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TURÁN NUMBER OF STRONG DIGRAPHS FORBIDDEN AT LEAST TWO TRIANGLES

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Abstract

Let $\mathcal{F}$ be a family of digraphs. A digraph D is $\mathcal{F}$-free if it has no isomorphic copy of any member of $\mathcal{F}$. The Turán number $ex(n, \mathcal{F})$ is the largest number of arcs of $\mathcal{F}$-free digraphs on n vertices. Bermond, Germa, Heydemann and Sotteau in 1980 [*Girth in digraphs*, J. Graph Theory, 4 (1980), 337–341] determined the Turán number of $\mathcal{C}_k$-free strong digraphs on n vertices for $k \geq 2$, where $\mathcal{C}_k = \{C_2, C_3, \ldots, C_k\}$ and C_i is a directed cycle of length $i \in \{2, 3, \ldots, k\}$. In this paper, we determine all Turán number of strong digraphs without $t \geq 2$ triangles, extending the previous result for the case $k = 3$.

Keywords: Turán number, strong digraph, triangle.

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1. Introduction

The *Turán-type problem* is one of the most basic and central topics in extremal graph theory, which involves the determination of the largest number of edges a graph may have if it contains no isomorphic copy of fixed graphs. This problem can be traced back to the work of Turán [18] in 1941, known as Turán's Theorem, which is a generalization of Mantel's Theorem due to Mantel [16] in 1907. Since then, the Turán-type problem has been widely investigated for undirected graphs, and there are many classic and significant results. Whereas, only a few such problems have been studied for digraphs. In the following context, we mainly consider the Turán-type problem in digraphs, and all digraphs considered have neither loops nor parallel arcs.

Let D be a digraph and $\mathcal{F}$ be a family of digraphs. We say D is $\mathcal{F}$*-free* if it contains no isomorphic copy of any member of $\mathcal{F}$. The *Turán number*, denoted by $ex(n, \mathcal{F})$, is defined to be the maximum number of arcs of $\mathcal{F}$-free digraphs on n vertices. An $\mathcal{F}$-free digraph D on n vertices is called a *Turán digraph* if its size attains $ex(n, \mathcal{F})$.

The investigation of digraph extremal problem was initiated by Brown and Harary [2], see, e.g., [3, 4] for more details. Let p and q be two positive integers. Denote by $\mathcal{D}_{p,q}$ the family of digraphs consisting of q distinct directed walks of length p with the same initial vertex and terminal vertex. Several results on $\mathcal{D}_{p,q}$-free digraphs with respect to different pairs of p and q have been given, for instance, see [12–14]. In 2021, Lyu [15] gave an extremal result regarding digraphs excluding an orientation of the diamond. A digraph D is *strong* if every vertex of D is reachable from every other vertex of D. There are also several results in term of the maximum size with given diameter and radius in strong digraphs and bipartite digraphs, respectively, see e.g., [6, 11].

Let $k \geq 2$ be an integer. Denote by $\mathcal{C}_k = \{C_2, C_3, \ldots, C_k\}$, where C_i is a directed cycle of length i for $i \in \{2, 3, \ldots, k\}$. In 1980, Bermond, Germa, Heydemann and Sotteau [1] determined the Turán number of $\mathcal{C}_k$-free strong digraphs.

Theorem 1 [1]**.** $ex(n, \mathcal{C}_k) = \frac{n^2+(3-2k)n+k^2-k-2}{2}$.

A *triangle* is a directed cycle of length three. The above theorem tells us that the Turán number of strong digraphs without triangles is equal to $\frac{n^2-n+4}{2}$, i.e., the special case for $k = 3$. In 2021, Chen and Chang [7] characterized all triangle-free strong digraphs on n vertices with size $\frac{n^2-n+4}{2}$ and proved such digraphs meet a conjecture proposed by Chudnovsky, Seymour and Sullivan [10]. The same authors [8] investigated the Turán number of triangle-free strong digraphs on n vertices with out-degree greater than one, and they showed that such Turán number is one of $\binom{n-1}{2} - 1$ and $\binom{n}{2} - 2$ by using critical properties in [7]. Additionally, they gave the exact value for $n = 7, 8, 9$. Moving on, Chen and Hou [9] further determined the exact Turán number for all $n \geq 10$.

Inspired by above work, we focus on the Turán-type problem in strong digraphs. In this paper, we present the exact Turán number of strong digraphs without $t \geq 2$ triangles, which extends the case $k = 3$ of Theorem 1. We first show that such Turán number is equal to $\binom{n}{2}$ if there are two arc disjoint triangles by constructing a desirable strong tournament. Let B_t be the union of $t \geq 2$ triangles sharing a unique common arc. We further determine the Turán number of B_t-free strong digraphs on n vertices.

Theorem 2.

$$ex(n, B_t) = \begin{cases} \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{if } t = 2, \\ \binom{n}{2}, & \text{if } t \geq 3. \end{cases}$$

The rest of this paper is organized as follows. In next section, we give some notation and auxiliary results. In Section 3, we will present our main results. Finally, we conclude this paper with a few remarks.

2. Notation

For a digraph D, we use $u \rightarrow v$ to mean $(u,v) \in A(D)$ and $u \nrightarrow v$ to mean $(u,v) \notin A(D)$. Every pair of distinct vertices u, v are *adjacent* if $u \rightarrow v$ or $v \rightarrow u$, otherwise they are *nonadjacent.* Let $N_D^+(v) = \{u \in V(D) : v \rightarrow u\}$ be the *out-neighborhood* of v and $N_D^-(v) = \{u \in V(D) : u \rightarrow v\}$ be the *in-neighborhood* of v. The cardinality of $N_D^+(v)$ (respectively, $N_D^-(v)$) is the *out-degree* $d_D^+(v)$ (respectively, *in-degree* $d_D^-(v)$) of v. Let $N_D(v) = N_D^+(v) \cup N_D^-(v)$ be the *neighborhood* of v and we denote $d_D(v) = |N_D(v)|$ to be the *degree* of v. The vertices in $N_D^+(v), N_D^-(v)$ and $N_D(v)$ are called *out-neighbors*, *in-neighbors* and *neighbors* of v, respectively. A vertex v of D is called a *sink* (respectively, *source*) if it has no out-neighbor (respectively, in-neighbor). Let $\Delta^+(D) = \max\{d_D^+(v) : v \in V(D)\}$ be the *maximum out-degree* of D and $\delta^+(D) = \min\{d_D^+(v) : v \in V(D)\}$ be the *minimum out-degree* of D. Analogously, we denote by $\Delta^-(D)$ the *maximum in-degree*, $\delta^-(D)$ the *minimum in-degree*, $\Delta(D)$ the *maximum degree* and $\delta(D)$ the *minimum degree* of D, respectively.

A *directed walk* (respectively, *directed path*) of length k in D is a sequence of vertices (respectively, distinct vertices) $x_0, x_1, \ldots, x_k$ such that $x_i \rightarrow x_{i+1}$ for $0 \leq i \leq k-1$. We call x_0 an *initial vertex* and x_k a *terminal vertex*, respectively. Both x_0 and x_k are called *end vertices.* A *directed cycle* of length k of D is a sequence of different vertices $x_0, x_1, \ldots, x_{k-1}$ such that $x_i \rightarrow x_{i+1}$ for $0 \leq i \leq k-2$ and $x_{k-1} \rightarrow x_0$. By $\langle x_0, x_1, \ldots, x_{k-1} \rangle$ we mean such a directed cycle.

A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. For any subdigraph H of D, we denote $N_H^+(v) = N_D^+(v) \cap V(H)$ and $N_H^-(v) = N_D^-(v) \cap V(H)$. Given a set $X \subseteq V(D)$, let $D[X]$ be the subdigraph induced by X in D. Let $D - X$ be the subdigraph $D[V(D) \backslash X]$. For any two disjoint vertex sets U and W, we denote $A(U, W) = \{(u, w) : u \in U \text{ and } w \in W\}$.

A digraph D is *acyclic* if there exists no directed cycle. Let $\beta(D)$ be the size of the smallest subset $X \subseteq A(D)$ such that $D \backslash X$ is acyclic. Note that every digraph D satisfies that $\beta(D) = 0$ if D is acyclic and $\beta(D) \geq 1$ if D is strong. A maximal strong subdigraph of D is called a *strong component* of D. If D is not a strong digraph, then D has at least two strong components. These strong components, denoted by $D_1, D_2, \ldots, D_h$, have an acyclic ordering such that $A(V(D_j), V(D_i)) = \emptyset$, where $h \geq 2$ and $1 \leq i < j \leq h$. Throughout this paper we label the strong components of a digraph D in accordance with this acyclic ordering.

A *tournament* on n vertices, denoted by T_n, is a digraph in which any two distinct vertices are adjacent. A digraph D is *transitive* if, for every pair $u \to v$ and $v \to w$ with $u \neq w$, we have $u \to w$. Notice that a tournament is transitive (*transitive tournament*) if and only if it is acyclic.

A classic result of Moon (Moon's Theorem [17]) states that any strong T_n is *pancyclic*, that is, it contains directed cycles of all lengths $3, 4, \ldots, n$. We need the following auxiliary lemmas, which will play important roles in the proof of our main results.

Lemma 3. *Any strong T_4 has a B_2.*

Proof. Denote by $V(T_4) = \{v_1, v_2, v_3, v_4\}$. By Moon's Theorem, there is a C_4 in any strong T_4, and we may assume without loss of generality that $C_4 = \langle v_1, v_2, v_3, v_4\rangle$. Up to isomorphism, we can further assume that $v_1 \to v_3$. If $v_2 \to v_4$, then $\langle v_1, v_2, v_4\rangle$ and $\langle v_1, v_3, v_4\rangle$ are two triangles with a unique common arc (v_4, v_1). Analogously, if $v_4 \to v_2$, then $\langle v_1, v_3, v_4\rangle$ and $\langle v_2, v_3, v_4\rangle$ are two triangles, and (v_3, v_4) is their common arc. Thereby, the lemma follows. ■

Lemma 4. *Let D be a B_2-free strong digraph on $n \geq 4$ vertices. Then $|A(D)| \leq \binom{n}{2} - 1$.*

Proof. On the contrary, we get $|A(D)| = \binom{n}{2}$. It implies that D is a strong tournament on n vertices. There must be a C_4 in D. Hence, D contains a strong T_4 since D is a tournament, which leads to a contradiction by Lemma 3. The lemma thus follows. ■

Lemma 5. *Let D be a B_2-free strong digraph on $n \geq 4$ vertices. Then $\delta(D) \leq n - 2$.*

Proof. By Lemma 4, we have $|A(D)| \leq \binom{n}{2} - 1$. Consequently, the following holds

$$\sum_{v \in V} d_D(v) = 2|A(D)| \leq 2\left(\binom{n}{2} - 1\right) = n^2 - n - 2.$$

This yields that $\delta(D) \leq n - 1 - \frac{2}{n}$. Since $n \geq 4$ and $\delta(D)$ is an integer, we get $\delta(D) \leq n - 2$. ■

3. MAIN RESULTS

Let Φ be a digraph on n vertices such that $V(\Phi) = \{v_1, v_2, \ldots, v_n\}$ and $A(\Phi)$ consists of all arcs (v_i, v_j) for $1 \leq i < j \leq n$. Observe that Φ is a transitive tournament.

3.1. Constructions of Γ and Θ

In this subsection, we will construct two strong digraphs Γ and Θ.

Construction of Γ. Let Γ be a digraph obtained from Φ by reversing the arc between its sink and source. That is, Γ is a digraph with vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and arc set $A(\Gamma) = (\{(v_i, v_j) : 1 \le i < j \le n\} \backslash (v_1, v_n)) \cup (v_n, v_1)$.

One can see that Γ is a strong tournament on n vertices. We now demonstrate that Γ contains no two arc disjoint triangles. Indeed, if there exist two arc disjoint triangles in Γ, then we must delete at least two arcs of Γ to guarantee that the resulting digraph is acyclic, implying that $\beta(\Gamma) \ge 2$. On the other hand, it is easily seen that $\Gamma \backslash (v_n, v_1)$ is acyclic, yielding that $\beta(\Gamma) = 1$. Hence, Γ contains no two arc disjoint triangles, and Γ is a Turán digraph of size $\binom{n}{2}$.

Construction of Θ. Let Θ be a digraph obtained from Φ by reversing all arcs of (v_i, v_{i+1}) for $i \in \{1, 2, \ldots, n-1\}$. In other words, Θ is a digraph with vertex set $V(\Theta) = \{v_1, v_2, \ldots, v_n\}$ and arc set $A(\Theta) = (\{(v_i, v_j) : 1 \le i < j \le n\} \backslash \{(v_i, v_{i+1}) : 1 \le i \le n-1\}) \cup \{(v_{i+1}, v_i) : 1 \le i \le n-1\}$.

Note that Θ is a strong tournament on n vertices. We shall illustrate that Θ has no isomorphic copy of B_t for $t \ge 3$. Suppose, on the contrary, that Θ contains a B_t for $t \ge 3$. There must exist a common arc, denoted by (u, v), of these t triangles. Clearly, $|N^+_\Theta(v) \cap N^-_\Theta(u)| \ge t \ge 3$. If $(u, v) \in \{(v_{i+1}, v_i) : 1 \le i \le n-1\}$, then $|N^+_\Theta(v) \cap N^-_\Theta(u)| \le 2$, which leads to a contradiction. It suffices to consider that $(u, v) \in A(\Theta) \backslash (\{(v_{i+1}, v_i) : 1 \le i \le n-1\})$. Assume that $u = v_\alpha$ and $v = v_\beta$, where $1 \le \alpha \le \beta - 2 \le n - 2$. By the construction of Θ, one can see that $N^-_\Theta(v_\alpha) = \{v_1, v_2, \ldots, v_{\alpha-2}, v_{\alpha+1}\}$ and $N^+_\Theta(v_\beta) = \{v_{\beta-1}, v_{\beta+1}, v_{\beta+2}, \ldots, v_n\}$. Observe that $N^+_\Theta(v) \cap N^-_\Theta(u) \subseteq \{v_{\alpha+1}\}$, which follows that $|N^+_\Theta(v) \cap N^-_\Theta(u)| \le 1$. This indicates that $(u, v) \notin A(\Theta) \backslash (\{(v_{i+1}, v_i) : 1 \le i \le n-1\})$, i.e., Θ contains no B_t for $t \ge 3$. Therefore, we obtain that for $t \ge 3$, $ex(n, B_t) = \binom{n}{2}$ and Θ is a B_t-free Turán digraph.

3.2. Upper bound of $ex(n, B_2)$

In this subsection, our main goal is to give the upper bound of $ex(n, B_2)$.

Theorem 6. *Let D be a B_2-free strong digraph on n vertices. Then*

$$|A(D)| \le \begin{cases} \dfrac{n^2 - 2n + 3}{2}, & \text{if } n \text{ is odd}, \\ \dfrac{n^2 - 2n + 2}{2}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. We proceed by induction on n. When $n = 3$, D is a triangle and $|A(D)| = 3$. When $n = 4$, by Lemma 4, $|A(D)| \le 5$ holds. Similarly, when $n = 5$, we also have $|A(D)| \leq 9$ by applying Lemma 4. Hence, we may assume that $n \geq 6$. Suppose Theorem 6 holds for all $|V(D)| < n$. Let v be a vertex of D of minimum degree $\delta(D)$. Then $d_D(v) = \delta(D) \leq n-2$ by Lemma 5. If $D-v$ is strong, then $D-v$ is a B_2-free strong digraph on $n-1$ vertices. One deduces that $|A(D)| = |A(D-v)| + d_D(v) \leq \frac{(n-1)^2-2(n-1)+3}{2} + n - 2 = \frac{n^2-2n+2}{2}$ and we are done.

As a consequence, it suffices to consider the case that $D-v$ is not strong. Then $D-v$ contains k strong components, denoted by $D_1, D_2, \ldots, D_k$, satisfying that $A(V(D_j), V(D_i)) = \emptyset$, where $k \geq 2$ and $1 \leq i < j \leq k$. Denote by $X_0 = V(D_2) \cup V(D_3) \cup \cdots \cup V(D_{k-1})$, $X_1 = V(D_1)$ and $X_k = V(D_k)$. In addition, let $D_0 = D[X_0]$. Notice that $N_D^+(v) \cap X_1 \neq \emptyset$ and $N_D^-(v) \cap X_k \neq \emptyset$ since D is strong.

It is convenient to write $\widetilde{D_0}$, $\widetilde{D_1}$ and $\widetilde{D_k}$ for $D[X_0 \cup \{v\}]$, $D[X_1 \cup \{v\}]$ and $D[X_k \cup \{v\}]$, respectively. Let $|X_0| = n_0$, $|X_1| = n_1$ and $|X_k| = n_k$.

We divide the rest of the proof into the following two parts.

Part A. $k = 2$, i.e., $D - v$ has exactly two strong components.

Proof. Note that $X_0 = \emptyset$. We have $n = n_1 + n_2 + 1$, and either $n_i \geq 3$ or $n_i = 1$, where $i \in \{1, 2\}$. Here to proceed with this case, we consider the following four cases.

Case 1. Both $\widetilde{D_1}$ and $\widetilde{D_2}$ are strong.

We divide the discussion into four cases.

Subcase 1.1. If both n_1 and n_2 are even, then n is odd. Hence, we have

$$\begin{aligned}|A(D)| &= \sum_{i=1,2} |A(D_i)| + |A(X_1, X_2)| + d_D(v) \\ &\leq \sum_{i=1,2} \frac{n_i^2 - 2n_i + 2}{2} + n_1 n_2 + n - 2 = \frac{n^2 - 2n + 3}{2}.\end{aligned}$$

Subcase 1.2. If n_1 is even and n_2 is odd, then n is even.

(1) If $|A(X_1, X_2)| \leq n_1 n_2 - 1$, then

$$\begin{aligned}|A(D)| &= \sum_{i=1,2} |A(D_i)| + |A(X_1, X_2)| + d_D(v) \\ &\leq \sum_{i=1,2} \frac{n_i^2 - 2n_i + 2}{2} + \frac{1}{2} + n_1 n_2 - 1 + n - 2 = \frac{n^2 - 2n + 2}{2}.\end{aligned}$$

(2) For $|A(X_1, X_2)| = n_1 n_2$, we will show that $d_D(v) \leq n - 3$. Since $N_D^+(v) \cap X_1 \neq \emptyset$, we obtain that $d_{D_1}^+(v) \geq 1$. Assume that $d_{D_1}^+(v) \geq 2$. We can find a B_2 since

$N_D^-(v) \cap X_2 \neq \emptyset$ and $|A(X_1, X_2)| = n_1 n_2$, a contradiction. Hence, $d_{D_1}^+(v) = 1$ holds. Analogously, $d_{D_2}^-(v) = 1$ holds. Denote by $N_{D_1}^+(v) = \{x\}$ and $N_{D_2}^-(v) = \{y\}$. We select a vertex $u \in X_1$ such that $x \to u$. Since $|A(X_1, X_2)| = n_1 n_2$, then $x \to y$ and $u \to y$. Observe that $v \nrightarrow u$ since $N_{D_1}^+(v) = \{x\}$ and $u \neq x$. One easily checks that $u \nrightarrow v$ since D is B_2-free. Similarly, there is a vertex $w \in X_2$ such that $w \to y$, $v \nrightarrow w$ and $w \nrightarrow v$. We thus obtain that $d_D(v) \leq n - 3$.

Thereby, we deduce that

$$\begin{aligned} |A(D)| &= \sum_{i=1,2} |A(D_i)| + |A(X_1, X_2)| + d_D(v) \\ &\leq \sum_{i=1,2} \frac{n_i^2 - 2n_i + 2}{2} + \frac{1}{2} + n_1 n_2 + n - 3 = \frac{n^2 - 2n + 2}{2}. \end{aligned}$$

Subcase 1.3. If n_1 is odd and n_2 is even, then n is even. Similar to the proof of Subcase 1.2, we can obtain that $|A(D)| \leq \frac{n^2-2n+2}{2}$. We omit the proof and leave it to the reader to verify.

Subcase 1.4. If both n_1 and n_2 are odd, then n is odd. Notice that both $|V(\widetilde{D_1})|$ and $|V(\widetilde{D_2})|$ are even. Thus, the following holds

$$\begin{aligned} |A(D)| &= \sum_{i=1,2} |A(\widetilde{D_i})| + |A(X_1, X_2)| \\ &\leq \sum_{i=1,2} \frac{(n_i+1)^2 - 2(n_i+1) + 2}{2} + n_1 n_2 = \frac{n^2 - 2n + 3}{2}. \end{aligned}$$

Case 2. $\widetilde{D_1}$ is strong and $\widetilde{D_2}$ is not strong.

Note that $n_1 \geq 3$. We divide the discussion into the following two cases.

Subcase 2.1. $n_2 \geq 3$. Since D is B_2-free, then

$$\begin{aligned} |A(D)| &= |A(\widetilde{D_1})| + |A(D_2)| + |A(X_1, X_2)| + d_{D_2}^-(v) \\ &\leq \frac{(n_1+1)^2 - 2(n_1+1) + 3}{2} + \frac{n_2^2 - 2n_2 + 3}{2} \\ &+ \ n_1 n_2 - (d_{D_2}^-(v) - 1) + d_{D_2}^-(v) \\ &= \frac{n^2 - 2n + 8 - 2n_2}{2} \leq \frac{n^2 - 2n + 2}{2} \end{aligned}$$

Subcase 2.2. $n_2 = 1$.

(1) If n_1 is even, then n is even. We have

$$\begin{aligned}|A(D)| &= |A(D_1)| + |A(X_1, X_2)| + d_D(v)\\ &\leq \frac{n_1^2 - 2n_1 + 2}{2} + n_1 + n - 2 = \frac{n^2 - 2n + 2}{2}.\end{aligned}$$

(2) If n_1 is odd, then n is odd. The following holds

$$\begin{aligned}|A(D)| &= |A(\widetilde{D_1})| + |A(X_1, X_2)| + d^-_{D_2}(v)\\ &\leq \frac{(n_1+1)^2 - 2(n_1+1) + 2}{2} + n_1 + 1 = \frac{n^2 - 2n + 3}{2}.\end{aligned}$$

Case 3. $\widetilde{D_1}$ is not strong and $\widetilde{D_2}$ is strong.

Analogous to the proof of Case 2, we omit the proof and leave it to the reader to verify.

Case 4. Both $\widetilde{D_1}$ and $\widetilde{D_2}$ are not strong.

Notice that $n \geq 6$, it suffices to consider the following three cases.

Subcase 4.1. If $n_1 \geq 3$ and $n_2 = 1$, then

$$\begin{aligned}|A(D)| &= |A(D_1)| + |A(X_1, X_2)| + d_D(v)\\ &\leq \frac{n_1^2 - 2n_1 + 3}{2} + n_1 - (d^+_{D_1}(v) - 1) + d^+_{D_1}(v) + 1\\ &= \frac{n^2 - 4n + 11}{2} \leq \frac{n^2 - 2n - 1}{2}.\end{aligned}$$

Subcase 4.2. If $n_1 = 1$ and $n_2 \geq 3$, then $|A(D)| \leq \frac{n^2-2n-1}{2}$ according to the calculations similar to Subcase 4.1.

Subcase 4.3. If both n_1 and n_2 are at least 3, then $n \geq 7$.

(1) When $d^+_{D_1}(v) = 1$, we obtain that

$$\begin{aligned}|A(D)| &= \sum_{i=1,2} |A(D_i)| + |A(X_1, X_2)| + d_D(v)\\ &\leq \sum_{i=1,2} \frac{n_i^2 - 2n_i + 3}{2} + n_1 n_2 - (d^-_{D_2}(v) - 1) + d^-_{D_2}(v) + 1\\ &= \frac{n^2 - 4n + 13}{2} \leq \frac{n^2 - 2n - 1}{2}.\end{aligned}$$

(2) When $d^+_{D_1}(v) \geq 2$, we consider two vertices x_1, x_2 of $N^+_{D_1}(v)$. If $A(\{x_1\}, X_2) = \emptyset$, then $|A(X_1, X_2)| \leq n_1 n_2 - d^-_{D_2}(v)$. For $A(\{x_1\}, X_2) \neq \emptyset$, we choose a vertex

$y_1 \in N^-_{D_2}(v)$ such that $x_1 \to y_1$. It is easily seen that $A(\{x_1\}, X_2\backslash\{y\}) = \emptyset$ and $x_2 \nrightarrow y_1$ since D is B_2-free. Thus, we obtain that $|A(X_1, X_2))| \leq n_1n_2 - d^-_{D_2}(v)$. Therefore, we have

$$\begin{aligned}|A(D)| &= \sum_{i=1,2} |A(D_i)| + |A(X_1, X_2)| + d_D(v)\\ &\leq \sum_{i=1,2} \frac{n_i^2 - 2n_i + 3}{2} + n_1n_2 - d^-_{D_2}(v) + d_D(v)\\ &\leq \frac{n^2 - 4n + 9 + 2d^+_{D_1}(v)}{2} \leq \frac{n^2 - 2n + 1}{2}.\end{aligned}$$

This completes the proof of Part A. □

Part B. $D - v$ has $k \geq 3$ strong components.

Proof. Note that $n = n_0 + n_1 + n_k + 1$ and $n_0 \geq 1$. We use R_0 (respectively, R_1 and R_k) and S_0 (respectively, S_1 and S_k) to denote $N^+_{D_0}(v)$ (respectively, $N^+_{D_1}(v)$ and $N^+_{D_k}(v)$) and $N^-_{D_0}(v)$ (respectively, $N^-_{D_1}(v)$ and $N^-_{D_k}(v)$). Let $r_i = |R_i|$ (respectively, $s_i = |S_i|$) for $i \in \{0, 1, k\}$.

We proceed with this case by considering the following four cases.

Case 1. Both $\widetilde{D_1}$ and $\widetilde{D_k}$ are strong.

Observe that both D_1 and D_k are strong. We divide the discussion into four cases.

Subcase 1.1. Both n_1 and n_k are even.

(1) $|A(R_1, S_k)| = r_1s_k$. We demonstrate that $d_{D_1}(v) \leq n_1-1$ and $d_{D_k}(v) \leq n_k-1$. First of all, we verify that $d_{D_1}(v) \leq n_1-1$. On the contrary, we have $d_{D_1}(v) = n_1$. It is obvious that $A(R_1, S_1) \neq \emptyset$ since D_1 is strong. Let $a \in R_1$ and $b \in S_1$ such that $a \to b$. Select a vertex $c \in S_k$. As $|A(R_1, S_k)| = r_1s_k$, we have $a \to c$. There must be a B_2 in $D[\{v, a, b, c\}]$, contradicting the fact that D is B_2-free. This proves $d_{D_1}(v) \leq n_1 - 1$. Similarly, we can also get $d_{D_k}(v) \leq n_k - 1$. Let u be a vertex of X_0. If $u \to v$ (respectively, $v \to u$), then $a \nrightarrow u$ (respectively, $u \nrightarrow c$) because D is B_2-free. We thus have

$$\begin{aligned}|A(D)| &= \sum_{i=0,1,k} |A(D_i)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i, X_j)| + d_D(v)\\ &\leq \sum_{i=1,k} \frac{n_i^2 - 2n_i + 2}{2} + \binom{n_0}{2} + n_0(n_1 + n_k) + n_1n_k + n_1 + n_k - 2\\ &= \frac{n^2 - 3n + n_1 + n_k + 2}{2} \leq \frac{n^2 - 2n}{2}.\end{aligned}$$

(2) $|A(R_1, S_k)| \leq r_1 s_k - 1$. If $|A(\widetilde{D_1})| = \frac{(n_1+1)^2-2(n_1+1)+3}{2}$, then $d_{D_1}(v) \geq n_1$ since $|A(D_1)| \leq \frac{n_1^2-2n_1+2}{2}$, implying that $d_{D_1}(v) = n_1$. Since D_1 is strong, there exist $a \in R_1$ and $b \in S_1$ such that $a \to b$. Thus, $\langle v, a, b\rangle$ is a triangle. For any vertex $p \in S_0$, we have $a \nrightarrow p$ since D is B_2-free. Analogously, if $|A(\widetilde{D_k})| = \frac{(n_k+1)^2-2(n_k+1)+3}{2}$, then $d_{D_k}(v) = n_k$ holds. There is a triangle $\langle v, c, d\rangle$ such that $c \in R_k$ and $d \in S_k$. For any vertex $q \in R_0$, $q \nrightarrow d$ holds.

Due to D is B_2-free, for any vertex $x \in R_1$ (respectively, $y \in S_1$), x dominates at most one vertex of S_0 (respectively, y is dominated by at most one vertex of R_0).

We divide the discussion into the following four cases.

(i) If $|A(\widetilde{D_1})| = \frac{(n_1+1)^2-2(n_1+1)+3}{2}$ and $|A(\widetilde{D_k})| = \frac{(n_k+1)^2-2(n_k+1)+3}{2}$, then

$$\begin{aligned}|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v)\\ &\leq \sum_{i=1,k} \frac{(n_i+1)^2 - 2(n_i+1)+3}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k - 1\\ &= \frac{n^2-3n+n_1+n_k+4}{2} \leq \frac{n^2-2n+2}{2}.\end{aligned}$$

(ii) If $|A(\widetilde{D_1})| = \frac{(n_1+1)^2-2(n_1+1)+3}{2}$ and $|A(\widetilde{D_k})| \leq \frac{(n_k+1)^2-2(n_k+1)+3}{2} - 1$, then

$$\begin{aligned}|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v)\\ &\leq \sum_{i=1,k} \frac{(n_i+1)^2 - 2(n_i+1)+3}{2} - 1 + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k - 1 + 1\\ &= \frac{n^2-3n+n_1+n_k+4}{2} \leq \frac{n^2-2n+2}{2}.\end{aligned}$$

(iii) If $|A(\widetilde{D_1})| \leq \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1$ and $|A(\widetilde{D_k})| = \frac{(n_k+1)^2-2(n_k+1)+3}{2}$, similar to (ii), we also have $|A(D)| \leq \frac{n^2-2n+2}{2}$.

(iv) If $|A(\widetilde{D_1})| \leq \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1$ and $|A(\widetilde{D_k})| \leq \frac{(n_k+1)^2-2(n_k+1)+3}{2} - 1$, then the following holds

$$\begin{aligned}|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i,X_j)| + d_{D_0}(v)\\ &\le \sum_{i=1,k}\left(\frac{(n_i+1)^2-2(n_i+1)+3}{2}-1\right)\\ &+\binom{n_0}{2}+n_0(n_1+n_k)+n_1n_k-1+2\\ &= \frac{n^2-3n+n_1+n_k+4}{2}\le\frac{n^2-2n+2}{2}.\end{aligned}$$

Subcase 1.2. n_1 is even and n_k is odd.

(1) $|A(R_1,S_k)| = r_1s_k$. Let u be a vertex of X_0. Analogous to Subcase 1.1, if $u\to v$ (respectively, $v\to u$), then $a\nrightarrow u$ (respectively, $u\nrightarrow c$) since D is B_2-free, where $a\in R_1$ and $c\in S_k$. Thus, we obtain that

$$\begin{aligned}|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i,X_j)| + d_{D_0}(v)\\ &\le \sum_{i=1,k}\frac{(n_i+1)^2-2(n_i+1)+3}{2}-\frac{1}{2}+\binom{n_0}{2}+n_0(n_1+n_k)+n_1n_k\\ &= \frac{n^2-3n+n_1+n_k+5}{2}.\end{aligned}$$

Note that $n=n_0+n_1+n_k+1$ and $n_0\ge 1$ since $k\ge 3$. If $n_0=1$, then n is odd and $|A(D)|\le\frac{n^2-2n+3}{2}$. If $n_0\ge 2$, then $n\ge n_1+n_k+3$. This follows that $|A(D)|\le\frac{n^2-2n+2}{2}$.

(2) $|A(R_1,S_k)|\le r_1s_k-1$.

(i) $|A(\widetilde{D_1})|=\frac{(n_1+1)^2-2(n_1+1)+3}{2}$. Analogous to the above discussion, we have

$$\begin{aligned}|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i,X_j)| + d_{D_0}(v)\\ &\le \sum_{i=1,k}\frac{(n_i+1)^2-2(n_i+1)+3}{2}-\frac{1}{2}+\binom{n_0}{2}+n_0(n_1+n_k)+n_1n_k-1+1\\ &= \frac{n^2-3n+n_1+n_k+5}{2}.\end{aligned}$$

If $n_0=1$, then n is odd and $|A(D)|\le\frac{n^2-2n+3}{2}$. If $n_0\ge 2$, then $n\ge n_1+n_k+3$ and $|A(D)|\le\frac{n^2-2n+2}{2}$.

(ii) $|A(\widetilde{D_1})| \le \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1$. We consider the following two cases.

(a_1) If $n_0 = 1$, then

$$\begin{aligned}
&|A(D)| \\
&= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) \\
&\le \sum_{i=1,k} \frac{(n_i+1)^2 - 2(n_i+1) + 2}{2} - \frac{1}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k - 1 + 1 \\
&= \frac{n^2 - 3n + n_1 + n_k + 3}{2} = \frac{n^2 - 2n + 1}{2}.
\end{aligned}$$

(a_2) If $n_0 \ge 2$, then

$$\begin{aligned}
&|A(D)| \\
&= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) \\
&\le \sum_{i=1,k} \frac{(n_i+1)^2 - 2(n_i+1) + 2}{2} - \frac{1}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k - 1 + 2 \\
&= \frac{n^2 - 3n + n_1 + n_k + 5}{2} \le \frac{n^2 - 2n + 2}{2}.
\end{aligned}$$

Subcase 1.3. n_1 is odd and n_k is even.

Similar to the proof of Subcase 1.2, we omit specific calculations.

Subcase 1.4. Both n_1 and n_k are odd.

(1) $|A(R_1, S_k)| = r_1 s_k$. The following holds

$$\begin{aligned}
|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) \\
&\le \sum_{i=1,k} \frac{(n_i+1)^2 - 2(n_i+1) + 2}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k \\
&= \frac{n^2 - 3n + n_1 + n_k + 4}{2} \le \frac{n^2 - 2n + 2}{2}.
\end{aligned}$$

(2) $|A(R_1, S_k)| \le r_1 s_k - 1$.

(i) When $n_0 = 1$, we have $n = n_1 + n_k + 2$. Then we obtain that

$$\begin{aligned}|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i, X_j)| + d_{D_0}(v)\\ &\le \sum_{i=1,k} \frac{(n_i+1)^2 - 2(n_i+1) + 2}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k - 1 + 1\\ &= \frac{n^2 - 3n + n_1 + n_k + 4}{2} = \frac{n^2 - 2n + 2}{2}.\end{aligned}$$

(ii) When $n_0 \ge 2$, we have

$$\begin{aligned}|A(D)| &= \sum_{i=1,k} |A(\widetilde{D_i})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i, X_j)| + d_{D_0}(v)\\ &\le \sum_{i=1,k} \frac{(n_i+1)^2 - 2(n_i+1) + 2}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k - 1 + 2\\ &= \frac{n^2 - 3n + n_1 + n_k + 6}{2}.\end{aligned}$$

If $n_0 = 2$, then $n = n_1 + n_k + 3$ and n is odd, $|A(D)| \le \frac{n^2-2n+3}{2}$ holds. If $n_0 \ge 3$, then $n \ge n_1 + n_k + 4$, yielding that $|A(D)| \le \frac{n^2-2n+2}{2}$.

Case 2. $\widetilde{D_1}$ is strong and $\widetilde{D_k}$ is not strong.

We divide the discussions into $n_k = 1$ and $n_k \ge 3$.

Subcase 2.1. $n_k = 1$. Obviously, $n = n_0 + n_1 + 2$ holds and v is dominated by the vertex in D_k since D is strong.

(1) When n_1 is even.

(i) If $|A(\widetilde{D_1})| \le \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1$ and $|A(R_1, S_k)| = r_1 s_k$, then

$$\begin{aligned}|A(D)| &= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v)\\ &\le \frac{(n_1+1)^2 - 2(n_1+1) + 3}{2} - 1 + \binom{n_0}{2} + n_0(n_1+1) + n_1 + 1\\ &= \frac{n^2 - 3n + n_1 + 4}{2} \le \frac{n^2 - 2n + 1}{2}.\end{aligned}$$

(ii) If $|A(\widetilde{D_1})| \leq \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1$ and $|A(R_1, S_k)| \leq r_1 s_k - 1$, then

$$\begin{aligned}|A(D)| &= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v)\\ &\leq \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1 + \binom{n_0}{2} + n_0(n_1+1) + n_1 - 1 + 2 + 1\\ &= \frac{n^2-3n+n_1+6}{2}.\end{aligned}$$

If $n_0 = 1$, then n is odd since $n = n_1+3$. Thereby, we have $|A(D)| \leq \frac{n^2-2n+3}{2}$. If $n_0 \geq 2$, then $n \geq n_1 + 4$, implying that $|A(D)| \leq \frac{n^2-2n+2}{2}$.

(iii) If $|A(\widetilde{D_1})| = \frac{(n_1+1)^2-2(n_1+1)+3}{2}$, then there must exist a triangle passing through v in $\widetilde{D_1}$. Denote such triangle by $\langle v, a, b\rangle$ and let $X_k = \{c\}$. Since D is a B_2-free strong digraph, we thus have $a \nrightarrow c$. Let u be a vertex of X_0. Notice that if $u \rightarrow v$, then $a \nrightarrow u$. Moreover, c is dominated by at most one vertex of R_0 since D is a B_2-free. Therefore, we deduce that

$$\begin{aligned}|A(D)| &= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v)\\ &\leq \frac{(n_1+1)^2-2(n_1+1)+3}{2} + \binom{n_0}{2} + n_0(n_1+1) + n_1 - 1 + 1\\ &= \frac{n^2-3n+n_1+4}{2} \leq \frac{n^2-2n+1}{2}.\end{aligned}$$

(2) When n_1 is odd.

(i) If $|A(R_1, S_k)| = r_1 s_k$, then

$$\begin{aligned}|A(D)| &= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v)\\ &\leq \frac{(n_1+1)^2-2(n_1+1)+2}{2} + \binom{n_0}{2} + n_0(n_1+1) + n_1 + 1\\ &= \frac{n^2-3n+n_1+5}{2} \leq \frac{n^2-2n+2}{2}.\end{aligned}$$

(ii) If $|A(R_1, S_k)| \leq r_1 s_k - 1$ and $n_0 = 1$, then we have

$$\begin{aligned}|A(D)| &= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v)\\ &\le \frac{(n_1+1)^2 - 2(n_1+1) + 2}{2} + \binom{n_0}{2} + n_0(n_1+1) + n_1 - 1 + 1 + 1\\ &= \frac{n^2 - 3n + n_1 + 5}{2} \le \frac{n^2 - 2n + 2}{2}.\end{aligned}$$

(iii) If $|A(R_1, S_2)| \le r_1 s_2 - 1$ and $n_0 \ge 2$, then

$$\begin{aligned}|A(D)| &= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v)\\ &\le \frac{(n_1+1)^2 - 2(n_1+1) + 2}{2} + \binom{n_0}{2} + n_0(n_1+1) + n_1 - 1 + 1 + 2\\ &= \frac{n^2 - 3n + n_1 + 7}{2}.\end{aligned}$$

If $n_0 = 2$, then $n = n_1 + 4$ and n is odd. Consequently, we get $|A(D)| \le \frac{n^2-2n+3}{2}$. If $n_0 \ge 3$, then $n \ge n_1 + 5$, which follows that $|A(D)| \le \frac{n^2-2n+2}{2}$.

Subcase 2.2. $n_k \ge 3$. Let x be a vertex of R_1. Notice that such vertex exists since D is strong. As D is B_2-free, x dominates at most one vertex of $S_0 \cup S_2$. It is clear that $|A(X_1, X_k)| \le n_1 n_k - (s_k - 1)$. Analogously, for any vertex y of S_k, y is dominated by at most one vertex of $R_0 \cup R_1$.

(1) n_1 is even.

(i) If $|A(\widetilde{D_1})| \le \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1$ and $|A(R_1, S_k)| = r_1 s_k$, then

$$\begin{aligned}&|A(D)|\\ &= |A(\widetilde{D_1})| + \sum_{i=0,k} |A(D_i)| + \sum_{i\in\{0,1\},j\in\{0,k\},i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v)\\ &\le \frac{(n_1+1)^2 - 2(n_1+1) + 1}{2} + \frac{n_k^2 - 2n_k + 3}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k + 1\\ &= \frac{n^2 - 3n + n_1 - n_k + 7}{2} \le \frac{n^2 - 2n + 5 - 2n_k}{2} \le \frac{n^2 - 2n - 1}{2}.\end{aligned}$$

(ii) If $|A(\widetilde{D_1})| \leq \frac{(n_1+1)^2-2(n_1+1)+3}{2} - 1$ and $|A(R_1, S_k)| \leq r_1 s_k - 1$, then

$$\begin{aligned}
&|A(D)| \\
&= |A(\widetilde{D_1})| + \sum_{i=0,k} |A(D_i)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v) \\
&\leq \frac{(n_1+1)^2 - 2(n_1+1) + 1}{2} + \frac{n_k^2 - 2n_k + 3}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k + 2 \\
&= \frac{n^2 - 3n + n_1 - n_k + 9}{2} \leq \frac{n^2 - 2n + 7 - 2n_k}{2} \leq \frac{n^2 - 2n + 1}{2}.
\end{aligned}$$

(iii) If $|A(\widetilde{D_1})| = \frac{(n_1+1)^2-2(n_1+1)+3}{2}$, then there exists a triangle passing through v in $\widetilde{D_1}$. Similarly, we get that

$$\begin{aligned}
&|A(D)| \\
&= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v) \\
&\leq \frac{(n_1+1)^2 - 2(n_1+1) + 3}{2} + \frac{n_k^2 - 2n_k + 3}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k + 1 \\
&= \frac{n^2 - 3n + n_1 - n_k + 9}{2} \leq \frac{n^2 - 2n + 1}{2}.
\end{aligned}$$

(2) n_1 is odd. We obtain that

$$\begin{aligned}
&|A(D)| \\
&= |A(\widetilde{D_1})| + |A(D_0)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_{D_0}(v) + d_{D_k}(v) \\
&\leq \frac{(n_1+1)^2 - 2(n_1+1) + 2}{2} + \frac{n_k^2 - 2n_k + 3}{2} + \binom{n_0}{2} + n_0(n_1+n_k) + n_1 n_k + 2 \\
&= \frac{n^2 - 3n + n_1 - n_k + 10}{2} \leq \frac{n^2 - 2n + 2}{2}.
\end{aligned}$$

Case 3. $\widetilde{D_1}$ is not strong and $\widetilde{D_k}$ is strong.

Analogous to the proof of Case 2, we can also deduce that $|A(D)| \leq \frac{n^2-2n+3}{2}$ for n is odd and $|A(D)| \leq \frac{n^2-2n+2}{2}$ for n is even, respectively. We omit the proof and leave it to the reader to verify.

Case 4. Both $\widetilde{D_1}$ and $\widetilde{D_k}$ are not strong.

We divide the discussions into the following four cases.

Subcase 4.1. $n_1 = 1$ and $n_k = 1$.

Denote by $X_1 = \{x\}$ and $X_k = \{y\}$. Since D is strong, then $v \to x$ and $y \to v$.

(1) $x \to y$.

Since D is B_2-free, for any vertex u (respectively, w) of R_0 (respectively, S_0), we have $u \nrightarrow y$ (respectively, $x \nrightarrow w$). Thus, we obtain that $|A(D)| \le \binom{n-1}{2} + 2 = \frac{n^2-3n+6}{2} \le \frac{n^2-2n}{2}$ since $n \ge 6$.

(2) $x \nrightarrow y$.

Obviously, x dominates at most one vertex of S_0 and y is dominated by at most one vertex of R_0 since D is B_2-free. Hence, we have $|A(D)| \le \binom{n-1}{2} - 1 + 2 + 2 = \frac{n^2-3n+8}{2} \le \frac{n^2-2n+2}{2}$ since $n \ge 6$.

Subcase 4.2. $n_1 \ge 3$ and $n_k = 1$. We denote $X_k = \{y\}$ and it is clear that $y \to v$.

(1) There exists a vertex x of R_1 such that $x \to y$. Then the following holds

$$\begin{aligned}
|A(D)| &= \sum_{i=0,1} |A(D_i)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_D(v) \\
&\le \frac{n_1^2 - 2n_1 + 3}{2} + \binom{n_0}{2} + n_0(n_1+1) + n_1 - (r_1 - 1) + r_1 + 1 \\
&= \frac{n^2 - 3n + 9 - n_1}{2} \le \frac{n^2 - 2n}{2}.
\end{aligned}$$

(2) For any vertex x of R_1, $x \nrightarrow y$. We obtain that

$$\begin{aligned}
|A(D)| &= \sum_{i=0,1} |A(D_i)| + \sum_{i\in\{0,1\}, j\in\{0,k\}, i\neq j} |A(X_i, X_j)| + d_D(v) \\
&\le \frac{n_1^2 - 2n_1 + 3}{2} + \binom{n_0}{2} + n_0(n_1+1) + n_1 - r_1 + r_1 + 1 + 2 \\
&= \frac{n^2 - 3n + 11 - n_1}{2} \le \frac{n^2 - 2n + 2}{2}.
\end{aligned}$$

Subcase 4.3. $n_1 = 1$ and $n_k \ge 3$.

The proof is similar to that of Subcase 4.2.

Subcase 4.4. $n_1 \ge 3$ and $n_k \ge 3$. As $n = n_0 + n_1 + n_k + 1$, $n \ge 8$ holds.

(1) $r_1 = s_k = 1$.

Then, we have $\binom{n}{2} - |A(D)| \geq n_1 - 1 + n_k - 1 + n_0 - 2 = n - 5$. Thereby, we get $|A(D)| \leq \frac{n^2-3n+10}{2} \leq \frac{n^2-2n+2}{2}$.

(2) $r_1 = 1$ and $s_k \geq 2$.

It is clear that $\binom{n}{2} - |A(D)| \geq n_1 - 1 + n_k - s_k + (s_k - 1) + n_0 - 2 = n - 5$, implying that $|A(D)| \leq \frac{n^2-3n+10}{2} \leq \frac{n^2-2n+2}{2}$.

(3) $r_1 \geq 2$ and $s_k = 1$.

Similar to (2), $|A(D)| \leq \frac{n^2-2n+2}{2}$ holds, we omit the proof.

(4) $r_1 \geq 2$ and $s_k \geq 2$.

(i) $A(R_1, S_k) \neq \emptyset$. There must be a triangle $\langle v, x, y \rangle$ such that $x \in R_1$ and $y \in S_k$. Analogous to the above discussion, we have

$$\begin{aligned}
&|A(D)| \\
&= \sum_{i=0,1,k} |A(D_i)| + \sum_{i \in \{0,1\}, j \in \{0,k\}, i \neq j} |A(X_i, X_j)| + d_D(v) \\
&\leq \sum_{i=1,k} \frac{n_i^2 - 2n_i + 3}{2} + \binom{n_0}{2} + n_0(n_1 + n_k) + n_1 n_k - (r_1 - 1) - (s_k - 1) + r_1 + s_k \\
&= \frac{n^2 - 3n + 12 - n_1 - n_k}{2} \leq \frac{n^2 - 3n + 6}{2} \leq \frac{n^2 - 2n - 2}{2}.
\end{aligned}$$

(ii) $A(R_1, S_k) = \emptyset$. Since $r_1 \geq 2$ and $s_k \geq 2$, we thus deduce that

$$\begin{aligned}
|A(D)| &= \sum_{i=0,1,k} |A(D_i)| + \sum_{i \in \{0,1\}, j \in \{0,k\}, i \neq j} |A(X_i, X_j)| + d_D(v) \\
&\leq \sum_{i=1,k} \frac{n_i^2 - 2n_i + 3}{2} + \binom{n_0}{2} + n_0(n_1 + n_k) + n_1 n_k - r_1 s_k + r_1 + s_k + 2 \\
&= \sum_{i=1,k} \frac{n_i^2 - 2n_i + 3}{2} + \binom{n_0}{2} + n_0(n_1 + n_k) + n_1 n_k - (r_1 - 1)(s_k - 1) + 3 \\
&\leq \frac{n^2 - 3n + 12 - n_1 - n_k}{2} \leq \frac{n^2 - 2n - 2}{2}.
\end{aligned}$$

This completes the proof of Part B. □

Combining the proofs of Part A and Part B, we complete our proof of Theorem 6. ■

3.3. Construction of Ψ

We now show that the upper bound of Theorem 6 is tight by constructing a suitable B_2-free strong digraphs Ψ on $n \geq 3$ vertices.

Construction of Ψ. Let Ψ be the digraph obtained from Θ by deleting all arcs (v_{i+2}, v_i), where $i \in \{1, 2, \ldots, n\}$ and i is even.

Apparently, Ψ is a strong digraph on n vertices. Any arc $(u, v) \in A(\Psi)$ satisfies that $|N^+_\Psi(v) \cap N^-_\Psi(u)| \leq 1$, which indicates that Ψ is B_2-free. Moreover, one can deduce that $|A(\Psi)| = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor + 1$.

In summary, we conclude that $ex(n, B_2) = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor + 1$.

4. CONCLUDING REMARKS

Recall that $\mathcal{C}_k = \{C_2, C_3, \ldots, C_k\}$ and C_i is a directed cycle of length $i \in \{2, 3, \ldots, k\}$. Let B_1 be a triangle and B_t be the union of t triangles sharing a unique common arc for $t \geq 2$. In 1980, Bermond, Germa, Heydemann and Sotteau [1] gave the precise Turán number of $\mathcal{C}_k$-free digraphs on n vertices, namely, $ex(n, \mathcal{C}_k) = \frac{n^2+(3-2k)n+k^2-k-2}{2}$. Particularly, the Turán number of B_1-free strong digraph is exactly $\frac{n^2-n+4}{2}$, that is, $ex(n, B_1) = \frac{n^2-n+4}{2}$.

In this paper, we mainly obtain the Turán number of strong digraphs on n vertices forbidden $t \geq 2$ different B_1. We construct a strong tournament Γ containing no arc disjoint B_1. Additionally, we present a B_t-free strong tournament Θ for $t \geq 3$. Finally, we verify that the maximum size of B_2-free strong digraphs on n vertices is at most $\binom{n}{2} - \lfloor \frac{n}{2} \rfloor + 1$ and then show another strong digraph Ψ whose size reaches this upper bound. That is, we determine the Turán number of B_2-free strong digraphs on n vertices.

Unfortunately, we do not fully characterize the structure of B_2-free Turán digraphs and we even do not known the minimum out-degree of such Turán digraphs. It would be interesting to study whether the Turán number of B_2-free strong digraphs will decrease or not if we add a condition that the minimum out-degree is at least two.

It would be also interesting to study the Turán number of strong digraphs without $t \geq 2$ vertex- or arc disjoint B_1 with out-degree restriction.

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