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GENERALIZED TURÁN PROBLEMS FOR DISJOINT EVEN WHEELS, AND FOR DISJOINT BOWTIES

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Abstract

Given graphs T and F, the *generalized Turán number* $ex(n,T,F)$ is the maximum possible number of copies of T in an F-free graph on n vertices. Let W_n be the *wheel graph* obtained from a cycle C_{n-1} and an extra vertex v by joining v and all vertices of C_{n-1}. Let $\ell \cdot F$ be the graph consisting of ℓ vertex-disjoint copies of F. A graph consisting of two triangles which intersect in exactly one common vertex is called a *bowtie* and denoted by F_2.

In this paper, we determine the exact values of $ex(n, K_r, (\ell+1)\cdot W_{2k})$ for $4 \leq r \leq \ell+3$, and $ex(n, K_r, (\ell+1)\cdot F_2)$ for $3 \leq r \leq \ell+2$, and characterize all their extremal graphs.

Keywords: generalized Turán number, extremal graph, even wheel, bowtie.

2020 Mathematics Subject Classification: 05C35.

1. Introduction

We basically follow the most common graph-theoretical terminology and notation and for concepts not defined here we refer the reader to [2]. All graphs in this paper are simple, finite and undirected.

Let $G = (V,E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $e(G)$ to denote the number of edges of G and use $d(v)$ to denote the degree of v. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and let $G-S$ denote the subgraph induced by $V(G)\backslash S$. For simplicity, we write $E(S)$ and $e(S)$ for $E(G[S])$ and $e(G[S])$, respectively. For $v \in V(G)$, let $N(v,S)$ denote the set of neighbors of v in S, and let $\deg(v,S) = |N(v,S)|$. Let $G[S,T]$ denote the bipartite subgraph induced by the edges with one end in S and the other in T, and let $e(S,T) = e(G[S,T])$.

For any two vertex disjoint graphs G_1 and G_2, let $G_1 \vee G_2$ denote the graph obtained from $G_1 \cup G_2$ by adding all edges between $V(G_1)$ and $V(G_2)$. Let $\mathcal{N}_r(G)$ denote the number of r-cliques in G. A graph G is called *edge-critical* if there exists an edge e in G such that $\chi(G-e) < \chi(G)$, where $\chi(G)$ is the chromatic number of G. Let $T_r(n)$ denote the *Turán graph*, the complete r-partite graph on n vertices with r partition classes, each of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

For a graph F, we say a graph G is F-free if G does not contain a copy of F as a subgraph. The *Turán number* of F, denoted by $ex(n,F)$, is the maximum possible number of edges in an F-free graph on n vertices. In 1941, Turán [17] proved that $T_r(n)$ is the unique extremal graph of $ex(n, K_{r+1})$. In 2015, Füredi and Gunderson determined the Turán number of odd cycles.

Theorem 1 (Füredi and Gunderson [6]). *For $k \geq 2$ and $n \geq 4k-2$,*

$$ex(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Let T, F be two graphs. The *generalized Turán number* $ex(n,T,F)$ is the maximum possible number of copies of T in an F-free graph on n vertices. The study of generalized Turán problems was initiated by Alon and Shikheman [1], there are many results focus on the generalized Turán problems, see e.g. [9, 10, 13, 22].

Let $\ell \cdot F$ be the graph consisting of ℓ vertex-disjoint copies of F. In 1959, Erdős and Gallai [4] determined the Turán number of matchings, i.e., $ex(n, (\ell+1)\cdot K_2) = \max\{\binom{2\ell+1}{2}, (n-\ell)\ell + \binom{\ell}{2}\}$ for $n \geq 2\ell+1$. Recently in [11], Hou, Yang and Zeng determined the value of $ex(n, K_3, (\ell+1)\cdot C_{2k+1})$ for $\ell \geq 1$, $k \geq 1$. Zhang, Chen, Győri and Zhu [20] determined the value of $ex(n, K_r, (\ell+1)\cdot K_r)$ for $r \geq 3$, $\ell \geq 1$.

Let $k \geq 2$ and $p_1, \ldots, p_k \geq 1$ be integers. The *generalized theta graph* $\Theta(p_1, \ldots, p_k)$ consists of a pair of end vertices joined by k internally disjoint paths of lengths $p_1, \ldots, p_k$, respectively. Recently, Gao, Wu and Xue [7] determined the value of $ex(n, K_r, (\ell+1)\cdot F)$ for the edge-critical generalized theta graphs F. Specially, C_{2k+1} is an edge-critical generalized theta graph.

Let W_n be the *wheel graph* obtained from a cycle C_{n-1} and an extra vertex v by joining v and all vertices of C_{n-1}. If n is odd then we call W_n *odd wheel*, and we call W_n *even wheel* if n is even. In 2013, Dzido determined the exact value of the Turán problem of even wheels.

Theorem 2 (Dzido [3]). *For $k \geq 3$ and $n \geq 6k-10$,*

$$ex(n, W_{2k}) = \left\lfloor \frac{n^2}{3} \right\rfloor.$$

In 2021, Yuan [19] determined the exact value of the Turán number for odd wheel. Xiao and Zamora [18] determined the value of $ex(n,(\ell+1)\cdot W_{2k+1})$. Recently, Hou, Li, Liu, Yuan and Zhang [12] determined the value of $ex(n,(\ell+1)\cdot F)$ for edge-critical graph F with $\chi(F)\geq 3$, which also implies the value of $ex(n,(\ell+1)\cdot W_{2k})$ as the even wheel W_{2k} is 4-edge-critical.

In 2020, Ma and Qiu extended the result of Simonovits [16] by considering the generalized Turán number of edge-critical graphs.

Theorem 3 (Ma and Qiu [14]). *Let F be an edge-critical graph with $\chi(F)=r+1>m\geq 2$ and n be sufficiently large. Then the Turán graph $T_r(n)$ is the unique graph attaining the maximum number of K_m's in an F-free graph on n vertices.*

In the same paper, they also prove a stability result.

Theorem 4 (Ma and Qiu [14]). *Let F be a graph with $\chi(F)=r+1>m\geq 2$. If G is an n-vertex F-free graph with $\mathcal{N}_m(G)\geq\mathcal{N}_m(T_r(n))-o(n^m)$, then G can be obtained from $T_r(n)$ by adding and deleting $o(n^2)$ edges.*

In this paper, we further study the function of $ex(n,K_r,(\ell+1)\cdot F)$ by considering the case $F=W_{2k}$. Our first main result is the following.

Theorem 5. *Let $\ell\geq 1$, $k\geq 2$, and n be sufficiently large. If $4\leq r\leq \ell+3$, then*

$$\begin{aligned} ex(n,K_r,(\ell+1)\cdot W_{2k}) &= \binom{\ell}{r}+\binom{\ell}{r-1}(n-\ell) \\ &\quad+\binom{\ell}{r-2}\left\lfloor\frac{(n-\ell)^2}{3}\right\rfloor+\binom{\ell}{r-3}\mathcal{N}_3(T_3(n-\ell)), \end{aligned}$$

and $K_\ell\vee T_3(n-\ell)$ is the unique extremal graph.

If $r\geq \ell+4$, then $ex(n,K_r,(\ell+1)\cdot W_{2k})=O\big(n^{2+\frac{1}{k-1}}\big)$.

A graph on $2k+1$ vertices consisting of k triangles which intersect in exactly one common vertex is called a *k-fan* and denoted by F_k. Specially, the F_2 is also called a *bowtie*. In 1995, Erdős, Füredi, Gould and Gunderson determined the value of $ex(n,F_k)$ and characterize the extremal graphs. We only list the case $k=2$ and its extremal graph for simplicity.

Theorem 6 (Erdős, Füredi, Gould and Gunderson [5]). *For $n\geq 5$,*

$$ex(n,F_2)=\left\lfloor\frac{n^2}{4}\right\rfloor+1.$$

The unique extremal graph is $T_2^+(n)$ which is obtained from $T_2(n)$ by adding one edge.

In 1976, Erdős and Sós determined the value of $ex(n, K_3, F_2)$.

Theorem 7 (Erdős and Sós [15]). *For all n,*

$$ex(n, K_3, F_2) = \begin{cases} n, & \text{for } n \equiv 0 \pmod 4, \\ n-1, & \text{for } n \equiv 1 \pmod 4, \\ n-2, & \text{for } n \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

Recently, Zhu, Chen, Gerbner, Győri, and Karim [21] extended it and determined the value of $ex(n, K_3, F_k)$ for $n \geq 4k^3$ and $k \geq 3$. In this paper, we determine the value of $ex(n, K_r, (\ell+1) \cdot F_2)$ for $r \geq 3$, which is our second main result.

Let $T_2^*(n)$ be the graph obtained from a bipartite Turán graph $T_2(n)$ by adding one edge to each its partition set, say v_1v_2 and u_1u_2, and then deleting the edges v_1u_2 and v_2u_1.

Theorem 8. *Let $\ell \geq 1$ and n be sufficiently large. If $3 \leq r \leq \ell + 2$, then*

$$\begin{aligned} ex(n, K_r, (\ell+1) \cdot F_2) = & \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) \\ & + \binom{\ell}{r-2} \left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor + \binom{\ell}{r-3}(n-\ell-4), \end{aligned}$$

and $K_\ell \vee T_2^*(n-\ell)$ is the unique extremal graph.

If $r \geq \ell + 3$, then $ex(n, K_r, (\ell+1) \cdot F_2) = O(n)$.

In Section 2, we prove Theorem 5. In Section 3, we prove Theorem 8.

2. PROOF OF THEOREM 5

To prove Theorem 5, we need the following results.

Theorem 9 (Gerbner, Methuku and Vizer [8]).

(i) *For any $r \geq 3$ and $k \geq 2$, we have $ex(n, K_r, C_{2k+1}) = O(n^{1+\frac{1}{k}})$.*

(ii) *If $r \leq \ell$, then $ex(n, K_r, \ell \cdot C_{2k+1}) = \Theta(n^2)$. If $r > \ell + 1$, then $ex(n, K_r, \ell \cdot C_{2k+1}) = O(n^{1+\frac{1}{k}})$.*

Lemma 10. *For any $r \geq 4$ and $k \geq 2$, we have*

$$ex(n, K_r, W_{2k}) = O\left(n^{2+\frac{1}{k-1}}\right).$$

Proof. Let G be a W_{2k}-free graph on n vertices. For any vertex $v \in V(G)$, $G[N(v)]$ does not contain a cycle on $2k-1$ vertices. Then

$$\begin{aligned}\mathcal{N}_r(G) &= \frac{\sum_v \mathcal{N}_{r-1}(G[N(v)])}{r} \le \frac{\sum_v ex(d(v), K_{r-1}, C_{2k-1})}{r} \\ &\le \frac{n}{r} ex\big(n, K_{r-1}, C_{2k-1}\big).\end{aligned}$$

By Theorem 9(i), we have $\mathcal{N}_r(G) = O\big(n^{2+\frac{1}{k-1}}\big)$ as required. ■

Lemma 11. *Let $r \ge 4, k \ge 2$ and c be a constant. Assume that G is a W_{2k}-free graph on n vertices. For sufficiently large n, we have*

$$\mathcal{N}_3(G) + c\mathcal{N}_r(G) \le \mathcal{N}_3(T_3(n)),$$

and the equality holds if and only if G is isomorphic to $T_3(n)$.

Proof. Let G_n be a W_{2k}-free graph on n vertices such that $\mathcal{N}_3(G_n) + c\mathcal{N}_r(G_n)$ is maximum. By Lemma 10, we have $\mathcal{N}_r(G_n) = o(n^3)$. Since $T_3(n)$ is W_{2k}-free and $\mathcal{N}_r(T_3(n)) = 0$ and by the choice of G_n, $\mathcal{N}_3(T_3(n)) \le \mathcal{N}_3(G_n) + c\mathcal{N}_r(G_n)$, it follows that $\mathcal{N}_3(G_n) \ge \mathcal{N}_3(T_3(n)) - o(n^3)$. By Theorem 4, there is a spanning tripartite subgraph (say G'_n) of G_n which is almost balanced by deleting $o(n^2)$ edges. Let (V_1, V_2, V_3) be the partition of G'_n.

Define

$$f(n) = \mathcal{N}_3(G_n) + c\mathcal{N}_r(G_n) - \mathcal{N}_3(T_3(n)). \tag{1}$$

Clearly $f(n) \ge 0$. We will show that if G_n contains a K_r with $r \ge 4$, then $f(n-1) - f(n) > 1$ for sufficiently large n.

For all distinct $i, j \in \{1,2,3\}$, let $L_i^j = \big\{v \in V_i \mid \deg(v, V_j) \ge \big(1 - \frac{1}{100k}\big)|V_j|\big\}$. For all distinct $i, j, t \in \{1,2,3\}$, let $L_i = \big\{v \in V_i \mid \deg(v, V_j) \ge \big(1 - \frac{1}{100k}\big)|V_j|$ and $\deg(v, V_t) \ge \big(1 - \frac{1}{100k}\big)|V_t|\big\}$. Let $L = L_1 \cup L_2 \cup L_3$, and let $S = V(G_n) \setminus L$.

Claim 12. *For different $i, j \in \{1,2,3\}$ and $n \ge n_1$, where n_1 is a sufficiently large integer, $|L_i^j| \ge \big(1 - \frac{1}{120}\big)|V_i|$.*

Proof. By contradiction, without loss of generality, we may suppose that $|L_1^2| = x|V_1|$ with $x < 1 - \frac{1}{120}$. Since deleting an edge of G_n can destroy at most $n-2$ triangles, it follows that deleting $o(n^2)$ edges will destroy $o(n^3)$ triangles. Recall that $\mathcal{N}_3(G_n) \ge \mathcal{N}_3(T_3(n)) - o(n^3)$. Thus

$$\mathcal{N}_3(G'_n) \ge \mathcal{N}_3(G_n) - o(n^3) \ge \mathcal{N}_3(T_3(n)) - o(n^3).$$

On the other hand,

$$\begin{aligned}\mathcal{N}_3(G_n') &< |L_1^2||V_2||V_3| + \left(|V_1| - |L_1^2|\right)\left(1 - \frac{1}{100k}\right)|V_2||V_3| \\ &= \left(x + (1-x)\left(1 - \frac{1}{100k}\right)\right)|V_1||V_2||V_3| \\ &\leq \left(1 - \frac{1}{120 \cdot 100k}\right)\frac{n^3}{27} + o(n^3),\end{aligned}$$

a contradiction for $n \geq n_1$, where n_1 is a large integer. Thus the claim holds. □

It follows from Claim 12 that $|L_i| = \left|L_i^j \cap L_i^t\right| \geq \left|L_i^j\right| + \left|L_i^t\right| - |V_i| \geq \left(1 - \frac{1}{60}\right)|V_i|$. This implies that $|S| \leq \frac{1}{60}(|V_1| + |V_2| + |V_3|) = \frac{n}{60}$.

Claim 13. *For different $i, j, t \in \{1, 2, 3\}$ and $n \geq \max\{4k, n_1\}$, and for any set $T \subset L_i \cup L_j$ with $|T| \leq 2k$, it holds that $\left|\bigcap_{x \in T} N(x, L_t)\right| \geq k$.*

Proof. By the definition of T, each vertex in T has at most $\frac{1}{100k}|V_t|$ non-neighbors in L_t. Then

$$\left|\bigcap_{x \in T} N(x, L_t)\right| \geq |L_t| - \frac{2k}{100k}|V_t| \geq \left(1 - \frac{1}{60}\right)|V_t| - \frac{1}{50}|V_t| \geq k$$

for $n \geq 4k$. □

Claim 14. *For each $i \in \{1, 2, 3\}$ and $n \geq \max\{4k, n_1\}$, L_i is an independent set.*

Proof. Suppose not, we may assume that x_1x_2 is an edge in $G_n[L_1]$ without loss of generality. By Claim 13, we assume that $\{u_1, \ldots, u_{k-1}\} \subseteq N(x_1, L_2) \cap N(x_2, L_2)$. By Claim 13, we further assume that $\{v_1, \ldots, v_{k-2}\} \subseteq \left(\bigcap_{i=1}^{k-1} N(u_i, L_1)\right) \setminus \{x_1, x_2\}$. Thus $x_1u_1v_1 \cdots v_{k-2}u_{k-1}x_2x_1$ is a cycle of length $2k-1$. By Claim 13, we choose a common neighbor y of $u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-2}, x_1, x_2$ in L_3, but then the set $\{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-2}, x_1, x_2, y\}$ forms a copy of W_{2k} with center y, a contradiction. Thus the claim holds. □

Claim 15. $\delta(G_n) < \frac{3n}{5}$.

Proof. Suppose, by contradiction, that $\delta(G_n) \geq \frac{3n}{5}$. Recall that $|S| \leq \frac{n}{60}$. Thus for any vertex v in G_n we have $\deg(v, L) \geq \frac{3n}{5} - \frac{n}{60} = \frac{7n}{12}$. Let $\{v_1, v_2, v_3, v_4\}$ be the vertex set of a K_4 in G_n as G_n contains a K_r with $r \geq 4$. By Claim 14, each $L_i (i = 1, 2, 3)$ is an independent set of G_n. By symmetry, we may distinguish the following four cases.

Case 1. $v_1 \in S$, $v_2 \in L_1$, $v_3 \in L_3$, $v_4 \in L_2$. By Claim 13, we assume that $\{y_1, \ldots, y_{k-2}\} \subseteq N(v_2, L_3) \cap N(v_4, L_3) \setminus \{v_3\}$. Set $T = \{v_3, v_4, y_1, \ldots, y_{k-2}\}$. By

Claim 13, we further assume that $\{x_1,\ldots,x_{k-2}\} \subseteq (\bigcap_{v\in T} N(v,L_1)) \setminus \{v_2\}$. But then the set $T\cup\{x_1,\ldots,x_{k-2},v_1,v_2\}$ forms a copy of W_{2k} with center v_4 (see Figure 1, the thick solid lines form the cycle C_{2k-1} in W_{2k}), a contradiction.

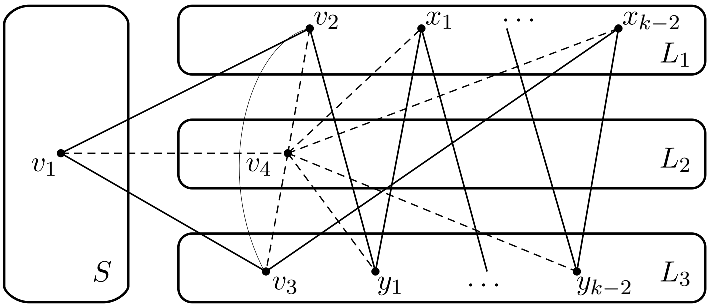

Figure 1. The illustration of Case 1.

Case 2. $\{v_1,\ v_2\} \subseteq S$, $v_3 \in L_1$, $v_4 \in L_2$. Recall that $\deg(v,L) \geq \frac{7n}{12}$ for any vertex v in G_n. This implies that $\deg(v,L_1\cup L_2) \geq \frac{7n}{12} - \lceil\frac{n}{3}\rceil > \frac{n}{5}$ for each $v\in S$. Without loss of generality, we may further assume that $\deg(v_1,L_1) > \frac{n}{10}$. Note that $\deg(v_4,L_1) \geq \big(1-\frac{1}{100k}\big)|V_1| - \frac{1}{60}|V_1| \geq \big(1-\frac{1}{30}\big)\left\lfloor\frac{n}{3}\right\rfloor$. It follows that v_1 and v_4 have a common neighbor x_1 in L_1. By Claim 13, we assume that $\{y_1,\ldots,y_{k-2}\} \subseteq N(x_1,L_3)\cap N(v_3,L_3)\cap N(v_4,L_3)$. Set $T=\{v_4,y_1,\ldots,y_{k-2}\}$. By Claim 13, we further assume that $\{x_2,\ldots,x_{k-2}\} \subseteq \big(\bigcap_{v\in T} N(v,L_1)\big)\setminus\{v_3,x_1\}$. But then the set $T\cup\{x_1,\ldots,x_{k-2},v_1,v_2,v_3\}$ forms a copy of W_{2k} with center v_4 (see Figure 2, the thick solid lines form the cycle C_{2k-1} in W_{2k}), a contradiction.

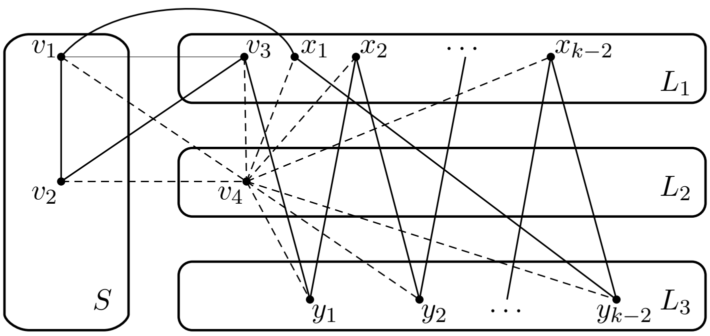

Figure 2. The illustration of Case 2.

Case 3. $\{v_1,v_2,v_3\} \subseteq S$, $v_4\in L$. Without loss of generality, we may assume that $v_4 \in L_2$. Note that for each $v_i (i=1,2,3)$ we have $\deg(v_i,L_1\cup L_3) > \frac{n}{5}$. Without of loss of generality, we may further assume that $\deg(v_1,L_1) > \frac{n}{10}$ and

$\deg(v_2, L_1) > \frac{n}{10}$. Note that $\deg(v_4, L_1) \geq (1 - \frac{1}{100k})|V_1| - \frac{1}{60}|V_1| \geq \left(1 - \frac{1}{30}\right) \left\lfloor \frac{n}{3} \right\rfloor$. It follows that v_1 and v_4 have a common neighbor x_1 in L_1. Similarly, v_2 and v_4 have a common neighbor x_2 in L_1. By Claim 13, we assume that $\{y_1, \ldots, y_{k-2}\} \subseteq N(x_1, L_3) \cap N(x_2, L_3) \cap N(v_4, L_3)$. Set $T = \{v_4, y_1, \ldots, y_{k-2}\}$. By Claim 13, we further assume that $\{x_3, \ldots, x_{k-1}\} \subseteq \left(\bigcap_{v \in T} N(v, L_1)\right) \setminus \{x_1, x_2\}$. But then the set $T \cup \{x_1, \ldots, x_{k-1}, v_1, v_2\}$ forms a copy of W_{2k} with center v_4 (see Figure 3, the thick solid lines form the cycle C_{2k-1} in W_{2k}), a contradiction.

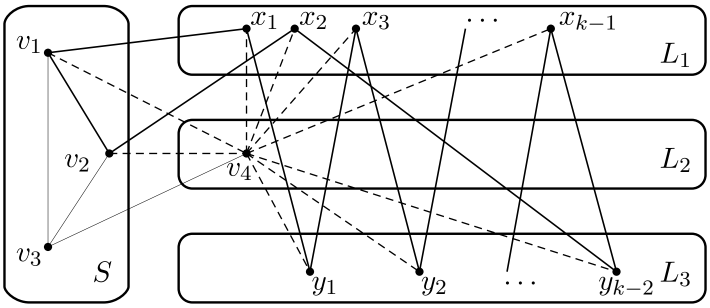

Figure 3. The illustration of Case 3.

Case 4. $\{v_1, v_2, v_3, v_4\} \subseteq S$. Since each $v_i (i = 1, 2, 3, 4)$ has degree at least $\frac{7n}{12}$ in L, it follows that $\sum_{i=1}^{4} \deg(v_i, L) \geq \frac{7n}{3}$. If every vertex in L has at most two neighbors in $\{v_1, v_2, v_3, v_4\}$, then $\sum_{i=1}^{4} \deg(v_i, L) < 2n$, a contradiction. Hence there exists a vertex x in L which is adjacent to at least three vertices in $\{v_1, v_2, v_3, v_4\}$. We may assume that the set $\{x, v_1, v_2, v_3\}$ forms a copy of K_4, and by Case 3 we are done. Thus the claim holds. □

By Claim 15, there exists a vertex $v \in G_n$ such that $d(v) < \frac{3n}{5}$. Since G_n is W_{2k}-free, $G_n[N(v)]$ is C_{2k-1}-free. By Theorem 1, the number of edges in $G_n[N(v)]$ is at most $\frac{1}{4}(d(v))^2$. By Theorem 9(i), the number of copies of $(r-1)$-cliques in $G_n[N(v)]$ is $O\left((d(v))^{1+\frac{1}{k-1}}\right) \leq \frac{n^2}{50c}$ for all $n \geq n_2$, where n_2 is a sufficiently large integer. If we delete v from G_n, it will destroy at most $\frac{1}{4}(d(v))^2$ triangles and $\frac{n^2}{50c}$ copies of r-cliques. Let $G' = G_n - v$. By the definition of $f(n)$, we have

$$
\begin{aligned}
& f(n-1) - f(n) \\
& \geq \mathcal{N}_3(G') + c\mathcal{N}_r(G') - \mathcal{N}_3(T_3(n-1)) - (\mathcal{N}_3(G_n) + c\mathcal{N}_r(G_n) - \mathcal{N}_3(T_3(n))) \\
& \geq \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor - (\mathcal{N}_3(G_n) - \mathcal{N}_3(G')) - (c\mathcal{N}_r(G_n) - c\mathcal{N}_r(G')) \\
& \geq \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor - \frac{1}{4}(\frac{3n}{5})^2 - \frac{n^2}{50} > 1
\end{aligned}
$$

for all $n \geq n_3$, where n_3 is a sufficiently large integer.

Let $n_4 = \max\{n_1, 4k, n_2, n_3\}$. For $n \geq n_4$, we conclude that if G_n contains a K_r, then

$$f(n-1) - f(n) > 1. \tag{2}$$

Claim 16. *For any positive integer $n' \geq n_4$, if $G_{n'}$ is K_r-free, then G_n is K_r-free for all $n \geq n'$.*

Proof. Suppose not, and let n^* be the smallest integer after n' satisfies G_{n^*} contains a K_r. Hence G_{n^*-1} is K_r-free. By (2) we have

$$0 \leq f(n^*) < f(n^*-1) - 1 = \mathcal{N}_3(G_{n^*-1}) - \mathcal{N}_3(T_3(n^*-1)) - 1.$$

Since W_{2k} is 4-edge-critical, we have $\mathcal{N}_3(G_{n^*-1}) \leq \mathcal{N}_3(T_3(n^*-1))$ by Theorem 3, then $f(n^*) < 0$, a contradiction. Thus the claim holds. □

Then there exists an integer $n_5 \geq n_4$ such that G_n is K_r-free. Otherwise, G_i contains a K_r for each $i \geq n_4$. Let $N > \binom{n_4}{3} + c\binom{n_4}{r} + n_4$. Then by (2) and (1),

$$\begin{aligned} 0 \leq f(N) &< f(N-1) - 1 < f(N-2) - 2 < \cdots < f(n_4) - (N - n_4) \\ &< \binom{n_4}{3} + c\binom{n_4}{r} - (N - n_4) < 0, \end{aligned}$$

a contradiction. Thus by Claim 16, G_n is K_r-free for $n \geq n_5$. Since W_{2k} is 4-edge-critical, by Theorem 3 we have $\mathcal{N}_3(G_n) + c\mathcal{N}_r(G_n) \leq \mathcal{N}_3(T_3(n))$ for all $n \geq n_5$, and the equality holds if and only if G_n is isomorphic to $T_3(n)$. This completes the proof of Lemma 11. ■

Now we prove Theorem 5.

Proof of Theorem 5. Let G be an $(\ell+1) \cdot W_{2k}$-free graph on n vertices that maximizes $\mathcal{N}_r(G)$. We distinguish two cases.

Case 1. $4 \leq r \leq \ell + 3$. Let L be a smallest set in $V(G)$ such that $G' = G - L$ is W_{2k}-free. Then $|L| \leq \ell |W_{2k}|$. Define

$$L_1 = \Big\{ v \in L \mid ((2k-1)\ell + 1) \cdot C_{2k-1} \subseteq G'[N(v) \cap V(G')] \Big\}$$

and $L_2 = L \setminus L_1$.

Claim 17. $|L_1| = \ell$.

Proof. Suppose first that $|L_1| \geq \ell + 1$, and let $\{v_1, \ldots, v_{\ell+1}\} \subseteq L_1$. We can recursively find $\ell + 1$ disjoint copies of W_{2k} such that each one is from $G[\{v_i\} \cup (N(v_i) \cap V(G'))]$ for $i = 1, \ldots, \ell + 1$. Indeed, assume we have found $j \leq \ell$ disjoint copies of W_{2k}. Pick a vertex in L_1 we have not selected, say v_{j+1}. By the definition of L_1, $G'[N(v_{j+1}) \cap V(G')]$ contains at least $(2k-1)\ell+1$ vertex disjoint copies of C_{2k-1}, then there are at least $(2k-1)\ell + 1 - (2k-1)j$ unused vertex disjoint copies of C_{2k-1} in $G'[N(v_{j+1}) \cap V(G')]$. Thus we can find the $(j+1)$-th copy of W_{2k}.

Suppose now that $|L_1| \le \ell - 1$. Since G' is W_{2k}-free, by Theorem 3, $\mathcal{N}_3(G') \le \frac{(n-|L|)^3}{27} + o(n^3)$. Then the r-cliques R in $G - L_2$ can be divided to three cases.

- $|R \cap G'| \le 2$. The number of this kind of r-cliques is $O(n^2)$.
- $|R \cap G'| = 3$. The number of this kind of r-cliques is at most $\binom{\ell-1}{r-3}\frac{(n-|L|)^3}{27} + o(n^3)$.
- $|R \cap G'| \ge 4$. The number of this kind of r-cliques is $O(n^{2+\frac{1}{k-1}})$ by Lemma 10.

For any vertex $v \in L_2$, $G'[N(v) \cap V(G')]$ is $((2k-1)\ell+1) \cdot C_{2k-1}$-free by definition of L_2. By Theorem 9(ii), the number of i-cliques in $G'[N(v) \cap V(G')]$ is $O(n^2)$. Hence, the number of r-cliques consisting of the vertex v, i vertices in $V(G')$ and $r-1-i$ vertices in $L - v$ is $O(n^2)$. Then

$$\begin{aligned}\mathcal{N}_r(G) &\le \binom{\ell-1}{r-3}\frac{(n-|L|)^3}{27} + o(n^3)\\ &< \binom{\ell}{r-3}\frac{(n-\ell)^3}{27} + o(n^3) = \mathcal{N}_r(K_\ell \vee T_3(n-\ell))\end{aligned}$$

for sufficiently large n, contradicting the choice of G. Thus the claim holds. □

Claim 18. $|L_2| = 0$.

Proof. Suppose not, and let $v \in L_2$. By the definition of L, there is a copy, say S, of W_{2k} containing v in $G - (L \setminus \{v\})$. Since there are exactly ℓ vertices in L_1 from Claim 17, we can recursively find ℓ vertex disjoint copies of W_{2k} in $G - V(S)$ similarly as in the proof of Claim 17. Together these copies with S form $\ell+1$ vertex disjoint copies of W_{2k}, a contradiction. Thus the claim holds. □

By Claims 17 and 18, we have $L = L_1$. By Theorem 2 and Lemma 11, we have

$$\begin{aligned}&\mathcal{N}_r(G)\\ &\le \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \binom{\ell}{r-2}e(G') + \binom{\ell}{r-3}\mathcal{N}_3(G') + \sum_{i=0}^{r-4}\binom{\ell}{i}\mathcal{N}_{r-i}(G')\\ &= \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \binom{\ell}{r-2}e(G') + \frac{\binom{\ell}{r-3}}{r-3}\sum_{i=0}^{r-4}(\mathcal{N}_3(G') + c_i\mathcal{N}_{r-i}(G'))\\ &\le \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \binom{\ell}{r-2}\left\lfloor\frac{(n-\ell)^2}{3}\right\rfloor + \binom{\ell}{r-3}\mathcal{N}_3(T_3(n-\ell))\\ &= \mathcal{N}_r(K_\ell \vee T_3(n-\ell)),\end{aligned}$$

where $c_i = (r-3)\binom{\ell}{i}/\binom{\ell}{r-3}$ and the equality holds if and only if $G = K_\ell \vee T_3(n-\ell)$.

Case 2. $r \geq \ell + 4$. By the similar analysis as in Claim 17, we can obtain that $|L_1| \leq \ell$ and the number of copies of r-cliques containing vertices in L_2 is $O(n^2)$. Since $r \geq \ell + 4$, it follows that $r - i \geq 4$ for each $i \in \{0, \ldots, |L_1|\}$. By Lemma 10, we obtain that

$$\mathcal{N}_r(G - L_2) \leq \sum_{i=0}^{|L_1|} \binom{|L_1|}{i} \mathcal{N}_{r-i}(G') \leq \sum_{i=0}^{|L_1|} \binom{|L_1|}{i} ex(n, K_{r-i}, W_{2k}) = O\left(n^{2+\frac{1}{k-1}}\right).$$

Hence, $ex(n, K_r, (\ell+1) \cdot W_{2k}) = O(n^{2+\frac{1}{k-1}})$. Thus the proof of Theorem 5 is complete. ■

3. Proof of Theorem 8

In this section we will prove Theorem 8. First we prove the following useful lemmas.

The *book graph* B_t is the graph consisting of $t - 2 \geq 1$ triangles, all sharing one edge. We call the vertices of degree two of a book graph the *page vertices*.

Lemma 19. *Let $c > 0$ be a constant and let G be an F_2-free graph on n vertices. For sufficiently large n, we have*

$$e(G) + c\mathcal{N}_3(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + c(n-4),$$

and the equality holds if and only if G is isomorphic to $T_2^*(n)$.

Proof. Assume that G is an F_2-free graph on n vertices such that $e(G) + c\mathcal{N}_3(G) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + c(n-4)$. By Theorem 7 we have $\mathcal{N}_3(G) \leq n$. If $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$, then by Theorem 6, G is isomorphic to $T_2^+(n)$. But then $e(G) + c\mathcal{N}_3(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1 + c\left\lceil \frac{n}{2} \right\rceil < \left\lfloor \frac{n^2}{4} \right\rfloor + c(n-4)$ for sufficiently large n, a contradiction. Thus

$$\left\lfloor \frac{n^2}{4} \right\rfloor - 4c \leq e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor, \ n - 4 \leq \mathcal{N}_3(G) \leq n. \tag{3}$$

Claim 20. *G is K_4-free.*

Proof. Suppose, otherwise, that G contains a K_4. Set $V(K_4) = S$. Since $G - S$ is F_2-free, by Theorem 6, $e(G - S) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 1$. Since $e(G) \geq \left\lfloor \frac{n^2}{4} \right\rfloor - 4c$, it follows that $e(S, V(G) \setminus S) \geq \left\lfloor \frac{n^2}{4} \right\rfloor - 4c - \left(\left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 1 \right) - 6 = 2n - 11 - 4c$. On the other hand, every vertex in $G - S$ is adjacent to at most one vertex in S as G is F_2-free, but then $e(S, V(G) \setminus S) \leq n - 4$, a contradiction. □

Since G is F_2-free, any two books B_1, B_2 of G satisfy that $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $V(B_1) \cap V(B_2) = \emptyset$. Let $B_1, \ldots, B_t$ be all vertex disjoint book graphs in G such that each B_i has page vertices as large as possible. Since each $B_i (i = 1, \ldots, t)$ contains exactly $|B_i| - 2$ triangles and by Claim 20, $B_1 \cup \cdots \cup B_t$ contains exactly $\sum_{i=1}^{t} |B_i| - 2t$ triangles. It follows that $\mathcal{N}_3(G) = \sum_{i=1}^{t} |B_i| - 2t$. Since $\sum_{i=1}^{t} |B_i| \leq n$ and by (3), we have $t \leq 2$.

If $t = 1$, then by Claim 20, $e(B_1) = 2|B_1| - 3$. By the choice of B_i, all triangles in G are contained in B_1. Since $\mathcal{N}_3(G) \geq n - 4$, $G - V(B_1)$ has at most two vertices, it follows that $e(G) \leq 2|B_1| - 3 + 2|B_1| + 1 \leq 4n - 2$, contradicting (3).

Thus we have $t = 2$. Then $B_1 \cup B_2$ contains at most $n - 4$ triangles. By (3) we obtain that $\mathcal{N}_3(G) = n - 4$ and $|B_1| + |B_2| = n$. Recall that $e(G) + c\mathcal{N}_3(G) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + c(n-4)$. Then $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor$. Let $V(B_1) = \{x_1, x_2\} \cup S_1$ and $V(B_2) = \{y_1, y_2\} \cup S_2$, where S_i is the set of page vertices of B_i. Clearly $|S_i| \geq 2$ for each $i = 1, 2$. Since G is K_4-free, we have $e(B_1) + e(B_2) = 2n - 6$. Since G is F_2-free, it follows that $e(\{x_1, x_2\}, \{y_1, y_2\}) \leq 2$. Thus

$$
\begin{aligned}
& e(\{x_1, x_2\}, S_2) + e(\{y_1, y_2\}, S_1) + e(S_1, S_2) \\
& = e(G) - (e(B_1) + e(B_2)) - e(\{x_1, x_2\}, \{y_1, y_2\}) \geq \left\lfloor \frac{n^2}{4} \right\rfloor - (2n - 6) - 2.
\end{aligned} \tag{4}
$$

Note that $|S_1| + |S_2| = n - 4$. It follows that $e(S_1, S_2) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor - 2n + 4$ and the equality holds if and only if $|S_1|$ is almost equal to $|S_2|$. We further claim that $e(\{x_1, x_2\}, S_2) + e(\{y_1, y_2\}, S_1) = 0$. Otherwise, let $z \in S_2$ such that $x_1 z \in E(G)$ without loss of generality. Since G is F_2-free, z is non-adjacent to any of S_1, it follows that

$$
\begin{aligned}
& e(\{x_1, x_2\}, S_2) + e(\{y_1, y_2\}, S_1) + e(S_1, S_2) \\
& \leq \left\lfloor \frac{n^2}{4} \right\rfloor - 2n + 4 - |S_1| + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor - 2n + 3,
\end{aligned}
$$

contradicting (4). Hence G is isomorphic to $T_2^*(n)$. The proof of Lemma 19 is complete. ■

Lemma 21. *Let c be a constant, and let G be an F_2-free graph on n vertices such that G contains a K_4. For sufficiently large n, we have*

$$
e(G) + c\mathcal{N}_4(G) < \left\lfloor \frac{n^2}{4} \right\rfloor .
$$

Proof. Let G be an F_2-free graph on n vertices such that $K_4 \subseteq G$ and $e(G) + c\mathcal{N}_4(G)$ is maximum. It suffices to show $e(G) + c\mathcal{N}_4(G) < \left\lfloor \frac{n^2}{4} \right\rfloor$. By contradiction, suppose that $e(G) + c\mathcal{N}_4(G) \geq \left\lfloor \frac{n^2}{4} \right\rfloor$. Define $f(n) = e(G) + c\mathcal{N}_4(G) - \left\lfloor \frac{n^2}{4} \right\rfloor$. Then $f(n) \geq 0$. Since G is F_2-free, it follows that any two K_4's in G cannot intersect, implying that the number of the copies of K_4 in G is at most $\left\lfloor \frac{n}{4} \right\rfloor$. Thus $e(G) \geq \left\lfloor \frac{n^2}{4} \right\rfloor - O(n)$. By Theorem 4, G has a bipartite spanning subgraph G' which is almost balanced by deleting $o(n^2)$ edges. Then $e(G') \geq \left\lfloor \frac{n^2}{4} \right\rfloor - o(n^2)$. Let (V_1, V_2) be the partition of G'. Define

$$L_1 = \left\{ v \in V_1 \mid \deg(v, V_2) \geq \left(1 - \frac{1}{1000}\right) |V_2| \right\},$$

$$L_2 = \left\{ v \in V_2 \mid \deg(v, V_1) \geq \left(1 - \frac{1}{1000}\right) |V_1| \right\},$$

and $S = (V_1 \setminus L_1) \cup (V_2 \setminus L_2)$.

Claim 22. *For each $i = 1, 2$ and $n \geq n_1$, where n_1 is a sufficiently large integer, $|L_i| \geq \left(1 - \frac{1}{500}\right)|V_i|$. Consequently, for each $v \in L_i$ we have $\deg(v, L_{3-i}) \geq 0.49n$.*

Proof. By contradiction, suppose that $|L_1| = x|V_1|$ with $x < 1 - \frac{1}{500}$ without loss of generality. Then

$$\begin{aligned} e(G') &< |L_1||V_2| + (|V_1| - |L_1|) \left(1 - \frac{1}{1000}\right) |V_2| \\ &= \left(x + (1 - x)\left(1 - \frac{1}{1000}\right)\right) |V_1||V_2| \leq \left(1 - \frac{1}{500 \times 1000}\right) \frac{n^2}{4} + o(n^2), \end{aligned}$$

contradicting $e(G') \geq \left\lfloor \frac{n^2}{4} \right\rfloor - o(n^2)$ for $n \geq n_1$, where n_1 is a large integer. Thus $|L_i| \geq \left(1 - \frac{1}{500}\right)|V_i|$ for each $i = 1, 2$.

For each $v \in L_i$, we have $\deg(v, L_{3-i}) \geq \left(1 - \frac{1}{1000}\right)|V_{3-i}| - \frac{1}{500}|V_{3-i}| \geq 0.49n$. Hence the claim holds. □

Claim 23. *$\delta(G) < 0.26n$.*

Proof. Suppose, by contradiction, that $\delta(G) \geq 0.26n$. By Claim 22, we have $|S| = \sum_{i=1}^{2} |V_i \setminus L_i| \leq 0.002n$. Then $\deg(v, L_1 \cup L_2) \geq 0.26n - 0.002n = 0.258n$ for any $v \in V(G)$. Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$ as G contains a K_4. If $\{v_1, v_2\} \subseteq L_i$, then there is a common neighbor $u \notin \{v_1, v_2, v_3, v_4\}$ in L_{3-i}, and $\{u, v_1, v_2, v_3, v_4\}$ forms a copy of F_2. Hence each $L_i (i = 1, 2)$ contains at most one vertex of $\{v_1, v_2, v_3, v_4\}$. By symmetry, we may distinguish the following two cases.

Case 1. $\{v_1, v_2\} \subseteq S$, $v_3 \in L_1$, $v_4 \in L_2$. Since $\deg(v_1, L_1 \cup L_2) \geq 0.258n$, by the average principle, we may assume that $\deg(v_1, L_2) \geq 0.129n$ without loss of generality. By Claim 22, we have $|N(v_1, L_2 \setminus \{v_4\}) \cap N(v_3, L_2 \setminus \{v_4\})| \geq$

$0.129n + 0.49n - 1 - |V_2| > 0$. Let $u \in N(v_1, L_2 \setminus \{v_4\}) \cap N(v_3, L_2 \setminus \{v_4\})$. Then $\{v_1, v_2, v_3, v_4, u\}$ forms a copy of F_2, a contradiction.

Case 2. $\{v_1, v_2, v_3\} \subseteq S$. If there exists a vertex in $\{v_1, v_2, v_3\}$, say v_1, such that $N(v_1, L_1 \setminus \{v_4\}) \neq \emptyset$ and $N(v_1, L_2 \setminus \{v_4\}) \neq \emptyset$. Recall that $\deg(v_1, L_1 \cup L_2) \geq 0.258n$. Without loss of generality, we may assume that $u_1 \in N(v_1, L_1 \setminus \{v_4\})$ and $\deg(v_1, L_2) \geq 0.129n$. By Claim 22, we have $|N(v_1, L_2 \setminus \{v_4\}) \cap N(u_1, L_2 \setminus \{v_4\})| \geq 0.129n + 0.49n - |V_2| - 1 > 0$. Let $u_2 \in N(v_1, L_2 \setminus \{v_4\}) \cap N(u_1, L_2 \setminus \{v_4\})$. But then $\{v_1, v_2, v_3, u_1, u_2\}$ forms a copy of F_2, a contradiction.

Hence each vertex in $\{v_1, v_2, v_3\}$ has no neighbors in one of $L_1 \setminus \{v_4\}, L_2 \setminus \{v_4\}$. Then there are at least two of $\{v_1, v_2, v_3\}$, say v_1, v_2, such that the neighbors of them in $L \setminus \{v_4\}$ are all in $L_1 \setminus \{v_4\}$. Recall that $\deg(v, L_1 \cup L_2) \geq 0.258n$ for any vertex v of G. Then $|N(v_1, L_1 \setminus \{v_4\}) \cap N(v_2, L_1 \setminus \{v_4\})| \geq 0.516n - |L_1| - 1 > 0$. Let $u \in N(v_1, L_1 \setminus \{v_4\}) \cap N(v_2, L_1 \setminus \{v_4\})$. But then $\{v_1, v_2, v_3, v_4, u\}$ forms a copy of F_2, a contradiction. Thus the claim holds. □

By Claim 23, there exists a vertex $v \in V(G)$ such that $d(v) < 0.26n$. If we delete v from G, it will destroy at most $0.26n$ edges and at most one copy of K_4 as G is F_2-free. Let $G^* = G - v$. Then

$$\begin{aligned}
&\left(e(G^*) + c\mathcal{N}_4(G^*) - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor\right) - f(n) \\
&= (e(G^*) - e(G)) + c(\mathcal{N}_4(G^*) - \mathcal{N}_4(G)) - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \left\lfloor \frac{n^2}{4} \right\rfloor \\
&\geq \frac{2n-2}{4} - 0.26n - c \geq 0.23n + 1
\end{aligned}$$

for sufficiently large n. By Theorem 6, $e(G^*) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$. This implies that $0.23n \leq c\mathcal{N}_4(G^*)$. Let T_1 be the vertex set of all disjoint K_4 in G^* and let $T_2 = V(G^*) \setminus T_1$. Then we have $\frac{0.92n}{c} \leq |T_1| = 4\mathcal{N}_4(G^*) \leq n$. Since G is F_2-free, it follows that the edges between any two K_4's are at most four, implying that $e(T_1) \leq \frac{3}{2}|T_1| + 4\binom{\frac{|T_1|}{4}}{2}$. By Theorem 6 we have $e(T_2) \leq \left\lfloor \frac{(n-1-|T_1|)^2}{4} \right\rfloor + 1$. Note that for any vertex in T_2 and any copy of K_4 in T_1, there is at most one edge between them. It follows that $e(T_1, T_2) \leq \mathcal{N}_4(G^*)|T_2| = \frac{|T_1|}{4}(n - 1 - |T_1|)$. Hence,

$$\begin{aligned}
e(G^*) &= e(T_1) + e(T_2) + e(T_1, T_2) \\
&\leq \frac{3}{2}|T_1| + 4\binom{\frac{|T_1|}{4}}{2} + \left\lfloor \frac{(n-1-|T_1|)^2}{4} \right\rfloor + 1 + \frac{|T_1|}{4}(n - 1 - |T_1|) \\
&\leq \frac{n^2}{4} + \frac{|T_1|^2}{8} - \frac{n|T_1|}{4} + o(n^2) \leq \frac{n^2}{4} + \frac{|T_1|(n-2n)}{8} + o(n^2) \\
&\leq \left(\frac{1}{4} - \frac{0.92}{8c}\right) n^2 + o(n^2),
\end{aligned}$$

contradicting $e(G^*) > e(G) - 0.26n \geq \left\lfloor \frac{n^2}{4} \right\rfloor - o(n^2)$ for sufficiently large n. The proof of Lemma 21 is complete. ■

Now we prove Theorem 8.

***Proof of Theorem* 8.** Let G be an $(\ell+1)\cdot F_2$-free graph on n vertices that maximizes $\mathcal{N}_r(G)$. We distinguish two cases.

Case 1. $3 \leq r \leq \ell+2$. Let L be the smallest set in $V(G)$ such that $G' = G - L$ is F_2-free. Then $|L| \leq \ell|F_2|$. Define

$$L_1 = \left\{ v \in L \mid (4\ell+2)\cdot K_2 \subseteq G'[N(v)\cap V(G')] \right\}$$

and $L_2 = L \setminus L_1$.

Claim 24. $|L_1| = \ell$.

Proof. Suppose first that $|L_1| \geq \ell+1$, and let $\{v_1, \ldots, v_{\ell+1}\} \subseteq L_1$. We can recursively find $\ell+1$ disjoint copies of F_2 such that each one is from $G[\{v_i\} \cup (N(v_i)\cap V(G'))]$ for $i = 1, \ldots, \ell+1$. Indeed, assume we have found $j \leq \ell$ disjoint copies of F_2. Pick a vertex in L_1 we have not selected, say v_{j+1}. By the definition of L_1, $G'[N(v_{j+1})\cap V(G')]$ contains at least $4\ell+2$ vertex disjoint edges, then there are at least $4\ell+2-4j$ unused vertex disjoint edges in $G'[N(v_{j+1})\cap V(G')]$. Thus we can find the $(j+1)$-th copy of F_2.

Suppose now that $|L_1| \leq \ell - 1$. Since G' is F_2-free, by Theorem 6 we have $e(G') \leq \left\lfloor \frac{(n-|L|)^2}{4} \right\rfloor + 1$. By Theorem 7, the number of triangles in G' is $O(n)$. Since G' is F_2-free, it follows that any two K_4's in G' cannot intersect, implying that the number of the copies of K_4 in G' is $O(n)$. Note that G' is K_5-free, the r-cliques R in $G - L_2$ can be divided to three cases.

- $|R \cap G'| \leq 1$. The number of this kind of r-cliques is $O(n)$.
- $|R \cap G'| = 2$. The number of this kind of r-cliques is at most $\binom{\ell-1}{r-2}\left(\left\lfloor \frac{(n-|L|)^2}{4} \right\rfloor + 1\right)$.
- $|R \cap G'| = 3$ or 4. The number of this kind of r-cliques is $O(n)$.

For any vertex $v \in L_2$, $G'[N(v)\cap V(G')]$ is $(4\ell+2)\cdot K_2$-free by definition of L_2. By Erdős-Gallai matching theorem, the number of edges in $G'[N(v)\cap V(G')]$ is $O(n)$. Therefore, the number of r-cliques consisting of the vertex v, i vertices in $V(G')$ and $r-1-i$ vertices in $L - v$ is $O(n)$ for each $i = 1, 2, 3, 4$. Then

$$\begin{aligned}\mathcal{N}_r(G) &\leq \binom{\ell-1}{r-2}\left\lfloor \frac{(n-|L|)^2}{4} \right\rfloor + O(n)\\ &< \binom{\ell}{r-2}\left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor \leq \mathcal{N}_r(K_\ell \vee T_2^*(n-\ell))\end{aligned}$$

for sufficiently large n, contradicting the choice of G. Thus the claim holds. $\square$

Claim 25. $|L_2| = 0$.

Proof. Suppose not, and let $v \in L_2$. By the definition of L, there is a copy, say S, of F_2 containing v in $G - (L \setminus \{v\})$. Since there are exactly ℓ vertices in L_1 from Claim 24, we can recursively find ℓ vertex disjoint copies of F_2 in $G - V(S)$, similarly as in the proof of Claim 24. Together these copies with S form $\ell + 1$ vertex disjoint copies of F_2, a contradiction. Thus the claim holds. $\square$

By Claims 24 and 25, we obtain that $L = L_1$.

Claim 26. $e(G') \leq \left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor$.

Proof. Suppose, otherwise, that $e(G') = \left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor + 1$ and G' is isomorphic to $T_2^+(n-\ell)$ by Theorem 6. Clearly G' is K_4-free. Then

$$\begin{aligned}
&\mathcal{N}_r(G)\\
&\leq \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \binom{\ell}{r-2}\left(\left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor + 1\right) + \binom{\ell}{r-3}\left\lceil \frac{n-\ell}{2} \right\rceil\\
&< \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \binom{\ell}{r-2}\left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor + \binom{\ell}{r-3}(n-\ell-4)\\
&= \mathcal{N}_r(K_\ell \vee T_2^*(n-\ell))
\end{aligned}$$

for sufficiently large n, contradicting the choice of G. Thus the claim holds. $\square$

By Claim 26 and Lemma 21, we have $e(G') + c\mathcal{N}_4(G') \leq \left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor$ for any constant c and sufficiently large n. Therefore by Lemma 19 we have

$$\begin{aligned}
&\mathcal{N}_r(G)\\
&\leq \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \binom{\ell}{r-2}e(G') + \binom{\ell}{r-3}\mathcal{N}_3(G') + \binom{\ell}{r-4}\mathcal{N}_4(G')\\
&= \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \frac{\binom{\ell}{r-2}}{2}(e(G') + c_1\mathcal{N}_3(G') + e(G') + c_2\mathcal{N}_4(G'))\\
&\leq \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \frac{\binom{\ell}{r-2}}{2}\left(\left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor + c_1(n-\ell-4) + \left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor\right)\\
&= \binom{\ell}{r} + \binom{\ell}{r-1}(n-\ell) + \binom{\ell}{r-2}\left\lfloor \frac{(n-\ell)^2}{4} \right\rfloor + \binom{\ell}{r-3}(n-\ell-4)\\
&= \mathcal{N}_r(K_\ell \vee T_2^*(n-\ell)),
\end{aligned}$$

where $c_1 = 2\binom{\ell}{r-3}/\binom{\ell}{r-2}$ and $c_2 = 2\binom{\ell}{r-4}/\binom{\ell}{r-2}$, and the equality holds if and only if $G = K_\ell \vee T_2^*(n-\ell)$.

Case 2. $r \geq \ell + 3$. By a similar analysis as in Claim 24, we can obtain that $|L_1| \leq \ell$, and the number of copies of r-cliques containing vertices in L_2 is $O(n)$. Recall that the number of copies of K_3's and K_4's in G' is $O(n)$. Note that G' is K_5-free. Then

$$\mathcal{N}_r(G - L_2) \leq \binom{|L_1|}{r-3}\mathcal{N}_3(G') + \binom{|L_1|}{r-4}\mathcal{N}_4(G') \leq O(n).$$

Hence, $ex(n, K_r, (\ell+1)\cdot F_2) = O(n)$. The proof of Theorem 8 is complete. ■

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