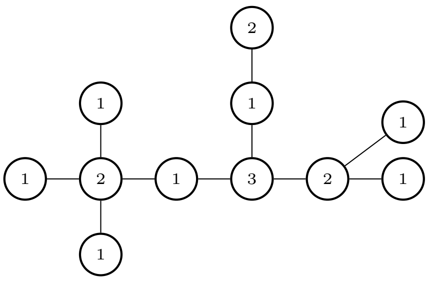
(2) T *is* 5-χ_ρ*-critical, then* T *is isomorphic to* T_7 *or* T_8*.*

Proof. (1) In Table 4, we list some structural properties of T_4, T_5 and T_6 which we use in the proof. Let T be a 4-χ_ρ-critical tree in $\mathcal{D}_5(1,0)$. Without loss of generality, we can assume that T does not contain T_4, T_5 or T_6 as they are 4-χ_ρ-critical trees in $\mathcal{D}_5(1,0)$. Then T has the property that degree of at least one of the centers is two, as T_4 is not a subtree.

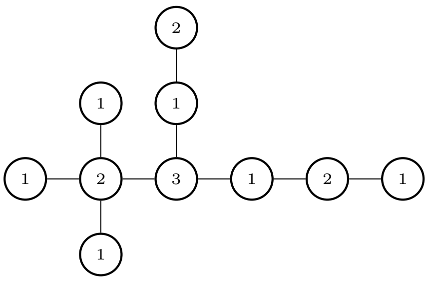
Suppose that v_1 has degree two. Then there can be at most one degree three neighbor to v_2, as T_5 is not a subtree. Then assign the packing coloring to the tree $T \cap T_5$ as illustrated in Figure 13(a). The tree T can be constructed from $T \cap T_5$ by attaching pendant vertices or paths of length two at v_2. The packing coloring shown in Figure 13(a) can be extended to entire T by coloring the newly added pendant vertices, which are at a distance two from v_2, with color 2, and coloring all other newly added vertices with color 1. Thus, a packing coloring for T is obtained using only three colors. This contradicts 4-χ_ρ-criticality of T.

Tree	Properties
T_4	Degree of both centers is three.
T_5	Two degree three vertices adjacent to center v_2.
T_6	One degree three vertex adjacent to center v_1.

Table 4. Properties of T_4, T_5, and T_6.



(a) A 3-packing coloring.



(b) A 3-packing coloring.

Figure 13. Subtrees of T_5 and T_6.

Suppose that v_2 has degree two. Then there can be no degree three neighbor to v_1, as T_6 is not a subtree. Then assign the packing coloring to the tree $T \cap T_6$ as illustrated in Figure 13(b). The tree T can be constructed from $T \cap T_6$ by attaching pendant vertices or paths of length two at v_1. The packing coloring shown in Figure 13(b) can be extended to entire T by coloring the newly added pendant vertices, which are at a distance two from v_1, with color 2, and coloring all other newly added vertices with color 1. Thus, a packing coloring for T is obtained using only three colors. This contradicts 4-χ_ρ-criticality of T.

Thus, if $T \in \mathcal{D}_5(1,0)$ is 4-χ_ρ-critical tree, then T is isomorphic to T_4, T_5 or T_6.

(2) In Table 5, we list some structural properties of T_7 and T_8 which we use in the proof.

Tree	Properties
T_7	Degree of both centers is four. One vertex of degree two adjacent to v_1. Two vertices of degree three adjacent to v_2.
T_8	Degree of both centers is four. One vertex of degree three adjacent to v_1.

Table 5. Properties of T_7 and T_8.

Let T be a 5-χ_ρ-critical tree in $\mathcal{D}_5(1,0)$. Without loss of generality, we can assume that T does not contain T_7 or T_8, as indicated by Lemma 3.9. Suppose that there is a center in T that has degree less than 4, while the other properties of T_7 and T_8 may hold. If v_1 has degree less than 4, then assign the packing coloring: $v_1 \to 1$, $v_2 \to 4$, non-center neighbors of $v_1 \to \{2,3\}$, distance two pendant vertices of $v_2 \to \{2,3\}$ and remaining vertices with 1. If v_2 has degree less than 4, then assign the packing coloring: $v_1 \to 4$, $v_2 \to 1$, non-center neighbors of $v_2 \to \{2,3\}$, large vertex $\to 2$, distance two pendant vertices (except the leaves at the large vertex) of $v_1 \to \{2,3\}$ and remaining vertices with 1. This ensures a packing coloring of T with less than 5 colors.

Now assume that both the centers have degree four. Furthermore, assume that there are no degree three vertices adjacent to v_1, otherwise T_8 will be a subtree of T.

Suppose that there are no degree two vertices adjacent to v_1, then assign the following packing coloring to T: $v_1 \to 2$, $v_2 \to 4$, large vertex $\to 3$, distance two pendant vertices from v_2 (except for the leaves of v_1) $\to \{2,3\}$ and remaining vertices with 1. Suppose there is a degree two vertex adjacent to v_1, then there can be at most one degree three vertex adjacent to v_2, otherwise T_7 will be a subtree. In this case, assign the following packing coloring to T: $v_1 \to 3$, $v_2 \to 4$, large vertex $\to 2$, distance two pendant vertices of v_1 and v_2 (except for the leaves of the large vertex and degree three vertex, as well as pendant vertices of v_1, v_2) $\to 2$, the degree three neighbor of $v_2 \to 2$ and remaining vertices with 1.

In all of the above cases, if T does not contain T_7 or T_8, then T can be packing colored using 4 colors. This contradicts 5-χ_ρ-criticality of T. ■

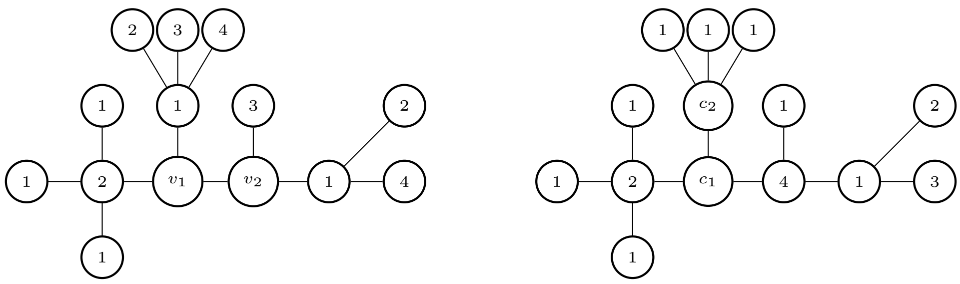
Case (2b). Packing coloring of $T \in \mathcal{D}_5(2,0)$. Let $T \in \mathcal{D}_5(2,0)$ be any tree. By Observation 3.6, there exists an optimal packing coloring such that the pair of large vertices is colored using $1,2$ or $2,3$.

Lemma 3.11. *Let $T \in \mathcal{D}_5(2,0)$. Suppose there is an optimal packing coloring for T such that a large vertex is colored 1, then there is an optimal packing coloring with one of the large vertices receiving a unique color.*

Proof. By Observation 3.6, let f be an optimal packing coloring of $T \in \mathcal{D}_5(2,0)$ which assigns the colors $1,2$ to the pair of large vertices.

Assume that the pendant vertices of the large vertex colored with 1 are assigned colors $2,3,4$ under f. If any of these three colors assigned to the leaves is unique in T, then a recoloring of T can achieve the required optimal packing coloring. Suppose there is another vertex, w, in T that is also colored with 4 under f. Then w must be present as a pendant vertex located at distance two from v_2. We can assume, without loss of generality, that w must be adjacent to a vertex in T colored with 1 under f. If not, there is an optimal coloring that is nearly identical to f but assigns the color 1 to w. Similarly, w must have a sibling colored with 2 under f, as v_2 cannot be colored 2 under f. Additionally, T has a vertex colored with 3 at distance of at most three from w. Refer Figure 14(a) for the optimal coloring f.

Note that both the centers, v_1, v_2 cannot be assigned colors $1,2,3,4$ under f, so they receive two unique colors in T, say c_1, c_2, respectively. Now, consider the following packing coloring f'. Large vertex colored 1 under $f \to c_2$ and its leaves $\to 1$, the center $v_2 \to 4$, all neighbors of v_2 (except v_1) $\to 1$, all the pendant vertices at distance two from v_2 (except for leaves of v_1) $\to \{2,3\}$ and for the remaining vertices follow the packing coloring f.



(a) An optimal coloring f. (b) An optimal packing coloring f'.

Figure 14. Two packing colorings of the same tree.

We conclude that f' is an optimal packing coloring that utilizes the same colors as f and also achieves the required condition. ∎

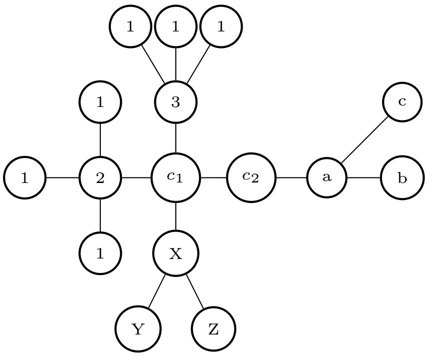
With the next result, we shall obtain a structural property of trees in $\mathcal{D}_5(2,0)$, which will restrict the class $\mathcal{D}_5(2,0)$ to look out for χ_ρ-critical trees.

Lemma 3.12. *Let $T \in \mathcal{D}_5(2,0)$. Suppose there exists a degree three vertex adjacent to the center v_1 (which is not v_2), then there is an optimal packing coloring such that one of the large vertices, which is not v_1 or v_2, receives a unique color.*

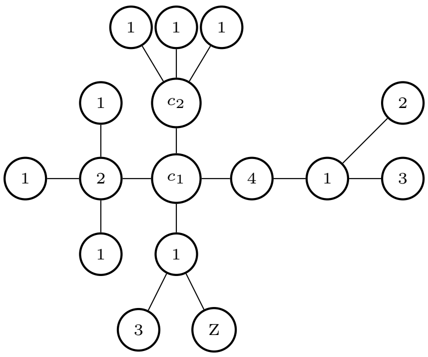
Proof. By Observation 3.6 and Lemma 3.11, we may assume that under an optimal packing coloring f, both large vertices of T receive colors $2, 3$.

Consider the vertices which are colored X, Y and Z as shown in Figure 15(a). If one of them is a unique color in T, then we can obtain another packing coloring by interchanging that unique color with 3 on the large vertex, to obtain the required packing coloring. So we can assume that $X = 1$ and $Y = 4$ and they are not unique in the optimal packing coloring f of T. This ensures that v_1 receives a unique color c_1. Now, running over a similar argument as in Lemma 3.11, we can deduce that $a = 1$, $b = 4$ and $c = 2$. Moreover, v_2 is assigned a unique color c_2 under f.

Further, we can construct an optimal packing coloring f' of T such that one of the large vertices receives a unique color in T. The packing coloring f' of T. Large vertex colored 3 under $f \to c_2$, center $v_2 \to 4$, all neighbors of v_2 (except v_1)$\to 1$, $Y = 3$, all the pendant vertices at distance two from v_2 (except for leaves of v_1) $\to \{2, 3\}$ respectively and for the remaining vertices follow the packing coloring f.



(a) An optimal packing coloring f.



(b) An optimal packing coloring f'.

Figure 15. Two packing colorings of the same tree.

Therefore, for $T \in \mathcal{D}_5(2,0)$, whenever there exists a degree three vertex adjacent to the center v_1, then there is an optimal packing coloring such that one of the large vertices receive a unique color in T. ■

Aided by Lemmas 3.12 and 3.7, we can focus on those trees $T \in \mathcal{D}_5(2,0)$ such that there are no degree three vertices adjacent to v_1. There are two χ_ρ-critical trees in the class of $\mathcal{D}_5(2,0)$ which do not have a degree three vertex adjacent to v_1. We name them T_9, T_{10}, and describe them below.

Though the tree T_9 is not 4-χ_ρ-critical in $\mathcal{D}_5$, by its minimality in $\mathcal{D}_5(2,0)$, it is 4-χ_ρ-critical in $\mathcal{D}_5(2,0)$. It remains to establish that T_{10} is 5-χ_ρ-critical in $\mathcal{D}_5(2,0)$.

Lemma 3.13. *The tree T_{10} is a 5-χ_ρ-critical tree in the class of $\mathcal{D}_5(2,0)$. Moreover, let T be any tree in $\mathcal{D}_5(2,0)$ which does not contain a degree three vertex adjacent to v_1. Suppose T is 5-χ_ρ-critical in $\mathcal{D}_5(2,0)$, then T is isomorphic to T_{10}.*

Consider the tree T_9 in Figure 16. It contains the tree T_2 as a subtree. Therefore, we can conclude that $\chi_\rho(T_9) = 4$. Note that T_9 is the minimal tree in the class of $\mathcal{D}_5(2,0)$.

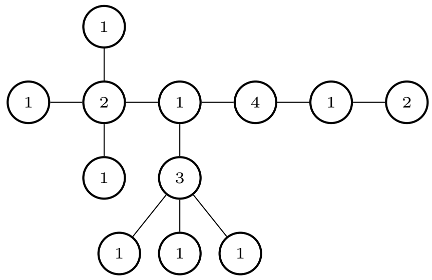


Figure 16. Tree T_9.

Consider the tree T_{10} in Figure 17. We want to show that $\chi_\rho(T_{10}) \geq 5$. If the center v_1 or a large vertex is colored 1, then we require minimum 5 colors for packing coloring of T_{10}. Assume that v_1 and the large vertices are not colored 1, in fact, they receive three different colors which are not 1. Suppose the center v_2 receives color 1, then one of the two neighbors of v_2 must receive an unused color. Therefore, we get $\chi_\rho(T_{10}) \geq 5$ and the packing coloring given in Figure 17 shows that $\chi_\rho(T_{10}) = 5$.

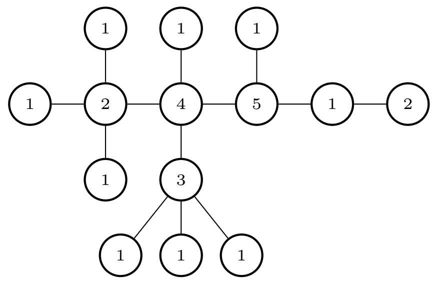


Figure 17. Tree T_{10}.

Proof. Any subtree of T_{10} in $\mathcal{D}_5(2,0)$ can only be obtained by removing the leaf at v_1 or v_2. If leaf at v_1 is removed, then recolor as: $v_1 \to 1, v_2 \to 4$ and the rest same as in Figure 17. If leaf at v_2 is removed, then recolor as: $v_2 \to 1$, degree two neighbor of $v_2 \to 2$, the leaf at distance two from v_2 (which is not leaf of v_1) $\to 3$ and the rest same as in Figure 17. Therefore, T_{10} is 5-χ_ρ-critical tree in the class of $\mathcal{D}_5(2,0)$.

Let $T \in \mathcal{D}_5(2,0)$ be any 5-χ_ρ-critical tree with no degree three vertex adjacent to v_1. Without loss of generality, we may assume that T does not contain T_{10}. Note that T_{10} has the property that the degree of v_1 is 4 and degree of v_2 is 3. If v_1 has degree less than 4 in T, then assign the packing coloring: $v_1 \to 1$, $v_2 \to 4$, large vertices $\to \{2,3\}$, distance two pendant vertices of $v_2 \to \{2,3\}$

and remaining vertices $\to 1$. If v_2 has degree less than 3 in T, then assign the packing coloring: $v_2 \to 1$, large vertices $\to \{2,3\}$, distance two pendant vertices of v_1, v_2 (except for leaves of v_1, v_2) $\to \{2,3\}$ and remaining vertices $\to 1$. Note that, for the above coloring, we rely on the fact that v_1 does not have a degree three neighbor. This is a contradiction to T being 5-χ_ρ-critical in $\mathcal{D}_5(2,0)$. Thus, T is isomorphic to T_{10}. ■

Theorem 3.14. *Let T be a tree of diameter 5. Let v_1, v_2 be the two centers of T. Let m be the number of degree three vertices adjacent to v_1. Let L_1 and L_2 be the total number of large-degree vertices adjacent to v_1 and v_2, respectively (excluding v_1 and v_2 themselves). Suppose that $L_1 > 0$ and $L_2 = 0$, then*

$$\chi_\rho(T) = \begin{cases} (L_1-2)+5, & \text{if } T_{10} \subset T \text{ and } m=0,\\ (L_1-2)+4, & \text{if } T_{10} \not\subset T,\ T_9 \subset T \text{ and } m=0,\\ (L_1-1)+5, & \text{if none of the above and } (T_7 \subset T \text{ or } T_8 \subset T),\\ (L_1-1)+4, & \text{if none of the above and } (T_4 \subset T \text{ or } T_5 \subset T \text{ or } T_6 \subset T),\\ (L_1-1)+3, & \text{otherwise.} \end{cases}$$

Proof. Let $T \in \mathcal{D}_5(L_1, 0)$ be any tree. Using Lemmas 3.5, 3.7 and 3.12 we can find a subtree T' of tree T with at most 2 large vertices such that

$$\chi_\rho(T) = \begin{cases} \chi_\rho(T') + (L_2-2), & \text{if } T' \in \mathcal{D}_5(2,0),\\ \chi_\rho(T') + (L_1-1), & \text{if } T' \in \mathcal{D}_5(1,0). \end{cases}$$

Thus it is sufficient to prove that $\chi_\rho(T')$ satisfies the following:

$$\chi_\rho(T') = \begin{cases} 5, & \text{if } T_{10} \subset T' \text{ and } m=0,\\ 4, & \text{if } T_{10} \not\subset T,\ T_9 \subset T' \text{ and } m=0,\\ 5, & \text{if none of the above and } (T_7 \subset T' \text{ or } T_8 \subset T'),\\ 4, & \text{if none of the above } (T_4 \subset T' \text{ or } T_5 \subset T' \text{ or } T_6 \subset T'),\\ 3, & \text{otherwise.} \end{cases}$$

The lower bounds for each of the cases have been established by the discussion on T_4–T_{10}.

When $T' \in \mathcal{D}_5(2,0)$, by Lemma 3.12, we have $m = 0$. One good choice for packing coloring is to assign the centers $\to \{4,5\}$, the large vertices $\to \{2,3\}$, every pendant vertex at distance two from v_1 (except the leaves of large vertices and v_2) $\to 2$, every prendant vertex distance two from v_2 (except the leaves of v_1) $\to \{2,3\}$ and remaining vertices $\to 1$. Moreover, by Lemma 3.13, any T' that contains T_{10} has packing chromatic number 5. Further, when T_{10} is not contained as a subtree in T', then as T_9 is a minimal tree in $\mathcal{D}_5(2,0)$, we have $T_9 \subset T'$. Thus T' will have packing chromatic number 4.

When $T' \in \mathcal{D}_5(1,0)$, one good choice for packing coloring is to assign the two centers $\to \{4,5\}$, large vertex $\to 2$, assign every vertex distance two from the centers (except leaves of large vertex)$\to \{2,3\}$ and remaining vertices $\to 1$. Thus, for any tree $T' \in \mathcal{D}_5(1,0)$, we have $\chi_\rho(T') \leq 5$. By the criticality of trees T_4-T_8 in Proposition 3.10, the upper bound for the packing chromatic number of any tree in $\mathcal{D}_5(1,0)$ that does not contain T_7 or T_8 is 4, while for any tree that does not contain T_4 or T_5 or T_6 is 3. ∎

3.3. Packing coloring of $\mathcal{D}_5(>0,>0)$.

Next, let us consider the trees that have large-degree vertices adjacent to both centers. In Lemmas 3.5 and 3.7, the number of large vertices adjacent to the center v_1 plays a crucial role, while v_2 has zero large neighbors (other than v_1). By fixing the number of large neighbors adjacent to v_2, the proofs remains unchanged for both those lemmas. Therefore, the following observations are immediate.

Observation 3.15. *Let $T \in \mathcal{D}_5(L_1, L_2)$ with $L_1 \geq 3$ or $L_2 \geq 3$. Then there is an optimal packing coloring of T with one of the large vertices receiving a unique color.*

Observation 3.16. *Let $T \in \mathcal{D}_5(L_1, L_2)$ such that there exists an optimal packing coloring of T with a large neighbor w receiving a unique color. Then $\chi_\rho(T) = \chi_\rho(T \setminus \{w\}) + 1$.*

Observations 3.15 and 3.16 facilitates the reduction of trees when $T \in \mathcal{D}_5(L_1, L_2)$, where $L_1 \geq 3$ or $L_2 \geq 3$. This allows us to focus our attention on χ_ρ-critical trees within the subclasses $\mathcal{D}_5(1,1)$, $\mathcal{D}_5(1,2)$, $\mathcal{D}_5(2,1)$, and $\mathcal{D}_5(2,2)$.

Case (3a). Packing coloring of $T \in \mathcal{D}_5(1,1)$. There are exactly three χ_ρ-critical trees in $\mathcal{D}_5(1,1)$. We name them as T_{11}, T_{12}, T_{13}, and describe them below.

Consider the tree T_{11} in Figure 18. It contains the tree T_1 as a subtree. Thus, it follows from the packing coloring given in Figure 18 that $\chi_\rho(T_{11}) = 4$. For criticality, we consider subtrees of T_{11} which lie in $\mathcal{D}_5(1,1)$. These are obtained by removing pendant vertices at the centers. Clearly, when one pendant vertex of a center is removed, then that subtree can be packing colored using 3 colors. Therefore, T_{11} is 4-χ_ρ-critical in $\mathcal{D}_5(1,1)$.

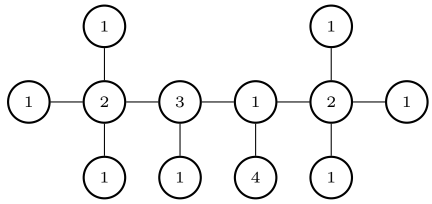


Figure 18. Tree T_{11}.

Consider the tree T_{12} in Figure 19. It contains the tree T_2 as a subtree. Thus, $\chi_\rho(T_{12}) = 4$. For criticality, we consider subtrees of T_{12} which lie in $\mathcal{D}_5(1,1)$. These are obtained by removing one or more leaves of the degree three neighbor of center. In any such case, the obtained subtree can be packing colored using 3 colors. Therefore, T_{12} is 4-χ_ρ-critical in $\mathcal{D}_5(1,0)$.

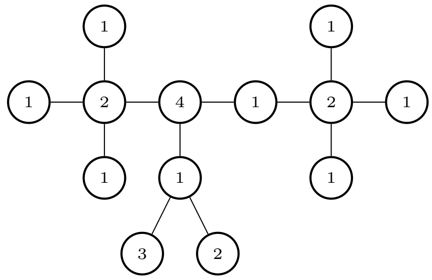


Figure 19. Tree T_{12}.

Consider the tree T_{13} in Figure 20. We want to show that $\chi_\rho(T_{13}) \geq 5$. If a center or a large vertex is colored 1, we need at least 5 colors. Suppose that these four vertices are given colors other than 1, they can be colored with a minimum of three different colors c_1, c_2, c_3. Now consider the degree three neighbor of the center. If it is not colored 1, then we require a color different from c_1, c_2, c_3 to color it. Suppose it is colored 1, then one of its leaf must receive a color different from c_1, c_2, c_3. In any case, T_{13} requires minimum 5 colors to be packing colored. Thus, $\chi_\rho(T_{13}) = 5$. Criticality of T_{13} is shown below.

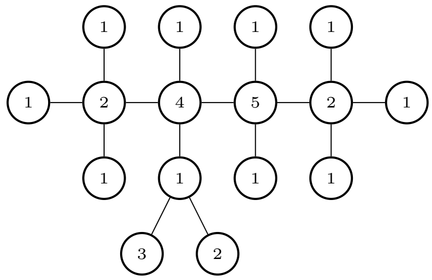


Figure 20. Tree T_{13}.

Lemma 3.17. *The tree T_{13} is 5-χ_ρ-critical in the subclass of $\mathcal{D}_5(1,1)$.*

Proof. We need to consider only subtrees of T_{13} that lie in $\mathcal{D}_5(1,1)$. Note that a subtree of T_{13} will still be in $\mathcal{D}_5(1,1)$ only if it is obtained by removing anyone or more of the leaves {b,c,e} in Figure 21. It is sufficient to provide a valid packing coloring using four colors or less for the subtrees obtained by removing one of those leaves. To obtain a valid packing coloring for a subtree of T_{13}, we will follow the same packing coloring given in Figure 20, with some minor changes listed in Table 6. Thus, T_{13} is 5-χ_ρ-critical in $\mathcal{D}_5$. ■

We now prove that T_{11}, T_{12} and T_{13} are the only χ_ρ-critical trees in $\mathcal{D}_5(1,1)$.

Proposition 3.18. *If $T \in \mathcal{D}_5(1,1)$ such that*

(1) *T is 4-χ_ρ-critical, then T is isomorphic to T_{11} or T_{12}.*

(2) *T is 5-χ_ρ-critical, then T is isomorphic to T_{13}.*

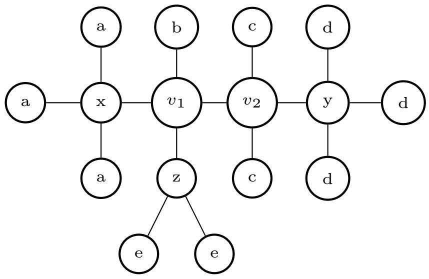


Figure 21. Tree T_{13} is 5-χ_ρ-critical in $\mathcal{D}_5(1,1)$.

pendant vertex removed	Change in packing coloring
b	$v_1 \to 1$, $v_2 \to 4$, z $\to 3$, e $\to 1$.
c	$v_2 \to 1$, c $\to 3$.
e	$v_2 \to 3$, e $\to 2$.

Table 6. Packing coloring using four colors.

Tree	Properties
T_{11}	Degree of both centers is three.
T_{12}	One degree three vertex adjacent to a center.

Table 7. Properties of T_{11} and T_{12}.

Proof. (1) Some structural properties of T_{11} and T_{12} are in Table 7.

Let T be a 4-χ_ρ-critical tree in $\mathcal{D}_5(1,1)$. Without loss of generality, we can assume that T does not contain T_{11} or T_{12} as they are 4-χ_ρ-critical trees in $\mathcal{D}_5(1,1)$. Then T has the property that degree of at least one of the centers is two, as T_{11} is not a subtree.

If v_2 has degree two, then there can be no degree three neighbor to v_1, as T_{12} is not a subtree. The maximal tree that can be obtained as $T \cap T_{12}$ can be packing colored using 3 colors, as in Figure 22. The tree T can be constructed from $T \cap T_{12}$ by attaching pendant vertices or paths of length two at v_1. The packing coloring shown in Figure 22 can be extended to entire T by coloring the newly added pendant vertices, which are at a distance of two from v_1, with color 2, and coloring all other newly added vertices with color 1. Thus, a packing coloring for T is obtained using only three colors. This contradicts 4-χ_ρ-criticality of T.

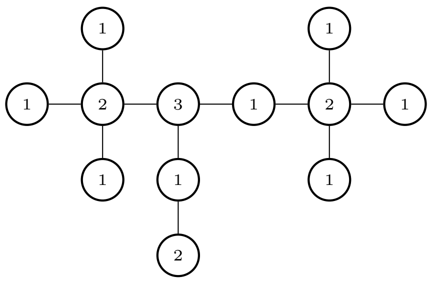


Figure 22. A packing 3-coloring.

An analogous argument holds when v_1 has degree two in T. Thus, if $T \in \mathcal{D}_5(1,1)$ is 4-χ_ρ-critical tree, then T is isomorphic to T_{11} or T_{12}.

(2) The tree T_{13} has the following properties:

(i) the degrees of both centers are equal to 4;

(ii) one of the centers has a neighbor of degree three.

Let T be a 5-χ_ρ-critical tree in $\mathcal{D}_5(1,1)$. Without loss of generality, we can assume that T does not contain T_{13}, as indicated by Lemma 3.17. Then let us suppose that there is a center in T that has degree less than 4. If v_1 has degree less than 4, then assign the following packing coloring to T: $v_1 \to 1$, $v_2 \to 4$, large vertices $\to 2$, non-large neighbor of $v_1 \to 3$, pendant vertices at distance two from v_2 (except leaves of v_1) $\to \{2,3\}$ and the remaining vertices $\to 1$. A similar type of packing coloring can be assigned when v_2 has degree less than 4. That is, T can be colored using less than 5 colors.

Now assume that both the centers have degree four. Furthermore, assume that there are no degree three vertices adjacent to v_1 nor v_2. Then assign the following packing coloring to T: $v_1 \to 3$, $v_2 \to 4$, large vertices $\to 2$, pendant vertices of distance two from v_1 and v_2 (except the leaves of v_1, v_2) $\to 2$ and remaining vertices with 1. That is, T can be packing colored using less than 5 colors.

Thus, if T does not contain T_{13}, then T can be packing colored using 4 colors. This contradicts that T is 5-χ_ρ-critical in $\mathcal{D}_5(1,1)$. ■

Case (3b). Packing coloring of $T \in \mathcal{D}_5(2,1)$. For any tree $T \in \mathcal{D}_5(1,2)$, we can interchange the labeling of the two centers and end up in the case of $\mathcal{D}_5(2,1)$. Thus it is sufficient to study the χ_ρ-critical trees in $\mathcal{D}_5(2,1)$ only. Moreover, by Observation 3.6, there is an optimal packing coloring which assigns colors $1,2$ or $2,3$ to the pair of large vertices adjacent to a center in T and the other large vertex in T is assigned the color $1, 2$ or 3. The next lemma ensures the existence of an optimal packing coloring which assigns colors $2,3$ to the pair of large vertices in T and assigns color 2 to the other large vertex in T.

Lemma 3.19. *Let $T \in \mathcal{D}_5(2,1)$. There exists an optimal packing coloring of T in which either a large vertex is assigned a unique color, or the large vertices adjacent to v_1 (excluding v_2) are colored with $2,3$ and the large vertex adjacent to v_2 (excluding v_1) is colored with 2.*

Proof. Let x, y be the two large neighbors of v_1 and z be the large neighbor of v_2. Let f be an optimal packing coloring of T and let the colors assigned to x, y, z be X, Y, Z, respectively. By Observation 3.6, we know that either $X = 2, Y = 1$ or $X = 2, Y = 3$, and Z $\in \{1,2,3\}$. Therefore it is sufficient to show that if the large vertices do not have a unique color assigned to them, then there exists an optimal packing coloring with $X = 2, Y = 3$ and $Z = 2$. Note that, whenever 1 is used to color the large vertex, either it will have a pendant vertex which is assigned a unique color or pendant vertices will be assigned the colors $2, 3, 4$. If a pendant vertex of a large vertex is assigned a unique color, then there is a recoloring which assigns the large vertex with that unique color, and its leaves with 1. Then the recoloring is the required optimal packing coloring.

Case. $X = 2$, $Y = 1$, $Z = 1$. In this case, as Y, Z are both 1, the centers will receive a unique color, say c_1, c_2, under f. As v_2 receives a unique color, we can assume that none of its neighbors receive color 2. Now consider the following packing coloring: $v_2 \to 4$, y $\to c_2$, leaves of y $\to 1$, z $\to 2$, leaves of z $\to 1$. In this packing coloring, a large vertex receives a unique color.

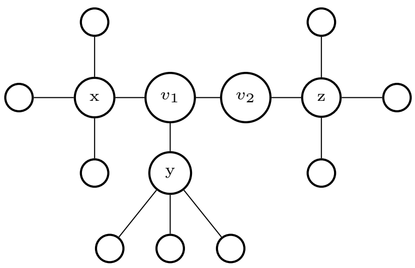


Figure 23. The minimal tree in $\mathcal{D}_5(2, 1)$.

Case. $X = 2$, $Y = 3$, $Z = 1$. Suppose that 4 is a unique color assigned to a pendant vertex of z in the tree T under f, then consider a packing coloring: one of the large neighbors of $v_1 \to 4$, the large neighbor of $v_2 \to 2$ and leaves of all large vertices $\to 1$. This is an optimal packing coloring with a large vertex receiving a unique color.

Assume that 4 is not unique in T. Therefore there exists a pendant vertex, say w, which is at a distance 2 from v_1 and is colored 4. We can further assume that the neighbour of w is 1 and there is a pendant vertex which is at distance two from w which is colored with 2 under f. Since X = 2, Y = 3 and one neighbor of v_1 is assigned 1 under f, this forces $f(v_1) = c_1$, which is a unique color. Moreover, v_2 also has a unique color, say c_2, in T under f, since Z = 1 and pendant vertices of z has colors 2, 3, 4. This is illustrated in Figure 24. Consider the following packing coloring: y $\to c_2$, $v_2 \to 4$, neighbor of $w \to 3$, w and its sibling $\to 1$, z $\to 2$, leaves of z $\to 1$. Since $c_2 \neq 1$ apriori, we can assume that all the non-large neighbors of v_2 can be colored with 1. Therefore there will be no conflict in assigning Z = 2. This is the required optimal packing coloring.

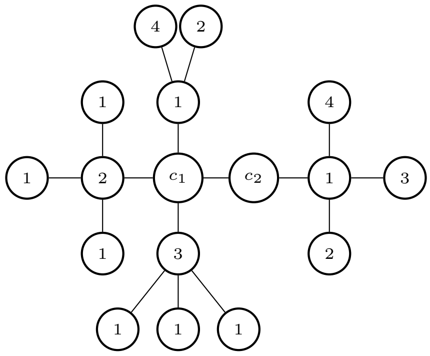


Figure 24. Tree $T \in \mathcal{D}_5(2,1)$, with $X = 2, Y = 3, Z = 1$.

Case. $X = 2$, $Y = 1$, $Z = 2$ or $Z = 3$. Suppose that 4 is a unique color in the tree T, then we can obtain another optimal packing coloring by recoloring y$\rightarrow$ 4. Assume that 4 is not unique in T. Therefore there exists a pendant vertex, say w, at distance two from v_2 which is also colored with 4. We can assume that the neighbour of w is colored 1, otherwise w can be recolored to 1. Also, as f is optimal, there is a pendant vertex which is colored 2, located at distance two away from w. Note that, v_2 cannot be colored using 1, 2, 3, 4 under f due to distance condition. So $f(v_2) = c_2$ is a unique color. Hence, we can assume that none of the neighbors of v_2 receive color 2. Moreover, as $Y = 1$, $f(v_1) = c_1$ is also a unique color.

When $Z = 2$, consider the following packing coloring: $v_1 \rightarrow 4$, $w \rightarrow 3$, y $\rightarrow c_1$, leaves of y $\rightarrow$ 1 and remaining are same as f. This is the required packing coloring.

When $Z = 3$, consider the following packing coloring: $v_1 \rightarrow 4$, z $\rightarrow$ 2, $w \rightarrow 3$, y$\rightarrow c_1$, leaves of y $\rightarrow$ 1 and the remaining are same as f. This is the required packing coloring.

Case. $X = 2$, $Y = 3$, $Z = 3$. This case cannot occur due to distance condition. ■

In the light of Lemma 3.19 and Observation 3.16, we can always assume that there exists a packing coloring of $T \in \mathcal{D}_5(2,1)$ such that the pair of large neighbors of v_1 receive 2, 3 while the large neighbor of v_2 receives 2. A similar reasoning of Lemma 3.12 can now be utilized to obtain the following.

Observation 3.20. *Let $T \in \mathcal{D}_5(2,1)$. If there exists a degree three vertex adjacent to the center v_1 (excluding v_2), then there is an optimal packing coloring such that one of the large vertices receive a unique color.*

Now Observation 3.20 helps restricting the class $\mathcal{D}_5(2,1)$ to look into for χ_ρ-critical trees. We can focus on those trees which do have degree three vertices adjacent to v_1. There are exactly two such χ_ρ-critical trees. We name them as T_{14}, T_{15}, and describe them below.

Though the tree T_{14} is not 4-χ_ρ-critical in $\mathcal{D}_5$, by its minimality in $\mathcal{D}_5(2,1)$, it is 4-χ_ρ-critical in $\mathcal{D}_5(2,1)$. It remains to establish that T_{15} is 5-χ_ρ-critical in $\mathcal{D}_5(2,1)$.

Lemma 3.21. *The tree T_{15} is a 5-χ_ρ-critical tree in the class of $\mathcal{D}_5(2,1)$. Moreover, let T be any tree in $\mathcal{D}_5(2,1)$ which does not contain a degree three vertex adjacent to v_1. Suppose T is 5-χ_ρ-critical in $\mathcal{D}_5(2,1)$, then T is isomorphic to T_{15}.*

Proof. Any subtree of T_{15} in $\mathcal{D}_5(2,1)$ can only be obtained by removing the leaf at v_1 or v_2. If leaf at v_1 is removed, then recolor as: $v_1 \to 1, v_2 \to 4$ and the rest same as in Figure 26. If leaf at v_2 is removed, then recolor as: $v_2 \to 1$ and the rest same as in Figure 26. Therefore, T_{15} is 5-χ_ρ-critical tree in the class of $\mathcal{D}_5(2,1)$.

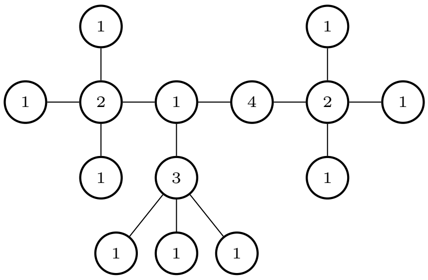


Figure 25. Tree T_{14}.

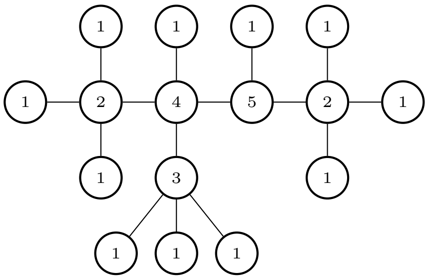


Figure 26. Tree T_{15}.

Consider the tree T_{14} in Figure 25. It contains the tree T_9 as a subtree. Thus $\chi_\rho(T_{14}) = 4$. Note that T_{14} is the minimal tree in the class of $\mathcal{D}_5(2,1)$.

Consider the tree T_{15} in Figure 26. It contains T_{10} as a subtree. Thus $\chi_\rho(T_{15}) = 5$.

Let $T \in \mathcal{D}_5(2,1)$ be any 5-χ_ρ-critical tree with no degree three neighbor adjacent to v_1. Without loss of generality, we may assume that T does not contain T_{15}. In T_{15}, the degree of center v_1 is four and degree of v_2 is three. In T, if v_1 has degree less than four, then consider the following packing coloring: $v_1 \to 1$, $v_2 \to 4$, pendant vertices at distance two from v_2 (except the neighbors of v_1 and leaves of large vertex adjacent to v_2) $\to \{2,3\}$, assign colors to the other vertices in $T \cap T_{15}$ as in Figure 26 and remaining vertices $\to 1$. If v_2 has degree less than three, since v_1 cannot have any degree three neighbors, then consider the following packing coloring: $v_2 \to 1$, pendant vertices of distance two from v_1 (except the leaves of large vertices) $\to 2$, assign colors to the other vertices in $T \cap T_{15}$ as in Figure 26 and remaining vertices $\to 1$. In any case, there is a packing coloring with four colors which contradicts the 5-χ_ρ-criticality of T in $\mathcal{D}_5(2,1)$. ■

Case (3c). Packing coloring of $T \in \mathcal{D}_5(2,2)$. Let $T \in \mathcal{D}_5(2,2)$ be any tree. By Observation 3.6, there exists an optimal packing coloring such that the pair of large vertices are colored using $1,2$ or $2,3$.

Lemma 3.22. *Let $T \in \mathcal{D}_5(2,2)$. There is an optimal packing coloring of T with a large vertex receiving a unique color or both pairs of large vertices receive colors $1,2$.*

Proof. Let f be an optimal packing coloring of $T \in \mathcal{D}_5(2,2)$. Then every pair of large vertices adjacent to a center is either assigned $1,2$ or $2,3$ under f. By distance criterion, both the pairs cannot be assigned $2,3$ under f. Let us suppose that f assigns $1,2$ to large neighbors of v_1 and $2,3$ to large neighbors of v_2. Then the large vertex which is assigned the color 1 under f must have three pendant vertices, which are colored $2,3,4$. Suppose 4 is uniquely assigned under f, then we can obtain an optimal packing coloring with large vertex receiving the unique color 4. Assume that 4 is not uniquely assigned in T. Then there is a pendant vertex, say w, which is at distance two from v_2 and is colored 4 under f. This forces the neighbor of w to be colored 1, and there exists a pendant vertex colored with 2 at distance two from w. Consider the packing coloring given by: large neighbor assigned 3 under $f \to 1$, leaves of large neighbor assigned 3 under $f \to 2,3,4$, $w \to 3$ and remaining is same as f.

Thus, there is an optimal packing coloring of $T \in \mathcal{D}_5(2,2)$ with both pairs of large vertices receiving colors $1,2$. ■

The subsequent lemma underscores the importance of having a three-degree neighbor adjacent to a center to determine the χ_ρ-critical trees within $\mathcal{D}_5(2,2)$.

Lemma 3.23. *Let $T \in \mathcal{D}_5(2,2)$. If T has a center which does not have a degree three neighbor, then there is an optimal packing coloring which assigns a unique color to one of the large vertices.*

Proof. Let us suppose that the center v_1 in T does not have a degree three neighbor. By Lemma 3.22, there exists an optimal packing coloring, say f, with both pairs of large vertices receiving colors $1,2$. Strategy is to perturb the packing coloring f in such a way that the color 4 is unique in tree, as it is already on the leaf of large vertices under f. Consider the following packing coloring, say f': large neighbor of v_1 assigned 1 under $f \to 3$, leaves of large neighbor of v_1 assigned 1 under $f \to 1$ and remaining as in f. Under f', 4 is assigned uniquely to a pendant vertex which is at distance two from v_2. Such an f' is valid packing coloring, because v_1 does not have degree three neighbors. Now, we can obtain an optimal packing coloring that assigns 4 to large vertex, which will be unique as required. ■

There is exactly one χ_ρ-critical tree in $\mathcal{D}_5(2,2)$ with both centers containing a degree three neighbor. We name it as T_{16}, and describe it below.

Consider the tree T_{16} in Figure 27. We want to show that $\chi_\rho(T_{16}) \geq 6$. Suppose none of the large vertices are colored with 1, then at least 3 different colors are required to color the large vertices. Then either both of the centers are assigned unused colors or one of the center is assigned 1 and other an unused color. In the former case, we are done. In the latter case, consider the degree three neighbor of the center colored with 1. It cannot receive any of the used colors, it must receive an unused color. Thus a minimum of 6 colors is required to color T_{16}.

Suppose one large neighbor of v_1 and another large neighbor of v_2 have been assigned 1, then a minimum of 5 more colors are required to color their neighbors. Thus, we are done in this case as well.

Suppose the pair of large neighbors of a center receive color 1, then a minimum of 5 more unused colors are required to color both sets of pendant vertices and the center.

Suppose that exactly one large vertex receives color 1, then four colors different from 1 are required to color its neighbors. Suppose the other center receives an unused color, then we are done. Else if the other center receives color 1, then one of its neighbors must receive an unused color. Thus a minimum of 6 colors is always required for packing coloring and $\chi_\rho(T_{16}) = 6$.

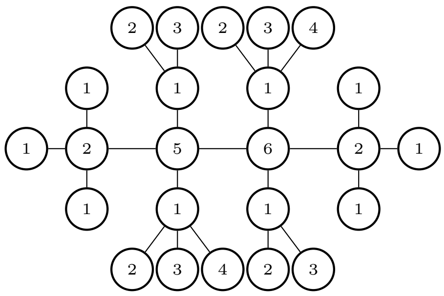


Figure 27. Tree T_{16}.

Observation 3.24. *T_{16} is the minimal tree in the class $\mathcal{D}_5(2,2)$ with both centers having a degree three neighbor. Therefore, T_{16} is 6-χ_ρ-critical in $\mathcal{D}_5(2,2)$ with both centers having a degree three neighbor.*

Now we are ready to state the characterization of packing chromatic number of trees in $\mathcal{D}_5(>0,>0)$.

Theorem 3.25. *Let T be a tree of diameter 5. Let v_1, v_2 be the two centers of T. Let m_1, m_2 be the number of degree three vertices adjacent to v_1, v_2, respectively. Let L_1 and L_2 be the total number of large-degree vertices adjacent to v_1 and v_2, respectively (excluding v_1 and v_2 themselves). Suppose that $L_1 > 0$ and $L_2 > 0$, then*

$$\chi_\rho(T) = \begin{cases} (L_1-2)+(L_2-2)+6, & \text{if } T_{16} \subset T, \\ (L_1-2)+(L_2-1)+5, & \text{if } T_{16} \not\subset T,\ T_{15} \subset T \text{ and } m_1 = 0, \\ (L_1-2)+(L_2-1)+4, & \text{if } T_{15}, T_{16} \not\subset T,\ T_{14} \subset T \text{ and } m_1 = 0, \\ (L_1-1)+(L_2-1)+5, & \text{if none of the above and } T_{13} \subset T, \\ (L_1-1)+(L_2-1)+4, & \text{if none of the above and } (T_{11} \subset T \text{ or} \\ & T_{12} \subset T), \\ (L_1-1)+(L_2-1)+3, & \text{otherwise.} \end{cases}$$

Proof. Let $T \in \mathcal{D}_5(L_1, L_2)$ be any tree, with $L_1 > 0$ and $L_2 > 0$. Using Observation 3.15, 3.16, 3.20 and Lemma 3.23 we can find a subtree T' of tree T with at most 2 large neighbors for each center such that

$$\chi_\rho(T) = \begin{cases} \chi_\rho(T') + (L_1-2) + (L_2-2), & \text{if } T' \in \mathcal{D}_5(2,2), \\ \chi_\rho(T') + (L_1-2) + (L_2-1), & \text{if } T' \in \mathcal{D}_5(2,1), \\ \chi_\rho(T') + (L_1-1) + (L_2-1), & \text{if } T' \in \mathcal{D}_5(1,1). \end{cases}$$

Thus it is sufficient prove that the packing chromatic number of T' satisfies

$$\chi_\rho(T') = \begin{cases} 6, & \text{if } T_{16} \subset T', \\ 5, & \text{if } T_{16} \not\subset T',\ T_{15} \subset T' \text{ and } m_1 = 0, \\ 4, & \text{if } T_{15}, T_{16} \not\subset T',\ T_{14} \subset T' \text{ and } m_1 = 0, \\ 5, & \text{if none of the above and } T_{13} \subset T', \\ 4, & \text{if none of the above and } (T_{11} \subset T' \text{ or } T_{12} \subset T'), \\ 3, & \text{otherwise.} \end{cases}$$

The lower bounds for each of the cases have been established by the discussion on T_{11}–T_{15} and by Observation 3.24. Moreover, for any tree T' containing T_{16}, extend the packing coloring of T_{16} in Figure 27, to T' by the following rule: every non-center neighbor to a center $\to 1$, every pendant vertex at distance two from center $\to \{2, 3\}$. This is a valid packing coloring which proves the upper bound when T' contains T_{16}.

When $T' \in \mathcal{D}_5(2,2)$ but $T_{16} \not\subset T'$, then by Lemma 3.23, we can reduce T' to the case in $\mathcal{D}_5(2,1)$. Suppose the number of degree three vertices in T' adjacent to v_1 is zero, then a good choice for packing coloring is to assign the centers

$\to 4, 5$, the large vertices $\to 2, 3$, every pendant vertex at distance two from v_1 (except leaves of large vertices and leaves of v_2) $\to 2$, every pendant vertex at distance two from v_2 (except leaves of large vertices and leaves of v_1) $\to 2, 3$ and remaining vertices $\to 1$. In this case, $\chi_\rho(T') \leq 5$. Moreover, by Lemma 3.21, if T' does not contain T_{15}, then $\chi_\rho(T') \leq 4$. As, any tree in $\mathcal{D}_5(2, 1)$ will contain T_{14}, T' will have packing chromatic number 4.

Now, suppose the number of degree three vertices in T' adjacent to v_1 is non-zero, then by Observation 3.20, we can reduce T' to the case in $\mathcal{D}_5(1, 1)$. A good choice of coloring in this case is to assign the centers $\to 4, 5$, the large vertices $\to 2$, every pendant vertex at distance two from v_1 (except leaves of large vertices and leaves of v_2) $\to 2, 3$, every pendant vertex at distance two from v_2 (except leaves of large vertices and leaves of v_1) $\to 2, 3$ and remaining vertices $\to 1$. Thus, we have $\chi_\rho(T') \leq 5$. By Proposition 3.18, we can infer that if T' does not contain T_{13}, then the upper bound for packing chromatic number is 4. Finally, if T' does not contain T_{11} or T_{12}, then 3 colors are sufficient for packing coloring T'. ■

Theorems 3.4, 3.14 and 3.25 utilize structural properties of the trees with diameter 5, which allow us to determine its packing chromatic number.

Remark 3.26. Suppose the adjacency list representation (refer to [14]) of a diameter 5 tree is given. Its centers can be determined as follows. Start by removing the pendant vertices, that is, remove all vertices that have adjacency list of size one. Repeating the above step once again, leaves a path, whose vertices are the two centers of the tree. Thus, centers can be identified in $O(n^2)$, where n is the number of vertices of the tree.

Moreover, we also check for containment of a diameter 5 tree in an another tree of diameter 5. Since the centers can be obtained using Remark 3.26 in polynomial time for both the trees, we can implement the Depth First Search (DFS) algorithm (refer to [14]) to compare the structure of the two trees and determine containment. Thus, for any tree of diameter 5, its packing chromatic number can be determined in polynomial time with respect to the number of vertices.

4. s-Shifted Packing Coloring of Infinite Path

In this section, we give an upper bound for the s-shifted packing chromatic number of an infinite path. An infinite path P_∞ is a tree with vertex set $V(P_\infty) = \mathbb{Z}$ and the edge set $E(P_\infty) = \{\{i, i+1\} \mid i \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{Z}$. The infinite path can be packing colored using the following packing coloring scheme

$$\ldots 1\ 2\ 1\ 3\ 1\ 2\ 1\ 3 \ldots.$$

Therefore, the packing chromatic number of the infinite path is $\chi_\rho(P_\infty) = 3$. For a positive integer s, fix the color palette $\mathbf{T}_s = \{s, s+1, \ldots\}$. Define the *s-shifted packing chromatic number* of any graph G, denoted by $\chi_\rho^s(G)$, as the least positive integer k, such that the graph G can be packing colored using the colors $\{s, s+1, \ldots, k\} \subset \mathbf{T}_s$.

For example, a 2-shifted packing coloring of path P_{18} is given by

$$2\ 3\ 5\ 2\ 4\ 6\ 2\ 3\ 7\ 2\ 4\ 5\ 2\ 3\ 6\ 2\ 4\ 7.$$

In the following, we will introduce a specialized packing coloring for paths, which will aid in providing an upper bound for the shifted packing coloring of P_∞.

Definition. A packing coloring of a path P_n is called cyclic, if it can be extended to a packing coloring for the cycle C_n and colors used in the packing coloring are from the set $\{1, \ldots, n-1\}$.

Example 4.1. The packing coloring of P_4 given by 1213 is cyclic while 1231 and 1214 are some packing colorings of P_4 which are not cyclic.

Observe that P_n, for $n \in \{1, 2, 3\}$, does not have a cyclic packing coloring.

Lemma 4.2. *Let $P \subset P_\infty$ be any induced finite subpath having at least four vertices. Suppose there is a packing coloring of P which is cyclic, then it can be extended to a packing coloring of P_∞.*

Proof. Without loss of generality, let $V(P) = \{1, \ldots, n\}$, $n \geq 4$. Let f be a cyclic packing coloring of P. Define a function $\tilde{f}$ for P_∞ given by

$$\tilde{f}(m) := f(x), \text{ where } m = x + in \text{ for some } i \in \mathbb{Z} \text{ and } x \in \{1, \ldots, n\}.$$

We prove that $\tilde{f}$ is a packing coloring. Let m_1, m_2 be any two distinct vertices of P_∞ such that $\tilde{f}(m_1) = \tilde{f}(m_2)$. Let $m_1 = x + i_1 n$ and $m_2 = y + i_2 n$, where x, y are vertices of P and i_1, i_2 are some integers. If $x = y$, then the distance between m_1 and m_2 in P_∞ is at least n. Since f is cyclic, we have $\mathrm{d}(m_1, m_2) \geq n > f(x)$. If $x \neq y$ and $i_1 = i_2$, then $\mathrm{d}(m_1, m_2) = \mathrm{d}(x, y) > f(x) = f(y)$, since f is a packing coloring of P. If $x \neq y$ and $i_1 \neq i_2$, then $\mathrm{d}(m_1, m_2) \geq \min\{\mathrm{d}(x, y), n - \mathrm{d}(x, y)\} > f(x) = f(y)$, since f is a packing coloring of the cycle on n-vertices. Thus $\tilde{f}$ is the required packing coloring of P_∞ which is induced by f. ■

Note that the 2-shifted packing coloring of P_{18} given above is cyclic. This is because, in cycle C_{18} with the same coloring, distance between any two vertices colored 2 is 3, distance between any two vertices colored $3, 4$ is 6 and distance between any two vertices colored $5, 6, 7$ is 9. By Lemma 4.2, it can be extended to a packing coloring of P_∞. Therefore, an upper bound for 2-shifted packing chromatic number of P_∞ is 7, that is, P_∞ can be packing colored using the colors from $\{2, \ldots, 7\}$. Sloper [13] showed that there is no packing coloring using only colors 2 through 6 in a path of length 35 or greater. Hence $\chi_\rho^2(P_\infty) = 7$.

We would like to provide an upper bound for the s-shifted packing chromatic number of P_∞ for any positive integer s. It is sufficient to provide a valid packing coloring with colors chosen from the modified color palette, $\mathbf{T}_s$. The strategy involves constructing a cyclic packing coloring for a finite length path that can seamlessly expand to an infinite path using Lemma 4.2. The coloring scheme for shifted packing coloring for a finite path is given in the following lemma.

Theorem 4.3. *Let $s \in \mathbb{Z}$ be a positive integer and let P be a path with $12(s+1)$ vertices such that $V(P) = \{1, \ldots, 12(s+1)\}$. Then*

$$\chi_\rho^s(P) \leq 3\left\lfloor \frac{s+1}{2} \right\rfloor + 4\left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{s}{6} \right\rfloor.$$

Proof. To establish an upper bound for the s-shifted packing chromatic number of P, it is sufficient to construct a packing coloring for P using colors from the palette $\mathbf{T}_s = \{s, s+1, \ldots\}$. Construct the packing coloring f as follows.

- Set $f(1 + i(s+1)) = s$ for all $i \in \{0, \ldots, 11\}$. This ensures that vertices of P are optimally packing colored using the color s.
- After each vertex colored s, there is a sequence of s uncolored vertices before encountering another vertex colored s under f. Refer to this group of s consecutive uncolored vertices as a **block** in P. Thus, there are 12 such blocks in P.
- For positive integers i, j, l satisfying $1 \leq j \leq j + l \leq 12$ and $1 \leq i \leq s$, the distance between the i^{th} vertex of the j^{th} block and i^{th} vertex of $(j+l)^{th}$ block is $l(s+1)$. See Figure 28 for reference.

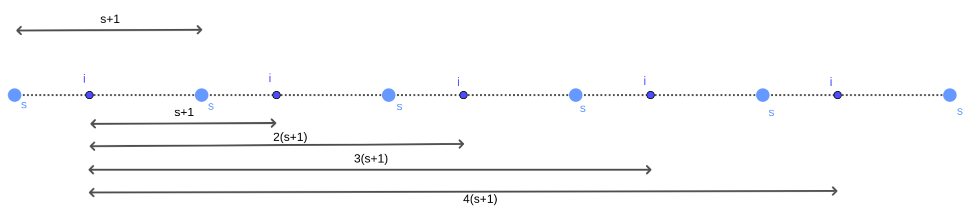


Figure 28. Distance between the i^{th} vertices.

• Coloring every second block.

For every $1 \leq j \leq 10$ and $1 \leq i \leq s$, the distance between the i^{th} vertex of j^{th} block and i^{th} vertex of $(j+2)^{nd}$ block is $2(s+1)$. Let $\mathbf{A} = \{s+1, \ldots, 2s+1\} \subset \mathbf{T}_s$.

To assign colors to vertices, partition $\mathbf{A}$ into two subsets $\mathbf{A}_1, \mathbf{A}_2$, of equal size and collect the remaining element of $\mathbf{A}$ in $\mathbf{A}'$, if any. That is,

$$\mathbf{A} = \mathbf{A}_1 \sqcup \mathbf{A}_2 \sqcup \mathbf{A}'$$

where $|\mathbf{A}_1| = |\mathbf{A}_2| = \left\lfloor \frac{s+1}{2} \right\rfloor$. If $\mathbf{A}'$ is non-empty, then assume that $\mathbf{A}' = \{2s+1\}$.

For $k \in \{1, 2\}$, in every alternate block, starting from the k^{th} block, assign the first $\left\lfloor \frac{s+1}{2} \right\rfloor$ uncolored vertices with colors from $\mathbf{A}_k$. If $s > 1$, then $\left\lfloor \frac{s+1}{2} \right\rfloor < s$ and hence there are $s - \left\lfloor \frac{s+1}{2} \right\rfloor = \left\lfloor \frac{s}{2} \right\rfloor$ uncolored vertices are remaining in each block.

• Coloring every third block.

For every $1 \leq j \leq 9$ and $1 \leq i \leq s$, the distance between the i^{th} vertex of j^{th} block and i^{th} vertex of $(j+3)^{rd}$ block is $3(s+1)$. Let $\mathbf{B} = \{2s+2, \ldots, 3s+2\} \cup \mathbf{A}' \subset \mathbf{T}_s$. To assign colors to uncolored vertices, partition $\mathbf{B}$ into three subsets $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$, of equal size and collect the remaining elements of $\mathbf{B}$ in $\mathbf{B}'$. That is,

$$\mathbf{B} = \mathbf{B}_1 \sqcup \mathbf{B}_2 \sqcup \mathbf{B}_3 \sqcup \mathbf{B}'$$

where $|\mathbf{B}_1| = |\mathbf{B}_2| = |\mathbf{B}_3| = \left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{s}{6} \right\rfloor$. If $\mathbf{B}'$ is non-empty, then assume that $\mathbf{B}'$ contains the largest elements of $\mathbf{B}$.

For $k \in \{1, 2, 3\}$, in every third block, starting from the k^{th} block, assign the first $\left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{s}{6} \right\rfloor$ many uncolored vertices with colors from $\mathbf{B}_k$. If $s > 5$, then $s - (|\mathbf{A}_1| + |\mathbf{B}_1|) = s - (\left\lfloor \frac{s+1}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{s}{6} \right\rfloor) = \left\lfloor \frac{s}{6} \right\rfloor$ uncolored vertices are remaining in each block.

• Coloring every fourth block.

For every $1 \leq j \leq 8$ and $1 \leq i \leq s$, the distance between the i^{th} vertex of j^{th} block and i^{th} vertex of $(j+4)^{th}$ block is $4(s+1)$. Let $\mathbf{C} = \{3s+3, \ldots, 4s+3\} \cup \mathbf{B}' \subset \mathbf{T}_s$. To assign colors to uncolored vertices, partition $\mathbf{C}$ into four subsets $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$, of size equal to $\left\lfloor \frac{s}{6} \right\rfloor$ and collect the remaining elements of $\mathbf{C}$ in $\mathbf{C}'$. That is,

$$\mathbf{C} = \mathbf{C}_1 \sqcup \mathbf{C}_2 \sqcup \mathbf{C}_3 \sqcup \mathbf{C}_4 \sqcup \mathbf{C}'$$

where $|\mathbf{C}_1| = |\mathbf{C}_2| = |\mathbf{C}_3| = |\mathbf{C}_4| = \left\lfloor \frac{s}{6} \right\rfloor$. Let the subsets $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ have the smallest $4 \times \left\lfloor \frac{s}{6} \right\rfloor$ elements of $\mathbf{C}$. Set $\mathbf{C}' = \mathbf{C} \setminus \cup_k \mathbf{C}_k$.

For $k \in \{1, 2, 3, 4\}$, in every fourth block, starting from the k^{th} block, assign the remaining $\left\lfloor \frac{s}{6} \right\rfloor$ many uncolored vertices with colors from $\mathbf{C}_k$. Thus we have $|\mathbf{A}_1| + |\mathbf{B}_1| + |\mathbf{C}_1| = (\left\lfloor \frac{s+1}{2} \right\rfloor) + (\left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{s}{6} \right\rfloor) + (\left\lfloor \frac{s}{6} \right\rfloor) = s$, for all positive $s \in \mathbb{Z}$. Therefore, the path P can be packing colored using colors from $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ for all positive $s \in \mathbb{Z}$.

For any positive integer s, the total number of colors used for packing coloring of P, excluding the color s, using the above coloring scheme is $2|\mathbf{A}_1| + 3|\mathbf{B}_1| +$

$4|\mathbf{C}_1| = 2(\lfloor\frac{s+1}{2}\rfloor) + 3(\lfloor\frac{s}{2}\rfloor - \lfloor\frac{s}{6}\rfloor) + 4(\lfloor\frac{s}{6}\rfloor) = 2(\lfloor\frac{s+1}{2}\rfloor) + 3(\lfloor\frac{s}{2}\rfloor) + \lfloor\frac{s}{6}\rfloor$. Therefore, the s-shifted packing chromatic number of P is bounded above by

$$\chi_\rho^s(P) \leq s + 2|\mathbf{A}_1| + 3|\mathbf{B}_1| + 4|\mathbf{C}_1| = 3\left\lfloor\frac{s+1}{2}\right\rfloor + 4\left\lfloor\frac{s}{2}\right\rfloor + \left\lfloor\frac{s}{6}\right\rfloor.$$

■

Let us illustrate our shifted packing coloring scheme for a finite path using an example.

Example 4.4. Consider $s = 7$. We want to provide a 7-shifted packing coloring for path P on $12 \times 8 = 96$ vertices using colors from the color palette $\mathbf{T}_7 = \{7, 8, \ldots\}$. Coloring schema is as follows.

- Let f be the packing coloring that we want to define as a function from $V(P) \mapsto \mathbf{T}_7$. Set $f(1 + 8i) = 7$ for all $i \in \{0, \ldots, 11\}$.
- There are 12 blocks in P with each block containing 7 uncolored vertices.
- We will represent the coloring f of P in 4 lines, where each line consists of 24 symbols/colors, with _ representing that the vertex is yet to be assigned a color under f. The leftmost vertex in a line follows immediately after the rightmost vertex in the preceding line.

7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _

7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _

7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _

7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _ 7 _ _ _ _ _ _ _

- For $1 \leq j \leq 10$ and $1 \leq i \leq 7$, the distance between the i^{th} vertex of the j^{th} block and the i^{th} vertex in the $j + 2^{nd}$ block is 16.
- We can assign colors in $\mathbf{A} = \{8, \ldots, 15\} \subset \mathbf{T}_7$, to i^{th} vertex in alternative blocks, for some $1 \leq i \leq \lfloor\frac{7+1}{2}\rfloor$. Partition $\mathbf{A}$ into $\mathbf{A}_1$ and $\mathbf{A}_2$, such that $|\mathbf{A}_1| = |\mathbf{A}_2| = \lfloor\frac{7+1}{2}\rfloor = 4$. Here, we have $\mathbf{A}_1 = \{8, 9, 10, 11\}$ and $\mathbf{A}_2 = \{12, 13, 14, 15\}$.
- For $j \in \{1, 2\}$, in every alternate block, starting from the j^{th} block, assign the first $\lfloor\frac{7+1}{2}\rfloor = 4$ vertices with colors from $\mathbf{A}_j$.

7 8 9 10 11 _ _ _ 7 12 13 14 15 _ _ _ 7 8 9 10 11 _ _ _

7 12 13 14 15 _ _ _ 7 8 9 10 11 _ _ _ 7 12 13 14 15 _ _ _

7 8 9 10 11 _ _ _ 7 12 13 14 15 _ _ _ 7 8 9 10 11 _ _ _

7 12 13 14 15 _ _ _ 7 8 9 10 11 _ _ _ 7 12 13 14 15 _ _ _

- For $1 \leq j \leq 9$ and $1 \leq i \leq 7$, the distance between the i^{th} vertex of the j^{th} block and the i^{th} vertex in the $j + 3^{rd}$ block is 24.

- We can assign colors in $\mathbf{B} = \{16, \ldots, 23\} \subset \mathbf{T}_7$, to i^{th} vertex in every third block, for some $\lfloor \frac{7+1}{2} \rfloor = 4 < i \leq 4 + \lfloor \frac{7}{2} \rfloor - \lfloor \frac{7}{6} \rfloor = 6$. Partition $\mathbf{B}$ into $\mathbf{B}_1$,$\mathbf{B}_2$,$\mathbf{B}_3$ and $\mathbf{B}'$, such that $|\mathbf{B}_1| = |\mathbf{B}_2| = |\mathbf{B}_3| = \lfloor \frac{7}{2} \rfloor - \lfloor \frac{7}{6} \rfloor = 2$. Here, we have $\mathbf{B}_1 = \{16, 17\}$, $\mathbf{B}_2 = \{18, 19\}$, $\mathbf{B}_3 = \{20, 21\}$ and $\mathbf{B}' = \{22, 23\}$.
- For $j \in \{1, 2, 3\}$, in every third block, starting from the j^{th} block, assign the first $\lfloor \frac{7}{2} \rfloor - \lfloor \frac{7}{6} \rfloor = 2$ uncolored vertices with colors from $\mathbf{B}_j$.

  7 8 9 10 11 16 17 _ 7 12 13 14 15 18 19 _ 7 8 9 10 11 20 21 _

  7 12 13 14 15 16 17 _ 7 8 9 10 11 18 19 _ 7 12 13 14 15 20 21 _

  7 8 9 10 11 16 17 _ 7 12 13 14 15 18 19 _ 7 8 9 10 11 20 21 _

  7 12 13 14 15 16 17 _ 7 8 9 10 11 18 19 _ 7 12 13 14 15 20 21 _
- For $1 \leq j \leq 8$ and $1 \leq i \leq 7$, the distance between the i^{th} vertex of the j^{th} block and the i^{th} vertex in the $j + 4^{th}$ block is 32.
- We can assign colors in $\mathbf{C} = \mathbf{B}' \cup \{24, \ldots, 32\} \subset \mathbf{T}_7$, to i^{th} vertex in every fourth block, for some $1 \leq i \leq 7$. Partition $\mathbf{C}$ into $\mathbf{C}_1$,$\mathbf{C}_2$,$\mathbf{C}_3$,$\mathbf{C}_4$ and $\mathbf{C}'$, such that $|\mathbf{C}_1| = |\mathbf{C}_2| = |\mathbf{C}_3| = |\mathbf{C}_4| = \lfloor \frac{7}{6} \rfloor = 1$. Here, we have $\mathbf{C}_1 = \{22\}$, $\mathbf{C}_2 = \{23\}$, $\mathbf{C}_3 = \{24\}$, $\mathbf{C}_4 = \{25\}$ and $\mathbf{C}' = \{26, \ldots, 32\}$.
- For $j \in \{1, 2, 3, 4\}$, in every fourth block, starting from the j^{th} block, assign the $\lfloor \frac{7}{6} \rfloor = 1$ uncolored vertex with color from $\mathbf{C}_j$.

  7 8 9 10 11 16 17 22 7 12 13 14 15 18 19 23 7 8 9 10 11 20 21 24

  7 12 13 14 15 16 17 25 7 8 9 10 11 18 19 22 7 12 13 14 15 20 21 23

  7 8 9 10 11 16 17 24 7 12 13 14 15 18 19 25 7 8 9 10 11 20 21 22

  7 12 13 14 15 16 17 23 7 8 9 10 11 18 19 24 7 12 13 14 15 20 21 25

Thus f is a packing coloring of the path P. The 7-shifted packing chromatic number of P is bounded above by 25.

In the following theorem, we give a packing coloring of infinite path using colors from the color palette $\mathbf{T}_s$, for some positive integer s. Hence, we will establish an upper bound for the s-shifted packing chromatic number of the infinite path P_∞. The coloring scheme for the packing coloring of P_∞ will use the similar idea as demonstrated in the example.

Theorem 4.5. *Let P_∞ be an infinite path. Let $s \in \mathbb{Z}$ be a positive integer. Then,*

$$\chi_\rho^s(P_\infty) \leq 3 \left\lfloor \frac{s+1}{2} \right\rfloor + 4 \left\lfloor \frac{s}{2} \right\rfloor + \left\lfloor \frac{s}{6} \right\rfloor .$$

Proof. The packing coloring obtained in Lemma 4.3 is a cyclic packing coloring. In the cycle with $12(s+1)$ vertices, the same coloring ensures that the distance between any two vertices colored s is $s+1$. Additionally, the distance between any two vertices colored with one of the colors from $\mathbf{A}$ is $2(s+1)$, from $\mathbf{B}$ is $3(s+1)$, and from $\mathbf{C}$ is $4(s+1)$. The required result follows by Lemma 4.2. ∎

5. Role of Large-Degree Vertices in Packing Coloring of a Tree

In this section, we obtain an upper bound on the packing chromatic number of a tree, based on the number of large-degree vertices. For a tree T, let $\mathcal{L}(T)$ denote the set of all large vertices in T, i.e.,

$$\mathcal{L}(T) = \{v \in V(T) \mid \deg(v) \geq 4\}.$$

To begin with, we obtain an upper bound for the packing chromatic number of a lobster tree in terms of the number of large-degree vertices using Theorem 4.5.

We set up the notations as follows. For a lobster tree T, let ω be its spine. Let $\mathcal{K} = \mathcal{L}(T) \setminus V(\omega)$ and let $V'(\omega)$ be the set of all vertices in the spine which is adjacent to at least one of the large vertices in $\mathcal{K}$, i.e.,

$$V'(\omega) = \{v \in V(\omega) \mid N(v) \cap \mathcal{K} \neq \emptyset\} \subset V(\omega).$$

Now define $\zeta = |\mathcal{K}| - |V'(\omega)|$.

Proposition 5.1. *Let T be any lobster tree and let ζ be as defined above. Then,*

$$\chi_\rho(T) \leq \zeta + 14.$$

Proof. Let T be a lobster with ω as the spine. Let $\mathcal{K}$ and $V'(\omega)$ be defined as earlier for lobster T. Assign the following coloring: pendant vertices adjacent to spine $\omega \to 1$, pendant vertices adjacent to vertices in $\mathcal{K} \to 1$, all the non-spine vertices having degree two or three $\to 1$, those pendant vertices adjacent to a non-spine vertex of degree two or degree three $\to \{2, 3\}$, the spine $\omega \to \{4, \ldots, 14\}$ (as spine is a path, we shall use Theorem 4.5) and finally it remains to assign colors to large vertices. For each vertex v in $V'(\omega)$, assign for at most one large vertex neighboring v with 2 and for the remaining large neighbors of v assign unique colors. This is a valid packing coloring for T and therefore, we obtain an upper bound for the packing chromatic number of T as $\chi_\rho(T) \leq \zeta + 14$. ■

In Figure 29, we illustrate the packing coloring for the lobster tree T as described in Proposition 5.1. For the tree T, $\zeta = |\mathcal{K}| - |V'(\omega)| = 4 - 3 = 1$. Consequently, we derive an upper bound for the packing chromatic number of T, which is $\chi_\rho(T) \leq 15$.

Remark 5.2. In [1], the authors provide an upper bound for the packing chromatic number of any lobster, T, expressed as an exponential function on the maximum amongst the number of large neighbors for a vertex on the lobster's spine. This bound is particularly effective for lobsters with a small number of large neighbors distributed across various vertices of the spine. In other words, for vertices $v \in V'(\omega)$, the quantity $|N(v) \cap \mathcal{K}|$ is bounded by sub-linear factors of

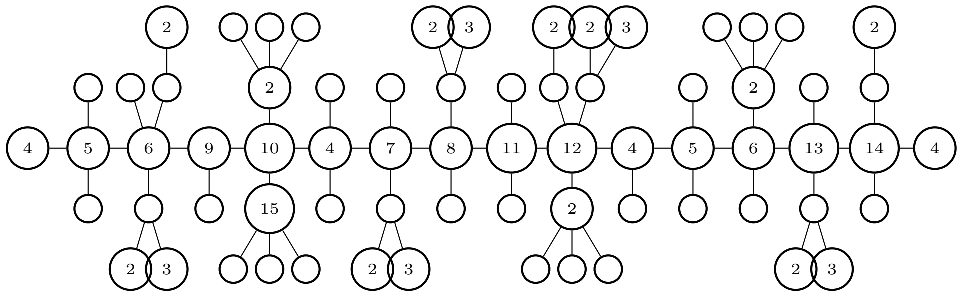


Figure 29. A packing coloring of lobster tree T.

$V(T)$. However, this bound is less effective when many large neighbors are adjacent to a single spine vertex while other spine vertices have fewer large neighbors. In such cases, the bound in Proposition 5.1 is more suitable.

The number of large vertices in any tree can also provide an upper bound for its packing chromatic number. This can be done with the support of a crucial result of Sloper, in [13], where they give a packing coloring for any binary tree using at most seven colors.

Theorem 5.3 (Theorem 15, [13]). *Any complete binary tree can be packing colored with* 7 *colors or fewer.*

We illustrate the packing coloring of complete binary tree of height three below.

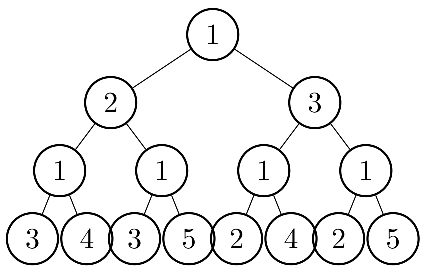


Figure 30. Packing coloring of complete binary tree of height three using Sloper's coloring.

Sloper's coloring schema utilizes color 6 at height five from the root and color 7 at height seven from the root. The complete coloring schema can be found in [13].

Note that binary trees do not contain any large-degree vertex. Using Theorem 5.3, we now establish an upper bound for the packing chromatic number, $\chi_\rho(T)$, based on the number of large-degree vertices it contains. Let $\eta = |\mathcal{L}(T)|$.

Theorem 5.4. *For any tree T, $\chi_\rho(T) \leq 2\eta + 7$.*

Proof. Let T be a tree. Pick any vertex $v \in V(T)$ and root the tree at v. Call the rooted tree T'. Identify all maximal vertex disjoint binary subtrees of T', i.e., subtrees in which every vertex has degree at most 3 and which are not contained in any larger such subtree. Equivalently, each connected component remaining after the removal of all the large vertices from the rooted tree, is a maximal binary subtree. Assign packing coloring to each of these subtrees using Theorem 5.3 with at most 7 colors. Now it remains to show that the packing coloring of any two distinct binary subtrees of T' do not conflict and we need to assign colors to the large vertices in T'.

Furthermore, in the packing coloring schema of a binary tree in Theorem 5.3, the minimum distance between the root and the first occurence of a vertex colored with a color from $\{1, \dots, 7\}$ is listed in Table 8.

Color	Distance from the root
1	0
2	1
3	1
4	3
5	3
6	5
7	7

Table 8. Minimum distances.

Observe that any two maximal binary subtrees B_1 and B_2 in T are at distance at least two apart. Moreover, if the distance between B_1 and B_2 is exactly two, then they are connected via a common large-degree vertex. By distance conditions listed in the table, there will be no conflicts between coloring both subtrees B_1, B_2 with the same coloring as in Theorem 5.3. Assign unique colors to all the large vertices in T'. If the parent of large vertex is again a large vertex, then both receive unique colors. If the parent of the large vertex is not a large vertex, then it is a part of some binary subtree.

The color assigned to this parent of large vertex, if it is 3 or 5, then there can be a conflict in the coloring. We modify the packing coloring such that every parent of a large vertex is given a unique color again. This results in a valid packing coloring for the tree and thus, we obtain the required bound. ■

While we believe that the above bound is sharp, we note that there are trees T for which $\chi_\rho(T)$ is a constant with η being unbounded. For example, a caterpillar where every vertex in the spine is a large-degree vertex can be packing colored using exactly 7 colors. In the following, we obtain a more general class of trees for which the packing chromatic number is bounded, with η being unbounded.

A tree T is said to be *r-sparse*, if for every $u \neq v \in \mathcal{L}(T)$, $\mathrm{d}(u, v) \geq r+1$. We show that trees that are r-sparse for sufficiently large r have packing chromatic number independent of η.

Corollary 5.5. *For $r \geq 8$, any r-sparse tree T has $\chi_\rho(T) \leq 8$.*

Proof. Utilize the packing coloring scheme illustrated in Theorem 5.4. Since the tree T is r-sparse, for $r \geq 8$, then all the large vertices can be assigned the color 8. Thus, we have $\chi_\rho(T) \leq 8$. ■

6. Bounded Width Trees

We have observed that the packing chromatic number is constant for caterpillars but unbounded for lobsters. Since lobsters include caterpillars, we identify a new class of trees in this section that encompasses caterpillars while maintaining a bounded packing chromatic number.

Let T be a tree. We define the *width* of T as follows. For a vertex v, let $T(v)$ denote the breadth-first tree of T starting at v. For $1 \leq i \leq n$, the i-th layer in $T(v)$ is the set of vertices in $T(v)$ that are at a distance of exactly i from v. Let w_v be the maximum number of non-leaf nodes in any layer of $T(v)$. Define the width of T as the minimum value of w_v, where the minimum is taken over all vertices v in T.

For $w \geq 1$, let $\mathcal{T}_w$ denote the set of all trees of width at most w. Observe that $\mathcal{T}_1$ is exactly the set of all caterpillars. Therefore, $\chi_\rho(\mathcal{T}_1) = 7$.

Examples of trees with widths of two and three are provided. Note that neither of these trees are lobster trees. Additionally, there exists a class of lobsters with unbounded width. Both the class of bounded-width trees and the class of lobsters include caterpillars, though they are not subsets of each other.

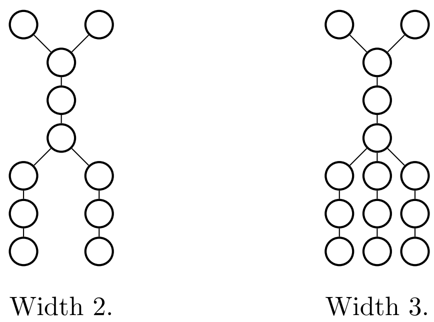


Figure 31.

Theorem 6.1. *For any $T \in \mathcal{T}_w$, we have $\chi_\rho(T) \leq 2^{2w+1}$.*

Proof. Let T be a tree of width w. Let v be a vertex in T such that the BFS tree of T rooted at v has at most w non-leaf vertices in any layer.

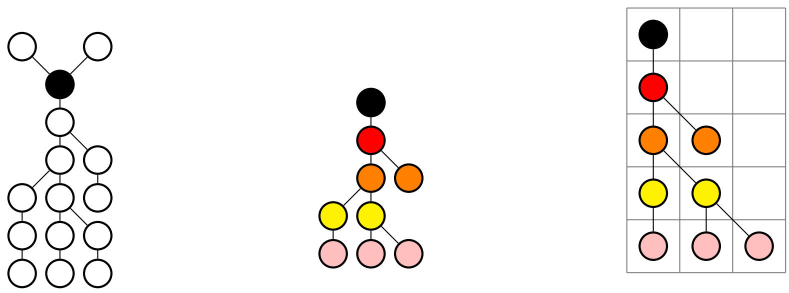
Firstly, we assign the color 1 to every leaf. Now, it is sufficient to provide a packing coloring of the non-leaf vertices of T, using colors from the palette $\{2, 3, \dots\}$.

Let d be the height (i.e., the maximum length of any root to leaf path) of T, excluding the leaves of T. The non-leaf vertices of T can be drawn into a $(d+1) \times w$ grid as follows.

Let $\mathcal{G}$ be the $(d+1) \times w$ grid, that is, $\mathcal{G}$ has $d+1$ rows, with exactly w squares. Place the root of the tree T on the top row leftmost entry. Next, place all the non-leaf vertices at layer i of T in the squares on $i+1$-th row in the order given by a BFS traversal starting from v, filling the leftmost squares first.

For example, consider the following width 3 tree with vertex shaded black as the root. The height of this tree after deleting the leaf nodes is 4. Therefore, we embed the rooted tree into a grid $\mathcal{G}$ of size 5×3 as illustrated.

For $1 \leq i \leq w$, let P_i denote the subgraph of T in the i-th vertical column of the grid $\mathcal{G}$. Note that each P_i comprises a collection of path graphs and is a subgraph of P_∞. Consequently, any packing coloring of P_∞ can naturally be restricted to obtain a packing coloring for each P_i. To maintain the distance property of the packing coloring in the tree T, we shall utilize Theorem 4.5 and color each P_i with a disjoint color palette.



A tree of Width 3. BFS Traversal of the rooted tree without leaves. Grid $\mathcal{G}$ containing the rooted tree.

Figure 32.

In order to obtain a packing coloring of T, use the following coloring schema. Let $s_1 = \chi_\rho^2(P_\infty) = 7$, and $s_i = \chi_\rho^{s_{i-1}+1}(P_\infty)$ for $i > 1$. Color P_1 with colors from $\{2, \dots, s_1\}$ treating it as a path. Similarly, color P_i with colors from $\{s_{i-1}+1, \dots, s_i\}$ treating it as a path. Note that, every non-leaf vertex of T is part of exactly one of the P_i's. Color every leaf vertex with 1. Let f be the resulting coloring of T. Consider any two non-leaf vertices u and v in T. Suppose $\mathrm{d}(u, v) = j$. If u and v do not lie in any P_i, clearly, u and v get different colors. If they lie in some P_i, then either the color of u and v is less than j or they have different colors. Thus f is a valid packing coloring of T. The number of colors required for this packing coloring is s_w, which can be obtained using a simple induction on Theorem 4.5. Observe that, we have

$$\chi_\rho^s(P_\infty) \leq 3\left\lfloor\frac{s+1}{2}\right\rfloor + 4\left\lfloor\frac{s}{2}\right\rfloor + \left\lfloor\frac{s}{6}\right\rfloor \leq 4s.$$

As a base case, we also have that $s_1 = 7 < 2^3$, and thus

$$s_w = \chi_\rho^{s_{w-1}}(P_\infty) \leq 4s_{w-1} \leq 2^{2w+1}$$

by induction. ∎

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