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ERRATUM TO "THE THICKNESS OF AMALGAMATIONS AND CARTESIAN PRODUCT OF GRAPHS" [DISCUSS. MATH. GRAPH THEORY 37 (2017) 561–572]

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Abstract

The authors explain and correct a mistake in [*The thickness of amalgamations and Cartesian product of graphs*, Discuss. Math. Graph Theory 37 (2017) 561–572]. The same mistake in [*The thickness of the Cartesian product of two graphs*, Canad. Math. Bull. 59 (2016) 705–720] is also corrected.

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In [3], the authors stated: "From a planar decomposition of $K_n \square P_2$, by contracting the edges from K_n^2 to a single vertex in every planar subgraphs, one can obtain a planar decomposition of K_{n+1}, so we have $\theta(K_n \square P_2) \geq \theta(K_{n+1})$". However, this is not true, because the graphs resulting from such contraction may not be planar. For example, Figure 1 exhibits a decomposition of $K_8 \square P_2$ into

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two planar subgraphs, but contracting either copy of K_8 makes at least one of the subgraphs non-planar. Consequently, Theorems 16 and 17 of [3] are not true, and they need to be replaced with weaker versions, which we will present here.

The following theorem is well-known.

Theorem 1 [1]**.** *The thickness of the complete graph* K_n *is* $\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$, *except that* $\theta(K_9) = \theta(K_{10}) = 3$.

For the thickness of $K_n \square P_m$ $(m \geq 2)$ we have a lower bound

$$\theta(K_n) \leq \theta(K_n \square P_m), \tag{1}$$

because K_n is a subgraph of $K_n \square P_m$. For upper bounds, from inequality (2) in [3] we have

$$\theta(K_n \square P_2) \leq \theta(K_{n+1}), \tag{2}$$

and for $m \geq 3$, from inequality (3) in [3] we have

$$\theta(K_n \square P_m) \leq \theta(K_{n+2} - e) \leq \theta(K_{n+2}). \tag{3}$$

Combining the lower and upper bounds we obtain the following results.

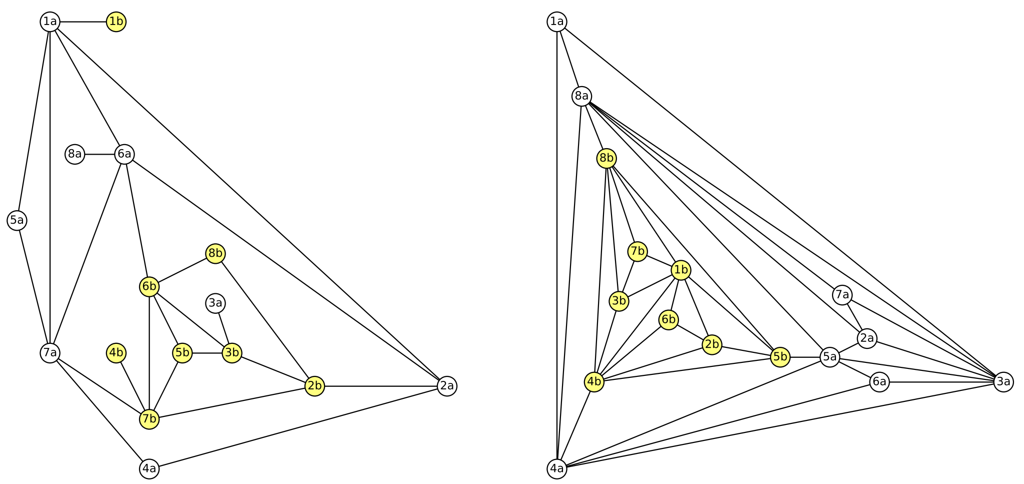

Figure 1. A planar decomposition of $K_8 \square P_2$. The numbers refer to the vertices of K_8. The letters and the colors refer to the vertices of P_2.

Theorem 2. *The thickness of the Cartesian product* $K_n \square P_2$ $(n \geq 2)$ *is*

$$\theta(K_n \square P_2) = \left\lfloor \frac{n+8}{6} \right\rfloor,$$

except that $\theta(K_9 \square P_2) = 3$ *and possibly when* $n = 6p + 4$ $(p \geq 2)$.

Proof. First we consider the general case that $n \notin \{8, 9\}$ and n is not of the form $6p+4$, where p is an integer. Then the lower bound (1) coincides with the upper bound (2), that is $\theta(K_n) = \theta(K_{n+1}) = \lfloor \frac{n+8}{6} \rfloor$, thus $\theta(K_n \Box P_2) = \lfloor \frac{n+8}{6} \rfloor$ as claimed.

Next we consider the remaining cases. If $n = 8$, by (1) we have $\theta(K_8 \Box P_2) \geq 2$. Figure 1 shows a decomposition of $K_8 \Box P_2$ into two planar subgraphs. Hence we have $\theta(K_8 \Box P_2) = 2 = \lfloor \frac{8+8}{6} \rfloor$ as claimed.

If $n = 4$, $K_4 \Box P_2$ is non-planar, and it is a subgraph of $K_8 \Box P_2$, thus $\theta(K_4 \Box P_2) = 2 = \lfloor \frac{4+8}{6} \rfloor$ as claimed.

If $n = 9$ or $n = 10$, again the lower bound (1) coincides with the upper bound (2); both are 3, thus $\theta(K_9 \Box P_2) = \theta(K_{10} \Box P_2) = 3$ as claimed. ■

We note that Theorem 2 leaves unknown the thicknesses of $K_{16} \Box P_2$, $K_{22} \Box P_2$, and so on.

Theorem 3. *The thickness of the Cartesian product $K_n \Box P_m$ $(n \geq 2, m \geq 3)$ is*

$$\theta(K_n \Box P_m) = \left\lfloor \frac{n+9}{6} \right\rfloor,$$

except that $\theta(K_3 \Box P_m) = 1$ and possibly when $n = 6p+3, 6p+4$ and $n = 8$ $(p \geq 2)$. Moreover, $\theta(K_8 \Box P_3) = 2$.

Proof. When $n \neq 7, 8$, from (1), (3) and Theorem 1, we obtain $\theta(K_n \Box P_m) = \theta(K_{n+2})$, except possibly when $n = 6p+3, 6p+4$ (p is a nonnegative integer).

When $n = 7$, we have $\theta(K_7) \leq \theta(K_7 \Box P_m) \leq \theta(K_9 - e)$, because both K_7 and $K_9 - e$ have thickness two, we have $\theta(K_7 \Box P_m) = 2$.

When $n = 3$, because $\theta(K_3) \leq \theta(K_3 \Box P_m) \leq \theta(K_5 - e)$ and both K_3 and $K_5 - e$ are planar graphs, we have $\theta(K_3 \Box P_m) = 1$.

When $n = 4$, $K_4 \Box P_m$ is non-planar and $\theta(K_4 \Box P_m) \leq \theta(K_6)$, so we have $\theta(K_4 \Box P_m) = 2$.

When $n = 8$ and $m = 3$, $K_8 \Box P_3$ is non-planar and has a decomposition into two planar subgraphs as shown in Figure 2, thus $\theta(K_8 \Box P_3) = 2$.

When $n = 9$, because $\theta(K_9) \leq \theta(K_9 \Box P_m) \leq \theta(K_{11})$ and both K_9 and K_{11} have thickness three, we have $\theta(K_9 \Box P_m) = 3$.

When $n = 10$, because $\theta(K_{10}) \leq \theta(K_9 \Box P_m) \leq \theta(K_{12})$ and both K_{10} and K_{12} have thickness three, we have $\theta(K_{10} \Box P_m) = 3$. ■

Let G be a connected graph, and let $v \notin G$. We denote by $G + v$ the graph obtained by connecting every vertex of G to the new vertex v. Furthermore, Lemma 2.3 in [2] states that the thickness of $G \Box K_2$ is equal to $\theta(G+v)$. However, the proof contains an error similar to one mentioned previously. In fact, it demonstrates that $\theta(G \Box K_2) \leq \theta(G+v)$. Theorem 2.4 in [2] is dependent on Lemma 2.3 in [2]; therefore, it cannot be considered valid.

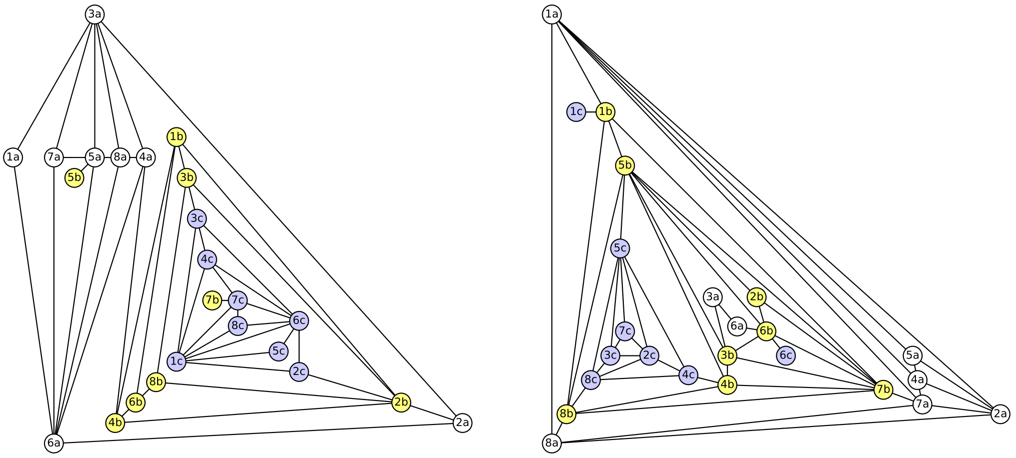

Figure 2. A planar decomposition of $K_8 \Box P_3$. The numbers refer to the vertices of K_8. The letters and the colors refer to the vertices of P_3.

For convenience we list here the edges of the decompositions in our figures.

Figure 1 left, 25 edges:

$(1a, 1b), (1a, 2a), (1a, 5a), (1a, 6a), (1a, 7a), (2a, 2b), (2a, 4a), (2a, 6a), (2b, 3b),$
$(2b, 7b), (2b, 8b), (3a, 3b), (3b, 5b), (3b, 6b), (4a, 7a), (4b, 7b), (5a, 7a), (5b, 6b),$
$(5b, 7b), ((6a, 6b), 6a, 7a), (6a, 8a), (6b, 7b), (6b, 8b), (7a, 7b).$

Figure 1 right, 39 edges:

$(1a, 3a), (1a, 4a), (1a, 8a), (1b, 2b), (1b, 3b), (1b, 4b), (1b, 5b), (1b, 6b), (1b, 7b),$
$(1b, 8b), (2a, 3a), (2a, 5a), (2a, 7a), (2a, 8a), (2b, 4b), (2b, 5b), (2b, 6b), (3a, 4a),$
$(3a, 5a), (3a, 6a), (3a, 7a), (3a, 8a), (3b, 4b), (3b, 7b), (3b, 8b), (4a, 4b), (4a, 5a),$
$(4a, 6a), (4a, 8a), (4b, 5b), (4b, 6b), (4b, 8b), (5a, 5b), (5a, 6a), (5a, 8a), (5b, 8b),$
$(7a, 8a), (7b, 8b), (8a, 8b).$

Figure 2 left, 47 edges:

$(1a, 3a), (1a, 6a), (1b, 2b), (1b, 3b), (1b, 4b), (1b, 6b), (1c, 2c), (1c, 3c), (1c, 4c),$
$(1c, 5c), (1c, 6c), (1c, 7c), (1c, 8c), (2a, 2b), (2a, 3a), (2a, 6a), (2b, 2c), (2b, 3b),$
$(2b, 4b), (2b, 8b), (2c, 6c), (3a, 4a), (3a, 5a), (3a, 7a), (3a, 8a), (3b, 3c), (3b, 8b),$
$(3c, 4c), (3c, 6c), (4a, 4b), (4a, 6a), (4a, 8a), (4b, 6b), (4c, 6c), (4c, 7c), (5a, 5b),$
$(5a, 6a), 5a, 7a), (5a, 8a), (5c, 6c), (6a, 7a), (6a, 8a), (6b, 8b), (6c, 7c), (6c, 8c),$
$(7b, 7c), (7c, 8c).$

Figure 2 right, 53 edges:

$(1a, 1b), (1a, 2a), (1a, 4a), (1a, 5a), (1a, 7a), (1a, 8a), (1b, 1c), (1b, 5b), (1b, 7b), (1b, 8b), (2a, 4a), (2a, 5a), (2a, 7a), (2a, 8a), (2b, 5b), (2b, 6b), (2b, 7b), (2c, 3c), (2c, 4c), (2c, 5c), (2c, 7c), (2c, 8c), (3a, 3b), (3a, 6a), (3b, 4b), (3b, 5b), (3b, 6b), (3b, 7b), (3c, 5c), (3c, 7c), (3c, 8c), (4a, 5a), (4a, 7a), (4b, 4c), (4b, 5b), (4b, 7b), (4b, 8b), (4c, 5c), (4c, 8c), (5b, 5c), (5b, 6b), (5b, 7b), (5b, 8b), (5c, 7c), (5c, 8c), (6a, 6b), (6b, 6c), (6b, 7b), (7a, 7b), (7a, 8a), (7b, 8b), (8a, 8b), (8b, 8c).$

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