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ON S-PACKING EDGE-COLORING OF GRAPHS WITH GIVEN EDGE WEIGHT

Jian Lu

School of Statistics and Applied Mathematics
Anhui University of Finance & Economics, Bengbu 233030, P.R. China
e-mail: lujianmath@163.com

and

Xiang-Feng Pan

School of Mathematical Sciences
Anhui University, Hefei 230601, P.R. China
e-mail: xfpan@ahu.edu.cn

Abstract

Let $S = (s_1, \dots, s_k)$ be a non-decreasing sequence of positive integers. A graph $G = (V(G), E(G))$ is said to be S-packing edge-colorable if $E(G)$ can be decomposed into disjoint sets $E_1, \dots, E_k$ such that for every $1 \leq i \leq k$, the distance between any two distinct edges in E_i is at least $s_i + 1$. The edge weight of G is defined as $ew(G) = \max\{d(u) + d(v) | uv \in E(G)\}$. A fork is the graph obtained from $K_{1,3}$ by subdividing an edge once. In 2023, Liu *et al.* proved that every subcubic multigraph is $(1, 2^7)$-packing edge-colorable. Based on the work of Liu *et al.*, we prove that every multigraph G with $ew(G) \leq 6$ is $(1, 2^7)$-packing edge-colorable, which confirms a conjecture of Yang and Wu (2022). In addition, we demonstrate that if G is a fork-free graph with $ew(G) \leq 6$, then G is $(1, 2^6)$-packing edge-colorable.

Keywords: S-packing edge-coloring, edge weight, fork-free graphs.

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1. Introduction

All graphs considered in this paper are finite and undirected. The *distance* between two vertices u and v in a graph $G = (V(G), E(G))$ is the length of a shortest path between u and v. For two edges e_1, e_2 in $E(G)$, the *distance* $d(e_1, e_2)$ between e_1 and e_2 is the distance between the corresponding vertices of e_1 and e_2 in the line graph of G.

Let $S = (s_1, s_2, \ldots, s_k)$ be a non-decreasing sequence of integers. An *S-packing edge-coloring* of G is a partition $E_1, E_2, \ldots, E_k$ of $E(G)$ such that for $1 \leq i \leq k$, $d(e_1, e_2) \geq s_i + 1$ for any two edges e_1 and e_2 in E_i. Note that if all $s_i = 1$ or all $s_i = 2$, then an S-packing edge-coloring is equivalent to a proper edge-coloring or a strong edge-coloring [2], respectively. In this paper, we are only concerned with the case where each $s_i \in \{1, 2\}$. For convenience, we use exponents to indicate identical components repeated in S, e.g., $(1, 1, 2, 2, 2) = (1^2, 2^3)$. And we write the color set of the $(1, 2^k)$-packing edge-coloring of G as $\{0, 1, 2, \ldots, k\}$, where 0 is the color that allows edges with distance at least 2 to be colored, and we collectively refer to colors 1 through k as the 2-colors, which are the colors allow edges with distance at least 3 to be colored.

The concept of S-packing edge-colorings is derived from its corresponding vertex counterpart, which was first proposed by Gastineau and Togni [3] as a logical extension of the packing chromatic number [4]. Fouquet and Vanherpe [1] proved that any subcubic graph admits a $(1^3, 2)$-packing edge-coloring. A spanning subgraph G' of G is called a 2-factor of G if each component of G' is a cycle. Gastineau and Togni [3] demonstrated that for each cubic graph G with a 2-factor, G is $(1^2, 2^5)$-packing edge-colorable.

In light of Vizing's [10] work $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, we know that if $\chi'(G) = \Delta(G)$, then G is said to be in class I; if $\chi'(G) = \Delta(G) + 1$, then G is said to be in class II, where $\chi'(G)$ and $\Delta(G)$ are the chromatic index and the maximum degree of G, respectively. Gastineau *et al.* [3] and Hocquard *et al.* [6] posed several conjectures of S-packing edge-coloring on subcubic graphs, especially in class I.

Conjecture 1. *If G is a simple subcubic graph, then G is*

(a) $(1^2,\ 2^4)$*-packing edge-colorable* [3];

(b) $(1, 2^7)$*-packing edge-colorable* [3];

(c) $(1^2,\ 2^3)$*-packing edge-colorable if G is in class* I [3];

(d) $(1, 2^6)$*-packing edge-colorable if G is in class* I [6].

In [6], Hocquard *et al.* made progress towards these conjectures by proving the following results.

Theorem 2 [6]**.** *If G is a simple subcubic graph, then G is*

(a) $(1^2,\ 2^5)$*-packing edge-colorable;*

(b) $(1, 2^8)$*-packing edge-colorable;*

(c) $(1^2,\ 2^4)$*-packing edge-colorable if G is in class* I;

(d) $(1, 2^7)$-*packing edge-colorable if* G *is in class* I.

Moreover, Hocquard *et al.* posed several problems of S-packing edge-coloring on planar graphs and bipartite graphs with $ew(G) \leq 5$, where $ew(G) = \max\{d(u) + d(v) : uv \in E(G)\}$ is called the *edge weight* of G.

Problem 3 [6]**.** If G is a simple bipartite graph with $ew(G) \leq 5$, then G is $(1, 2^4)$-packing edge-colorable.

Recently, Liu *et al.* [7, 8] proved Conjecture 1(a) and (b), and in particular, for Conjecture 1(b), they obtained a stronger conclusion.

Theorem 4 [8]**.** *If* G *is a connected subcubic graph with more than* 70 *vertices, then* G *is* $(1^2, 2^4)$-*packing edge-colorable.*

Theorem 5 [7]**.** *If* G *is a subcubic multigraph, then* G *is* $(1, 2^7)$-*packing edge-colorable.*

In [11], Yang and Wu solved Problem 3 and got a more favorable result. They showed that every simple graph G with $ew(G) \leq 5$ is $(1, 2^4)$-packing edge-colorable. In addition, they proved that every simple graph G with $ew(G) \leq 6$ is $(1, 2^8)$-packing edge-colorable and made the following conjecture.

Conjecture 6 [11]**.** *If* G *is a simple graph with* $ew(G) \leq 6$*, then* G *is* $(1, 2^7)$-*packing edge-colorable.*

In this paper, by proving the following result and combining it with Theorem 5, we prove Conjecture 6.

Theorem 7. *For every multigraph* G *with* $ew(G) \leq 6$ *and* $\Delta(G) \geq 4$*,* G *is* $(1, 2^7)$-*packing edge-colorable.*

The graph obtained from $K_{1,3}$ (usually called *claw*) by subdividing an edge once is called a *fork*. A graph is H-*free* if it does not contain H as an induced subgraph. In this paper, we also consider the S-packing edge-coloring on fork-free graphs with $ew(G) \leq 6$ and get the following result.

Theorem 8. *For every fork-free multigraph* G *with* $ew(G) \leq 6$*,* G *is* $(1, 2^6)$-*packing edge-colorable.*

The graphs G' and G'' in Figure 1 show that our results in Theorems 7–8 are sharp. In [6], Hocquard *et al.* showed that G' is $(1, 2^7)$-packing edge-colorable but not $(1, 2^6)$-packing edge-colorable. For the graph G'', it can be seen that G'' is $(1, 2^6)$-packing edge-colorable. Note that $|E(G'')| = 9$, the distance between any two edges in $E(G'')$ is at most two, and the maximum size of a matching in G'' is three. Hence, at most three edges in G'' can be colored with color 0, and the remaining six edges must be colored with different 2-colors. Therefore, G'' is not $(1, 2^5)$-packing edge-colorable.

G' G''

Figure 1. Sharpness examples of Theorems 7–8.

2. PRELIMINARIES

Let $G = (V(G), E(G))$ be a graph. For $v \in V(G)$, $N_G(v)$ is the set of all neighbors of v. We denote the degree of v by $d_G(v) = |N_G(v)|$. If $d_G(v) = d$, then v is called a d-vertex. When G is clear from the context, then we write $N(v)$ and $d(v)$ instead of $N_G(v)$ and $d_G(v)$, respectively. Let $\delta(G) = \min\{d(v)|v \in V(G)\}$. We denote by $N'(e)$ the set of edges with distance 1 to e and $N''(e)$ the set of edges with distance at most 2 to e. Clearly, $N'(e) \subseteq N''(e)$. For $V' \subseteq V(G)$, the graph induced by $V(G) \setminus V'$ is denoted as $G - V'$. If $V' = \{v\}$, we simplify $G - \{v\}$ to $G - v$.

To prove Theorem 7, we need the following two lemmas. Let $T_1, \ldots, T_n$ be n subsets of a set T. A subset $\{t_1, t_2, \ldots, t_n\} \subseteq T$ is called the system of distinct representatives of $\{T_1, \ldots, T_n\}$ if $t_i \in T_i$ and $t_i \neq t_j$ for $1 \leq i, j \leq n$.

Lemma 9 (Hall's marriage theorem [5]). *Let $T_1, \ldots, T_n$ be n subsets of a set T. A system of distinct representatives of $\{T_1, \ldots, T_n\}$ exists if and only if for all k, $1 \leq k \leq n$ and every subcollection of size k, $\{T_{i_1}, \ldots, T_{i_k}\}$, we have $|T_{i_1} \cup \cdots \cup T_{i_k}| \geq k$.*

Recall that a strong k-edge-coloring is a (2^k)-packing edge-coloring. Due to the work of Nakprasit [9] on strong edge-coloring of bipartite graphs, we have the following lemma.

Lemma 10 [9]. *Let G be a simple bipartite graph in which the vertices in one part have maximum degree 2. Then G is $(2^{2\Delta(G)})$-packing edge-colorable.*

From now on, a $(1, 2^k)$-*coloring* in this paper refers to $(1, 2^k)$-packing edge-coloring, where $k = 6$ or 7. A *partial coloring* of G is a coloring of any subset of $E(G)$ using the colors $\{0, 1, 2, \ldots, k\}$, such that for any two colored edges e_1 and e_2, if they are both colored with the color 0, then $d(e_1, e_2) \geq 2$; if they are both colored with a same 2-color, then $d(e_1, e_2) \geq 3$.

Let G_1 be a proper subgraph of G with a partial coloring φ. For any $e \in E(G) \setminus E(G_1)$, we denote by $A_\varphi(e)$ the set of 0-color and 2-colors that can be used to color e, and by $A^2_\varphi(e)$ the set of 2-colors that can be used to color e. And we write $A_\varphi(e)$ and $A^2_\varphi(e)$ as $A(e)$ and $A^2(e)$, respectively, if it is clear from the context. Obviously, if no edges in $N'(e)$ are colored 0 under φ, then $A(e) = A^2(e) \cup \{0\}$, and we can color e with color 0. Otherwise, $A(e) = A^2(e)$, we can only color e with a 2-color in $A^2(e)$. We say that φ is *finished* when we extend φ from G_1 to G.

3. PROOF OF THEOREM 7

Assume that Theorem 7 fails, and H is a counterexample with $|V(H)| + |E(H)|$ as small as possible. By the choice of H, H is a connected multigraph with $ew(H) \leq 6$ and any proper subgraph of H has a $(1, 2^7)$-coloring.

Claim 11. *H is simple.*

Proof. Suppose that there are k multiple edges $e_1, e_2, \ldots, e_k$ between two vertices u and v in H, where $k \geq 2$. Since $ew(H) \leq 6$, $k \leq 3$. If $k = 3$, then H is the graph with 2 vertices and 3 edges. Obviously, H is $(1, 2^7)$-colorable, a contradiction. Hence $k = 2$. Let $H_1 = H - e_1$. Then H_1 has a $(1, 2^7)$-coloring φ by the minimality of H. Note that there are at most 7 edges in $N''(e_1)$ as $ew(H) \leq 6$. Hence, $|A(e_1)| \geq 1$. If all the edges in $N''(e_1)$ are colored with different 2-colors under φ, then we can finish φ by coloring e_1 with color 0. Otherwise, we can finish φ by coloring e_1 with a 2-color in $A^2(e_1)$, a contradiction. ■

Claim 12. $\delta(H) \geq 2$.

Proof. Suppose that v is a 1-vertex in H with a neighbor u. By the minimality of H, $H_1 = H - v$ has a $(1, 2^7)$-coloring φ. Since $ew(H) \leq 6$, there are at most 6 edges in $N''(uv)$. Hence, $|A^2(uv)| \geq 1$, and we can finish φ by coloring uv with a 2-color in $A^2(uv)$, a contradiction. ■

Claim 13. *H has no adjacent 2-vertices.*

Proof. Suppose that u and v are two adjacent 2-vertices in H. Denote $N(u) = \{u_1, v\}$ and $N(v) = \{v_1, u\}$. Since $ew(H) \leq 6$, $d(v_1) \leq 4$ and $d(u_1) \leq 4$. Let $H_1 = H - \{u, v\}$. Then H_1 has a $(1, 2^7)$-coloring φ by the minimality of H.

If $0 \in A_\varphi(vv_1)$ and $0 \in A_\varphi(uu_1)$, then we first color vv_1 and uu_1 with color 0 and call this coloring ϕ_1. Observe that $|A^2_{\phi_1}(uv)| \geq 7 - ((d(v_1) - 1) + (d(u_1) - 1)) \geq 1$. Hence, we can finish ϕ_1 by coloring uv with a 2-color in $A^2_{\phi_1}(uv)$, a contradiction.

If only one of $A_\varphi(vv_1)$ and $A_\varphi(uu_1)$, say $A_\varphi(uu_1)$, does not contain color 0, then there must exist some edge $e \in N'(uu_1) \setminus \{uv\}$ such that $\varphi(e) = 0$. We first color vv_1 with 0 and call this coloring ϕ_2. Observe that $|A^2_{\phi_2}(uu_1)| \geq 2$ and $|A^2_{\phi_2}(uv)| \geq 2$. Hence, we can finish ϕ_2 by coloring uv and uu_1 with different 2-colors, a contradiction.

If $0 \notin A_\varphi(vv_1)$ and $0 \notin A_\varphi(uu_1)$, then there are two edges $e_1 \in N'(uu_1)\setminus\{uv\}$ and $e_2 \in N'(vv_1) \setminus \{uv\}$ such that $\varphi(e_1) = \varphi(e_2) = 0$. We first color uv with color 0 and call this coloring ϕ_3. Observe that $|A^2_{\phi_3}(uu_1)| \geq 2$ and $|A^2_{\phi_3}(vv_1)| \geq 2$. Hence, we can finish ϕ_3 by coloring uu_1 and vv_1 with different 2-colors, a contradiction. ■

Claim 14. *H has no 4-cycle that contains a 4-vertex.*

Proof. Suppose that $C_4 = v_1v_2v_3v_4v_1$ is a 4-cycle in H with $d(v_1) = 4$. Denote $N(v_1) = \{w_1, w_2, v_2, v_4\}$. Since $ew(H) \leq 6$ and $\delta(H) \geq 2$ (by Claim 12), $d(w_1) = d(w_2) = d(v_2) = d(v_4) = 2$. By Claim 13, $3 \leq d(v_3) \leq 4$.

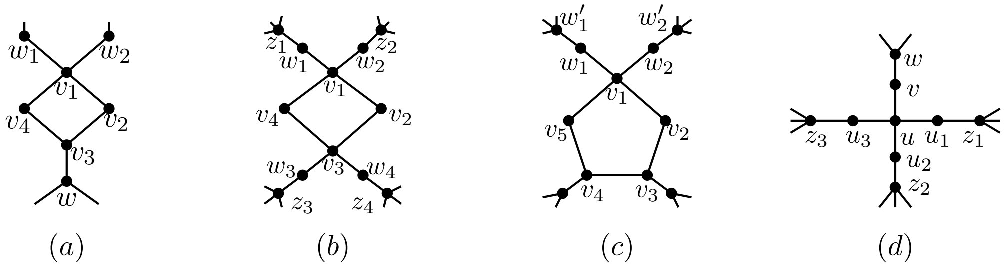

Figure 2. The reducible configurations of Claims 14–16.

Case 1. $d(v_3) = 3$. Denote $N(v_3) = \{w, v_2, v_4\}$ (see Figure 2(a)). Let $H_1 = H - v_1$. Then H_1 has a $(1, 2^7)$-coloring φ by the minimality of H. Next, we erase the color of v_3v_4 and v_3v_2 in H_1. If $\varphi(v_3w) = 0$, then we first color v_1v_4 with color 0 and call this partial coloring ϕ_1. Observe that $|A^2_{\phi_1}(e)| \geq 5$ for $e \in E(C_4) \setminus \{v_1v_4\}$ and $|A^2_{\phi_1}(v_1w_i)| \geq 2$ for $1 \leq i \leq 2$. Hence, we can finish ϕ_1 by sequentially coloring $v_1w_1, v_1w_2, v_1v_2,\ v_2v_3, v_3v_4$ with different 2-colors, a contradiction. If $\varphi(v_3w) \neq 0$, then we first color v_1v_2 and v_3v_4 with color 0 and call this partial coloring ϕ_2. Observe that $|A^2_{\phi_2}(v_1v_4)| \geq 4$, $|A^2_{\phi_2}(v_2v_3)| \geq 4$ and $|A^2_{\phi_2}(v_1w_i)| \geq 2$ for $1 \leq i \leq 2$. Hence, we can finish ϕ_2 by sequentially coloring $v_1w_1, v_1w_2, v_2v_3, v_1v_4$ with different 2-colors, a contradiction.

Case 2. $d(v_3) = 4$. Denote $N(v_3) = \{w_3, w_4, v_2, v_4\}$ (see Figure 2(b)). Let $H_2 = H - \{v_1, v_3\}$. Then H_2 has a $(1, 2^7)$-coloring ψ by the minimality of H. Since $ew(H) \leq 6$ and $\delta(H) \geq 2$, $d(w_3) = d(w_4) = 2$. By Claim 13, $w_3w_4 \notin E(H)$.

Subcase 2.1. $w_3 = w_1$. We first color v_3w_1 and v_1v_4 with color 0 and call this partial coloring ψ_1. Observe that $|A^2_{\psi_1}(v_1w_2)| \geq 3$, $|A^2_{\psi_1}(v_3w_4)| \geq 3$, $|A^2_{\psi_1}(v_1w_1)| \geq 6$, $|A^2_{\psi_1}(v_1v_2)| \geq 6$, $|A^2_{\psi_1}(v_2v_3)| \geq 6$ and $|A^2_{\psi_1}(v_3v_4)| \geq 6$. Hence, we can finish ψ_1 by sequentially coloring $v_1w_2, v_3w_4, v_1w_1, v_1v_2, v_2v_3, v_3v_4$ with different 2-colors, a contradiction.

Subcase 2.2. $w_3 \neq w_1$. By the symmetry of w_1, w_2, w_3 and w_4, we have $w_i \neq w_j$ for $1 \leq i \neq j \leq 4$. Recall that $d(w_i) = 2$ for $1 \leq i \leq 4$. Let z_i be the neighbor of w_i not in C_4 for $1 \leq i \leq 4$.

Subcase 2.2.1. $\varphi(w_i z_i) = 0$ for all $1 \leq i \leq 4$. We first color v_1v_4 with color 0 and call this partial coloring ψ_2. Since $d(w_i) = 2$ and $d(z_i) \leq 4$ for $1 \leq i \leq 4$, we have $|A^2_{\psi_2}(v_1w_1)| \geq 4$, $|A^2_{\psi_2}(v_1w_2)| \geq 4$, $|A^2_{\psi_2}(v_3w_3)| \geq 4$, $|A^2_{\psi_2}(v_3w_4)| \geq 4$, $|A^2_{\psi_2}(v_1v_2)| \geq 7$, $|A^2_{\psi_2}(v_2v_3)| \geq 7$ and $|A^2_{\psi_2}(v_3v_4)| \geq 7$. Hence, we can finish ψ_2 by sequentially coloring v_1w_1, v_1w_2, v_3w_3, v_3w_4, v_1v_2, v_2v_3, v_3v_4 with different 2-colors, a contradiction.

Subcase 2.2.2. $\varphi(w_j z_j) \neq 0$ for some $1 \leq j \leq 4$. Assume $\varphi(w_3z_3) \neq 0$ by symmetry. We first color v_3w_3 with color 0 and call this partial coloring ψ_3. Then we delete the color of w_2z_2 under ψ_3. Observe that $|A^2_{\psi_3}(v_1w_1)| \geq 3$, $|A^2_{\psi_3}(v_1w_2)| \geq 3$, $|A^2_{\psi_3}(v_1v_2)| \geq 6$, $|A^2_{\psi_3}(v_1v_4)| \geq 6$, $|A^2_{\psi_3}(v_2v_3)| \geq 5$, $|A^2_{\psi_3}(v_3v_4)| \geq 5$, $|A^2_{\psi_3}(v_3w_4)| \geq 2$ and $|A_{\psi_3}(w_2z_2)| \geq 2$.

Subcase 2.2.2.1. $0 \in A_{\psi_3}(w_2z_2)$. Note that $d(w_2z_2, v_1v_4) = 2$, hence we can color w_2z_2 and v_1v_4 with color 0 and then color v_3w_4, v_1w_1, v_1w_2, v_2v_3, v_3v_4, v_1v_2 with different 2-colors in order to finish ψ_3, a contradiction.

Subcase 2.2.2.2. $0 \notin A_{\psi_3}(w_2z_2)$. Then $\psi_3(z_2z_2') = 0$, where $z_2' \in N(z_2)\setminus\{w_2\}$. Hence, $|A^2_{\psi_3}(w_2z_2)| \geq 2$. We first color v_1w_2 with color 0. Recall that for $1 \leq i \leq 2$, $w_i \neq w_4$, and by Claim 13, $w_iw_4 \notin E(H)$. Thus $d(v_1w_i, v_3w_4) > 2$ for $1 \leq i \leq 2$ and $d(w_2z_2, v_2v_3) > 2$. If $A^2_{\psi_3}(w_2z_2) \cap A^2_{\psi_3}(v_2v_3) \neq \emptyset$, then we can color w_2z_2 and v_2v_3 with a same color in $A^2_{\psi_3}(w_2z_2) \cap A^2_{\psi_3}(v_2v_3)$ and then finish ψ_3 by sequentially coloring $v_3w_4, v_1w_1, v_3v_4, v_1v_4, v_1v_2$ with different 2-colors, a contradiction. If $A^2_{\psi_3}(w_2z_2) \cap A^2_{\psi_3}(v_2v_3) = \emptyset$, then $|A^2_{\psi_3}(w_2z_2) \cup A^2_{\psi_3}(v_2v_3)| \geq 7$. Let $T = \{w_2z_2, v_3w_4, v_1w_1, v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Observe that for any $T' \subseteq T$, $|\bigcup_{e\in T'} A^2_{\psi_3}(e)| \geq |T'|$. Hence, by Lemma 9, we can finish ψ_3 by coloring each edge in T with a different 2-color, which is a contradiction. ■

Claim 15. *H has no 5-cycle that contains a 4-vertex.*

Proof. Suppose otherwise that there is a 5-cycle $v_1v_2v_3v_4v_5v_1$ in H with $d(v_1) = 4$. Denote $N(v_1) = \{w_1, w_2, v_2, v_5\}$. Since $ew(H) \leq 6$, $d(w_1) = d(w_2) = d(v_2) = d(v_4) = 2$. Denote $N(w_1) = \{v_1, w_1'\}$ and $N(w_2) = \{v_1, w_2'\}$. Then $d(w_1') \leq 4$ and $d(w_2') \leq 4$ as $ew(H) \leq 6$. By Claim 13, $d(v_3) = d(v_4) = 3$ (see Figure 2(c)). Let $H_1 = H - \{v_1, v_2, v_5\}$. Then H_1 has a $(1, 2^7)$-coloring φ by the minimality of H. We erase the color of v_3v_4 in H_1 under φ. Observe that $|A_\varphi(v_1w_1)| \geq 3$, $|A_\varphi(v_1w_2)| \geq 3$, $|A_\varphi(v_1v_2)| \geq 5$, $|A_\varphi(v_1v_5)| \geq 5$, $|A_\varphi(v_5v_4)| \geq 4$, $|A_\varphi(v_3v_4)| \geq 2$ and $|A_\varphi(v_2v_3)| \geq 4$.

Case 1. $0 \notin A_\varphi(v_2v_3)$. Then $\varphi(v_3v_3') = 0$, where $v_3' \in N(v_3) \setminus \{v_2, v_4\}$. Denote $N(v_4) = \{v_3, v_5, v_4'\}$. Since $ew(H) \leq 6$, there are at most 7 colored edges in $N''(v_4v_4')$, including v_3v_3'. We first erase the color of v_4v_4' and recolor v_4v_4' with a 2-color not used on the edges in $N''(v_4v_4')$ to ensure that $\varphi(v_4v_4') \neq 0$. Then color v_4v_5 with color 0 and call this partial coloring ϕ. Next, if $0 \in A_\phi(v_1w_1)$, then we color v_1w_1 with color 0. Observe that $|A^2_\phi(v_3v_4)| \geq 2$, $|A^2_\phi(v_1w_2)| \geq 2$, $|A^2_\phi(v_2v_3)| \geq 4$, $|A^2_\phi(v_1v_5)| \geq 4$ and $|A^2_\phi(v_1v_2)| \geq 5$. Hence, we can finish ϕ by sequentially coloring $v_3v_4, v_1w_2, v_2v_3, v_1v_5, v_1v_2$ with different 2-colors, a contradiction. Hence, $0 \notin A_\phi(v_1w_1)$, which implies $\phi(w_1w_1') = 0$, where $w_1' \in N(w_1)\setminus\{v_1\}$. Then $|A^2_\phi(v_3v_4)| \geq 2$, $|A^2_\phi(v_1w_1)| \geq 3$, $|A^2_\phi(v_1w_2)| \geq 3$, $|A^2_\phi(v_2v_3)| \geq 4$, $|A^2_\phi(v_1v_5)| \geq 5$ and $|A^2_\phi(v_1v_2)| \geq 6$. Hence, we can finish ϕ by sequentially coloring v_3v_4, v_1w_1, v_1w_2, v_2v_3, v_1v_5, v_1v_2 with different 2-colors, a contradiction.

Case 2. $0 \in A_\varphi(v_2v_3)$. By symmetry of v_3 and v_4, we also have $0 \in A_\varphi(v_4v_5)$. We first color v_2v_3 and v_4v_5 with color 0 and call this partial coloring τ. If $0 \in A_\tau(v_1w_1)$, then we color v_1w_1 with color 0. Observe that $|A^2_\tau(v_3v_4)| \geq 1$, $|A^2_\tau(v_1w_2)| \geq 2$, $|A^2_\tau(v_1v_5)| \geq 4$ and $|A^2_\tau(v_1v_2)| \geq 4$. Hence, we can finish τ by sequentially coloring v_3v_4, v_1w_2, v_1v_5, v_1v_2 with different 2-colors, a contradiction. Hence, $0 \notin A_\tau(v_1w_1)$, which implies $\tau(w_1w_1') = 0$, where $w_1' \in N(w_1)\setminus\{v_1\}$. Then $|A^2_\tau(v_3v_4)| \geq 1$, $|A^2_\tau(v_1w_1)| \geq 3$, $|A^2_\tau(v_1w_2)| \geq 3$, $|A^2_\tau(v_1v_5)| \geq 5$ and $|A^2_\tau(v_1v_2)| \geq 5$. Hence, we can finish τ by sequentially coloring v_3v_4, v_1w_1, v_1w_2, v_1v_5, v_1v_2 with different 2-colors, a contradiction. ∎

Claim 16. *No* 2*-vertex in* H *is adjacent to a* 4*-vertex and a* 3*-vertex.*

Proof. Suppose that v is a 2-vertex in H adjacent to a 4-vertex u and a 3-vertex w. Denote $N(u) = \{v, u_1, u_2, u_3\}$. Since $ew(H) \leq 6$, $d(u_1) = d(u_2) = d(u_3) = 2$ by Claim 12. By Claim 13, $vu_i \notin E(H)$ and $u_iu_j \notin E(H)$ for $1 \leq i \neq j \leq 3$. Denote $N(u_i) = \{u, z_i\}$ for $1 \leq i \leq 3$, then $d(z_i) \leq 4$ as $ew(H) \leq 6$. By Claim 14, $z_i \neq z_j$ and $z_i \neq w$ for $1 \leq i \neq j \leq 3$. By Claim 15, $z_iw \notin E(H)$ and $z_iz_j \notin E(H)$ for $1 \leq i \neq j \leq 3$. Hence, the distance between any two edges in $\{u_1z_1, u_2z_2, u_3z_3, wv\}$ is 3 (see Figure 2(d)). Let $H_1 = H - N(u)$, then H_1 has a $(1, 2^7)$-coloring φ by the minimality of H. Observe that $|A_\varphi(vw)| \geq 2$, $|A_\varphi(uv)| \geq 6$, $|A_\varphi(u_iz_i)| \geq 2$ and $|A_\varphi(uu_i)| \geq 5$ for $1 \leq i \leq 3$. We first color uu_3 with color 0.

Case 1. $0 \in A_\varphi(wv)$. Then we color wv with color 0 and call this partial coloring ϕ.

Subcase 1.1. $0 \in \bigcup_{1\leq i\leq 3} A_\phi(u_iz_i)$. By symmetry, let $0 \in A_\phi(u_1z_1)$ and we color u_1z_1 with color 0. Observe that $|A^2_\phi(u_2z_2)| \geq 1$, $|A^2_\phi(u_3z_3)| \geq 1$, $|A^2_\phi(uu_1)| \geq 4$, $|A^2_\phi(uu_2)| \geq 4$ and $|A^2_\phi(uv)| \geq 5$. Hence, we can finish ϕ by sequentially coloring $u_2z_2, u_3z_3, uu_1, uu_2, uv$, a contradiction.

Subcase 1.2. $0 \notin \bigcup_{1\leq i\leq 3} A_\phi(u_iz_i)$. It follows that for each $1 \leq i \leq 3$, there exists some edge z_iz_i' with $\phi(z_iz_i') = 0$, where $z_i' \in N(z_i) \setminus \{u_i\}$. Hence, $|A^2_\phi(uv)| \geq 5$, $|A^2_\phi(uu_1)| \geq 5$, $|A^2_\phi(uu_2)| \geq 5$ and $|A^2_\phi(u_iz_i)| \geq 2$ for $1 \leq i \leq 3$.

If there exist two edges in $\{u_1z_1, u_2z_2, u_3z_3\}$, say u_1z_1 and u_2z_2, that satisfy $A^2_\phi(u_1z_1) \cap A^2_\phi(u_2z_2) \neq \emptyset$, then we can color u_1z_1 and u_2z_2 with the same color in $A^2_\phi(u_1z_1) \cap A^2_\phi(u_2z_2)$ and then finish ϕ by sequentially coloring u_3z_3, uu_1, uu_2, uv with different 2-colors, a contradiction.

If $A^2_\phi(u_iz_i) \cap A^2_\phi(u_jz_j) = \emptyset$ for $1 \leq i \neq j \leq 3$, then $|\bigcup_{1\leq i\leq 3} A^2_\phi(u_iz_i)| \geq 6$. Let $T = \{u_1z_1, u_2z_2, u_3z_3, uu_1, uu_2, uv\}$. Observe that for any $T' \subseteq T$, $|\bigcup_{e\in T'} A^2_\phi(e)| \geq |T'|$. Hence, by Lemma 9, we can finish ϕ by coloring each edge in T with a different 2-color, which is a contradiction.

Case 2. $0 \notin A_\varphi(wv)$. Then there exists some edge ww' with $\varphi(ww') = 0$, where $w' \in N(w) \setminus \{v\}$.

Subcase 2.1. $0 \in \bigcup_{1\leq i\leq 3} A_\phi(u_iz_i)$. By symmetry, let $0 \in A_\phi(u_1z_1)$. We color u_1z_1 with color 0 and call this partial coloring τ.

If $0 \notin A_\tau(u_2z_2)$, then there exists some edge z_2z_2' with $\tau(z_2z_2') = 0$, where $z_2' \in N(z_2) \setminus \{u_2\}$. Observe that $|A^2_\tau(u_3z_3)| \geq 1$, $|A^2_\tau(u_2z_2)| \geq 2$, $|A^2_\tau(wv)| \geq 2$, $|A^2_\tau(uu_1)| \geq 4$, $|A^2_\tau(uu_2)| \geq 5$ and $|A^2_\tau(uv)| \geq 6$. Hence, we can finish τ by sequentially coloring $u_3z_3, u_2z_2, wv, uu_1, uu_2, uv$, a contradiction.

If $0 \in A_\varphi(u_2z_2)$, then we color u_2z_2 with color 0. Observe that $|A^2_\tau(u_3z_3)| \geq 1$, $|A^2_\tau(wv)| \geq 2$, $|A^2_\tau(uu_1)| \geq 4$, $|A^2_\tau(uu_2)| \geq 4$ and $|A^2_\tau(uv)| \geq 6$. Hence, we can finish τ by sequentially coloring $u_3z_3, wv, uu_1, uu_2, uv$ with different 2-colors, a contradiction.

Subcase 2.2. $0 \notin \bigcup_{1\leq i\leq 3} A_\phi(u_iz_i)$. It is followed that for each $1 \leq i \leq 3$, there exists some edge z_iz_i' with $\varphi(z_iz_i') = 0$, where $z_i' \in N(z_i) \setminus \{u_i\}$. Hence, $|A^2_\varphi(vw)| \geq 2$, $|A^2_\varphi(uv)| \geq 6$, $|A^2_\varphi(uu_1)| \geq 5$, $|A^2_\varphi(uu_2)| \geq 5$ and $|A^2_\varphi(u_iz_i)| \geq 2$ for $1 \leq i \leq 3$. Let $T = \{u_1z_1, u_2z_2, u_3z_3, wv\}$. Note that if any two edges e_1 and e_2 in T have $A^2_\varphi(e_1) \cap A^2_\varphi(e_2) = \emptyset$, then $|\bigcup_{1\leq i\leq 3} A^2_\varphi(u_iz_i) \cup A^2_\varphi(wv)| \geq 8$, a contradiction. Therefore, there must exist at least two edges of T, say u_1z_1 and wv, that satisfy $A^2_\varphi(u_1z_1) \cap A^2_\varphi(wv) \neq \emptyset$. We can first color u_1z_1 and wv with the same color in $A^2_\varphi(u_1z_1) \cap A^2_\varphi(wv)$ and then finish φ by sequentially coloring $u_2z_2, u_3z_3, uu_1, uu_2, uv$, a contradiction. ■

Since $ew(H) \leq 6$ and $\delta(H) \geq 2$, then $\Delta(H) = 4$. Let v be a 4-vertex in H. Then each neighbor of v is a 2-vertex as $ew(H) \leq 6$ and $\delta(H) \geq 2$. Let w be a 2-vertex in $N(v)$. Then by Claims 13 and 16, each neighbor of w is a 4-vertex. It follows that H is a bipartite graph with one vertex part consisting of 2-vertices and the other vertex part consisting of 4-vertices. By Lemma 10, there exists a 2^8-coloring ϕ of H. By replacing a 2-color of ϕ to a 0-color, then we obtain a $(1, 2^7)$-coloring of H, a contradiction.

The proof is complete.

4. Proof of Theorem 8

Assume that Theorem 8 fails, and H is a counterexample with $|V(H)| + |E(H)|$ as small as possible. By the choice of H, H is a connected fork-free multigraph with $ew(H) \leq 6$ and any proper subgraph of H has a $(1, 2^6)$-coloring.

Claim 17. *H is simple.*

Proof. Suppose that there are k multiple edges $e_1, e_2, \ldots, e_k$ between two vertices u and v in H, where $k \geq 2$. Since $ew(H) \leq 6$, $k \leq 3$. If $k = 3$, then H is the graph with 2 vertices and 3 edges. Obviously, H is $(1, 2^6)$-colorable, a contradiction. Hence, $k = 2$. Let $H_1 = H - e_1$. Then H_1 has a $(1, 2^6)$-coloring φ by the minimality of H. Assume $d(u) \leq d(v)$, then $2 \leq d(u) \leq 3$ as $ew(H) \leq 6$.

Case 1. $d(u) = 2$. Then $d(v) \leq 4$ as $ew(H) \leq 6$. If $d(v) = 4$, denote $N(v) = \{v_1, v_2, u\}$. Then $d(v_1) \leq 2$ and $d(v_2) \leq 2$ as $ew(H) \leq 6$. Hence, there are at most 5 edges in $N''(e_1)$. If $d(v) \leq 3$, then there are at most 4 edges in $N''(e_1)$. Thus, we always have $|A^2_\varphi(e_1)| \geq 1$ and we can finish φ by coloring e_1 with a 2-color in $A^2_\varphi(e_1)$, a contradiction.

Case 2. $d(u) = 3$. Then $d(v) = 3$ as $ew(H) \leq 6$. Denote $N(v) = \{u, v_1\}$, $N(u) = \{v, u_1\}$. Then $d(v_1) \leq 3$ and $d(u_1) \leq 3$. If $v_1 = u_1$, then $|A^2_\varphi(e_1)| \geq 2$. Hence, we can finish φ by coloring e_1 with a 2-color in $A^2_\varphi(e_1)$, a contradiction. Thus, $v_1 \neq u_1$. If $u_1v_1 \in E(H)$, then there are at most 6 edges in $N''(e_1)$. Hence, $|A_\varphi(e_1)| \geq 1$ and we can finish φ by coloring e_1 with the color 0 or some 2-color in $A^2_\varphi(e_1)$, a contradiction. Therefore, $u_1v_1 \notin E(H)$. It follows that there are at most 7 edges in $N''(e_1)$. Denote by $N'(u_1)$ and $N'(v_1)$ the edges incident with u_1 and v_1, respectively.

If there exists two edges $e' \in N'(v_1)$ and $e'' \in N'(u_1)$ satisfy $\varphi(e') = \varphi(e'') = 0$, then $|A^2_\varphi(e_1)| \geq 1$ and we can finish φ by coloring e_1 with a 2-color in $A^2_\varphi(e_1)$, a contradiction.

If there is only one edge $e' \in N'(v_1) \cup N'(u_1)$ that satisfies $\varphi(e') = 0$ (assume $e' \in N'(v_1)$ by symmetry), then we first erase the color of e_2 and uu_1 under φ. Next, we recolor uu_1 with color 0 and call this partial coloring ψ. Observe that $|A^2_\psi(e_i)| \geq 2$ for each $1 \leq i \leq 2$. Hence, we can finish ψ by coloring e_1 and e_2 with different 2-colors, a contradiction.

If there is no edge $e' \in N'(v_1) \cup N'(u_1)$ that satisfies $\varphi(e') = 0$, then we first erase the color of vv_1, uu_1 and e_2 under φ. Next, we recolor both vv_1 and uu_1 with color 0 and call this partial coloring ϕ. Observe that $|A^2_\phi(e_i)| \geq 2$ for each $1 \leq i \leq 2$. Hence, we can also finish ϕ by coloring e_1 and e_2 with different 2-colors, a contradiction. ■

Claim 18. $\delta(H) \geq 2$.

Proof. Suppose that v is a 1-vertex in H with a neighbor u. By the minimality of H, $H_1 = H - v$ has a $(1, 2^6)$-coloring φ. Since $ew(H) \leq 6$, there are at most 6 edges in $N''(uv)$. Hence, $|A(uv)| \geq 1$. If all the edges in $N''(uv)$ are colored different 2-colors under φ, then we can finish φ by coloring vu with color 0. Otherwise, we can finish φ by coloring vu with a 2-color in $A^2(uv)$, which is a contradiction. ■

Claim 19. $\Delta(H) \leq 3$.

Proof. Since $ew(H) \leq 6$, we have $\Delta(H) \leq 4$ by Claim 18. Suppose to the contrary that v is a 4-vertex with $N(v) = \{v_1, v_2, v_3, v_4\}$. Then $d(v_i) = 2$ for $1 \leq i \leq 4$ as $ew(H) \leq 6$ and $\delta(H) \geq 2$. Note that H is fork-free, we consider the following two cases.

Case 1. $v_1v_2 \in E(H)$. Let $H_1 = H - v_1$. Then H_1 has a $(1, 2^6)$-coloring φ by the minimality of H. Observe that $|A^2(vv_1)| \geq 1$ and $|A^2(v_1v_2)| \geq 3$, hence we can finish φ by coloring vv_1 and v_1v_2 with different 2-colors, a contradiction.

Case 2. $v_1v_2 \notin E(H)$. By symmetry, we have $v_iv_j \notin E(H)$ for $1 \leq i \neq j \leq 4$. Denote $N(v_1) = \{v, w\}$. Then w is adjacent to at least two vertices in $N(v)\backslash\{v_1\}$. For otherwise there is a fork induced by v, v_1, w and the two vertices in $N(v)\backslash\{v_1\}$ not adjacent to w. By symmetry, we may assume $\{v_2w, v_3w\} \subseteq E(H)$. If $v_4w \notin E(H)$, then there is a fork induced by v, v_1, v_2, v_4 and z, where $z \in N(v_4) \setminus \{v\}$, a contradiction. Hence, $v_4w \in E(H)$. Therefore, $H \cong G_1$ (see Figure 3), and it can be seen that there is a $(1, 2^6)$-coloring of H, which is a contradiction. ■

Claim 20. *If $C_3 = v_1v_2v_3v_1$ is a 3-cycle in H, then each v_i is a 3-vertex.*

Proof. Suppose that v_1 is not a 3-vertex. Then by Claims 18 and 19, v_1 is a 2-vertex. Let $H_1 = H - v_1$. Then H_1 has a $(1, 2^6)$-coloring φ by the minimality of H. By Claim 19, $d(v_2) \leq 3$ and $d(v_3) \leq 3$. If $d(v_2) = 2$, then $|A^2(v_1v_3)| \geq 2$ and $|A^2(v_1v_2)| \geq 4$, hence we can finish φ by sequentially coloring v_1v_3 and v_1v_2 with different 2-colors, a contradiction.

Therefore, $d(v_2) = 3$, and by the symmetry of v_2 and v_3, $d(v_3) = 3$. Observe that $|A(v_1v_2)| \geq 2$ and $|A(v_1v_3)| \geq 2$. If no edges in $N'(v_1v_2)$ are colored 0 under φ, then we can finish φ by coloring v_1v_2 with color 0 and coloring v_1v_3 with a 2-color in $A^2(v_1v_3)$, a contradiction. If an edge in $N'(v_1v_2)$ is colored 0 under φ, then $|A^2(v_1v_2)| \geq 2$ and $|A^2(v_1v_3)| \geq 2$. Hence, we can finish φ by coloring v_1v_2 and v_1v_3 with different 2-colors, which is also a contradiction. ■

Claim 21. *H has no adjacent 2-vertices.*

Proof. Suppose that u and v are two adjacent vertices in H. Denote by $N(u) = \{u_1, v\}$ and $N(v) = \{u, v_1\}$. By Claim 20, $u_1 \neq v_1$, and by Claim 19, $d(u_1) \leq 3$, $d(v_1) \leq 3$. Let H_1 be the graph obtained from H by contracting uv. Obviously,

$ew(H_1) \leq 6$ and H_1 is simple and fork-free. Hence, H_1 has a $(1, 2^6)$-coloring φ by the minimality of H. If uu_1 and vv_1 are not colored 0 under φ, then we can finish φ by coloring uv with color 0, a contradiction. Otherwise, we can extend φ to H by coloring uv with a 2-color in $A^2(uv)$ as there are at most 6 edges in $N''(uv)$, which is also a contradiction. ∎

Claim 22. *H is claw-free.*

Proof. Suppose to the contrary that there is a claw in H. By Claim 19, we may assume that the claw is induced by $\{v\} \cup N(v)$, where $N(v) = \{v_1, v_2, v_3\}$. By Claims 18–19, $2 \leq d(v_i) \leq 3$ for $1 \leq i \leq 3$.

Case 1. $d(v_1) = 2$. Let $N(v_1) = \{v, w\}$. By Claim 21, $d(w) = 3$. Since H is fork-free, $wv_j \in E(H)$ for some $2 \leq j \leq 3$. We assume $wv_3 \in E(H)$ by symmetry. Denote $N(w) = \{z, v_1, v_3\}$. Then $d(z) \leq 3$ by Claim 19.

Subcase 1.1. $d(v_3) = 2$. Let $H_1 = H - \{v_1, v_3\}$. Then H_1 has a $(1, 2^6)$-coloring φ by the minimality of H. If $\varphi(vv_2) = 0$, then $|A_\varphi(wv_1)| \geq 4$, $|A_\varphi(wv_3)| \geq 4$, $|A_\varphi(vv_1)| \geq 3$ and $|A_\varphi(vv_3)| \geq 3$. Hence, we can finish φ by sequentially coloring vv_1, vv_3, wv_1 and wv_3, a contradiction. Therefore, $\varphi(vv_2) \neq 0$. By symmetry of vv_2 and wz, $\varphi(wz) \neq 0$. Then we first color vv_1 and wv_3 with color 0 and call this partial coloring τ. Observe that $|A^2_\tau(vv_3)| \geq 2$ and $|A^2_\tau(wv_1)| \geq 2$. Thus, we can finish τ by sequentially coloring v_1w, vv_3 with different 2-colors, a contradiction.

Subcase 1.2. $d(v_3) = 3$. Denote $N(v_3) = \{v, w, z_1\}$. Then $z_1v_2 \in E(H)$, for otherwise there is a fork induced by $\{v, z_1\} \cup N(v)$. Recall that $d(w) = 3$ and $N(w) = \{z, v_1, v_3\}$.

Subcase 1.2.1. $z = v_2$. If $d(z_1) = 3$, then there is a fork induced by $\{v, v_3, w, z_1, z_1'\}$, where $z_1' \in N(z_1) \setminus \{v_3, v_2\}$. Hence $d(z_1) = 2$. It follows that $H \cong G_2$ (see Figure 3), and it can be seen that there is a $(1, 2^6)$-coloring of G_2, a contradiction.

Subcase 1.2.2. $z = z_1$. If $d(v_2) = 3$, then there is a fork induced by $\{v_2', v_2, v, z_1, w\}$, where $v_2' \in N(v_2) \setminus \{v, z_1\}$. Hence $d(v_2) = 2$. It follows that $H \cong G_3$ (see Figure 3), and it can be seen that there is a $(1, 2^6)$-coloring of G_3, a contradiction.

Subcase 1.2.3. $z \notin \{v_2, z_1\}$. Then $zz_1 \in E(H)$, otherwise there is a fork induced by $\{v, v_3, z_1, w, z\}$. If $d(z) = 3$, we can find a fork induced by $\{v_1, v_3, w, z, z'\}$, where $z' \in N(z) \setminus \{w, z_1\}$, a contradiction. If $d(v_2) = 3$, we can find a fork induced by $\{v_1, v_3, v, v_2, v_2'\}$, where $v_2' \in N(v_2) \setminus \{v, z_1\}$, a contradiction. Hence, $d(z) = d(v_2) = 2$ and $H \cong G_4$ (see Figure 3), and it can be seen that there is a $(1, 2^6)$-coloring of G_4, a contradiction.

Case 2. $d(v_1) = 3$. By symmetry of v_1, v_2 and v_3, $d(v_2) = d(v_3) = 3$. Denote $N(v_1) = \{u_1, u_2\}$. Then u_2 must be adjacent to v_2 or v_3, for otherwise there is a fork induced by $N(v) \cup \{v, u_2\}$. We may assume $u_2v_3 \in E(H)$ by symmetry. Similarly, u_1 must be adjacent to v_2 or v_3.

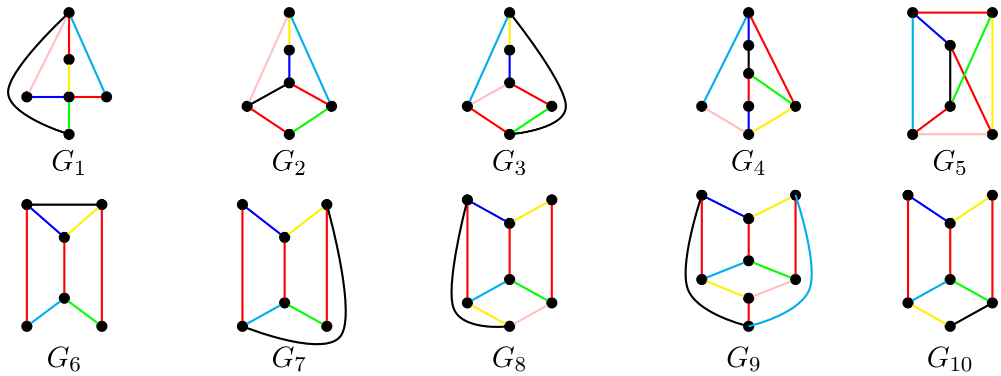

Figure 3. The graphs G_1–G_{10} and their corresponding $(1, 2^6)$-coloring, where the red color represents color 0.

Subcase 2.1. $u_1v_3 \in E(H)$. Since $d(v_2) = 3$, we have $v_2u_1 \in E(H)$ and $v_2u_2 \in E(H)$. For otherwise there is a fork induced by $N(v) \cup \{v, v_2'\}$, where $v_2' \in N(v_2) \setminus \{v, u_1, u_2\}$. Therefore, $H \cong G_5$ (see Figure 3), and it can be seen that there is a $(1, 2^6)$-coloring of G_5, a contradiction.

Subcase 2.2. $u_1v_3 \notin E(H)$. Then $u_1v_2 \in E(H)$. Since H has no adjacent 2-vertices (by Claim 21), if both u_1, u_2, v_2 and v_3 have no other neighbors except the vertices in $N(v) \cup N(v_1)$, then H is isomorphic to one of $G_5 - G_7$ (see Figure 3), and it can be seen that each G_i is $(1, 2^6)$-colorable for $5 \leq i \leq 7$, a contradiction. Hence, there is a vertex in $\{u_1, u_2, v_2, v_3\}$, say v_2, that has a neighbor $v_2' \notin N(v) \cup N(v_1)$. Then $v_2'v_3 \in E(H)$, for otherwise there is a fork induced by $N(v) \cup \{v, v_2'\}$.

If $v_2'u_1 \in E(H)$, then $d(u_2) = 2$ (as if $d(u_2) = 3$, there is a fork induced by $N(u_2) \cup \{u_2, v_2'\}$). Hence, $H \cong G_8$ (see Figure 3), and it can be seen that there is a $(1, 2^6)$-coloring of G_8, a contradiction.

Therefore, $v_2'u_1 \notin E(H)$. By symmetry of u_1 and u_2, $v_2'u_2 \notin E(H)$. If $d(v_2') = 3$, then $v_2''u_1 \in E(H)$ and $v_2''u_2 \in E(H)$, where $v_2'' \in N(v_2') \setminus \{v_2, v_3\}$. For otherwise there is a fork induced by $N(v_2') \cup \{v_2', u_1\}$ or $N(v_2') \cup \{v_2', u_2\}$. Hence, $H \cong G_9$ (see Figure 3), and there is a $(1, 2^6)$-coloring of G_9, a contradiction. Thus, $d(v_2') = 2$. By symmetry of v_2', u_1 and u_2, $d(u_1) = d(u_2) = 2$. Then $H \cong G_{10}$ (see Figure 3), and it can be seen that there is a $(1, 2^6)$-coloring of G_{10}, a contradiction. ∎

Claim 23. *H has no two 3-cycles share one common edge.*

Proof. Suppose that $v_1v_2v_3v_1$ and $v_2v_3v_4v_2$ are two 3-cycles in H with common edge v_2v_3. By Claim 20, $d(v_i) = 3$ for $1 \le i \le 4$. Note that $v_1v_4 \notin E(H)$, otherwise $H \cong K_4$ and hence H is $(1, 2^6)$-colorable, a contradiction. Denote $N(v_1) = \{w_1, v_2, v_3\}$ and $N(v_4) = \{w_2, v_2, v_3\}$. By Claim 19, $d(w_j) \le 3$ for $1 \le j \le 2$. Let $H_1 = H - \{v_1, v_2, v_3, v_4\}$. Then H_1 has a $(1, 2^6)$-coloring φ by the minimality of H. We first color v_1v_3 and v_2v_4 with color 0 and call this partial coloring ψ.

Case 1. $w_1 = w_2$. Observe that $|A^2_\psi(v_1w_1)| \ge 3$, $|A^2_\psi(v_4w_1)| \ge 3$, $|A^2_\psi(v_1v_2)| \ge 5$, $|A^2_\psi(v_3v_4)| \ge 5$ and $|A^2_\psi(v_2v_3)| \ge 6$. Hence, we can color $v_1w_1, v_4w_1, v_1v_2, v_3v_4, v_2v_3$ with different 2-colors in order to finish ψ, a contradiction.

Case 2. $w_1 \ne w_2$. Since w_1 and w_2 may be adjacent in H, we consider the following two subcases.

Subcase 2.1. $w_1w_2 \notin E(H)$. Since H is claw-free (by Claim 22), we can observe that there are at most five edges in $N''(v_jw_j)$ that are colored 2-colors for $1 \le j \le 2$. Thus, $|A^2_\psi(v_1w_1)| \ge 1$ and $|A^2_\psi(v_4w_2)| \ge 1$. Note that $|A^2_\psi(v_1v_2)| \ge 4$, $|A^2_\psi(v_3v_4)| \ge 4$ and $|A^2_\psi(v_2v_3)| \ge 6$. Therefore, we can finish ψ by sequentially coloring $v_1w_1, v_4w_2, v_1v_2, v_3v_4, v_2v_3$ with different 2-colors, a contradiction.

Subcase 2.2. $w_1w_2 \in E(H)$. Since H has no adjacent 2-vertices (by Claim 21), we may assume $d(w_2) = 3$. Let $z \in N(w_2) \setminus \{v_4, w_1\}$. Then $zw_1 \in E(H)$, otherwise, there is a claw induced by $N(w_2) \cup \{w_2\}$, which contradicts to Claim 22. Hence, $|A^2_\psi(v_1w_1)| \ge 2$, $|A^2_\psi(v_4w_2)| \ge 2$, $|A^2_\psi(v_1v_2)| \ge 4$, $|A^2_\psi(v_3v_4)| \ge 4$ and $|A^2_\psi(v_2v_3)| \ge 6$. Therefore, we can color $v_1w_1, v_4w_2, v_1v_2, v_3v_4, v_2v_3$ with different 2-colors in order to finish ψ, a contradiction. ∎

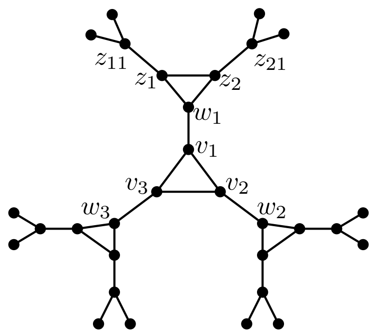

Figure 4. The edge coloring sequence of H.

By Claims 18–19, $2 \le \delta(H) \le \Delta(H) \le 3$. If $\Delta(H) = 2$, then H is a cycle. Hence, H is $(1, 2^6)$-colorable, a contradiction. Therefore, $\Delta(H) = 3$. Let v_1 be a 3-vertex in H with three neighbors v_2, v_3 and w_1. Since H is claw-free (by Claim

22), we may assume $v_2v_3 \in E(H)$. Then by Claim 20, $d(v_2) = d(v_3) = 3$. Let w_j be the neighbor of v_j that is not in the cycle $C = v_1v_2v_3v_1$ for $2 \leq j \leq 3$. Then $w_1 \neq w_2 \neq w_3$ by Claim 23. For a vertex $x \in V(H) \setminus V(C)$, the *distance* between x and the cycle C is denoted by $d(C, x) = \min\{d(x, v_i)|1 \leq i \leq 3\}$. For an edge $e = uv \in E(H) \setminus E(C)$, the *distance* between e and C is denoted by $d(C, e) = d(C, u) + d(C, v)$. In the below, we will give a $(1, 2^6)$-coloring of H in four steps.

Step 1. Color v_1w_1, v_2w_2 and v_3w_3 with the same color 0.

Step 2. Color the edges in $E(H) \setminus E(C)$ except the edges incident with the vertices in $\bigcup_{1\leq i\leq 3} N(w_i)$ according to their distance from the cycle C from far to near. That is, for two uncolored edges e_1 and e_2, if $d(C, e_1) > d(C, e_2)$, then we color e_1 before e_2; if $d(C, e_1) = d(C, e_2)$, then we randomly pick one of them to be colored first.

Denote by S the edge coloring sequence. Next, we will show that every edge in S can be colored. Let xy be an edge in S. Clearly, if there are at most 6 edges in $N''(xy)$ colored before xy, then we can color xy with a 2-color or a 0-color. To illustrate this, we discuss it in the following two cases.

Case 1. $d(C, x) = d(C, y)$. Then there are vertices $x_1 \in N(x) \setminus \{y\}$ and $y_1 \in N(y) \setminus \{x\}$ that satisfy $d(C, x_1) < d(C, x)$ and $d(C, y_1) < d(C, y)$. For any edge x_1x_2 incident with x_1, since $d(C, x_2) \leq d(C, x)$, $d(C, x_1x_2) = d(C, x_1) + d(C, x_2) < 2d(C, x) = d(C, x) + d(C, y) = d(C, xy)$. Similarly, for any edge y_1y_2 incident with y_1, $d(C, y_1y_2) < d(C, xy)$. Hence, the edges incident with x_1 and y_1 are colored after xy in S, as they are closer to C than xy. Note that $\Delta(H) \leq 3$ (by Claim 19), thus there are at most 6 edges in $N''(xy)$ that are colored before xy.

Case 2. $d(C, x) > d(C, y)$. Then there is at least one vertex $y_1 \in N(y) \setminus \{x\}$ that satisfy $d(C, y_1) < d(C, y)$, and hence the edges incident with y_1 are colored after xy in S. When $d(y) = 2$, obviously there are at most 6 edges in $N''(xy)$ that are colored before xy since $\Delta(H) \leq 3$. Hence, we only need to consider $d(y) = 3$. Denote $y_2 \in N(y) \setminus \{x, y_1\}$. Since H is claw-free (by Claim 22), y_2 is adjacent to y_1 or x. If y_2 is adjacent to x, then there are at most 6 edges that are colored before xy in S (three edges incident with y_2 and three edges incident with the vertex $t \in N(x) \setminus \{y, y_2\}$). Thus, we may assume y_2 is adjacent to y_1. Then $d(C, y_2) \leq d(C, y) < d(C, x)$, which implies yy_2 and the edges incident with y_1 are colored after xy in S. Note that there are at most five edges incident with the vertices in $N(x) \setminus \{y\}$, as H is claw-free and $\Delta(H) \leq 3$. Therefore, there are at most 6 edges in $N''(xy)$ that may be colored before xy in S (one edge incident with y_2 and five edges incident with the vertices in $N(x) \setminus \{y\}$).

Step 3. Color the edges incident with the vertices in $\bigcup_{1\leq i\leq 3} N(w_i) \setminus \{v_i\}$.

Based on the symmetry of w_1, w_2 and w_3, it is clear that if the edges incident with the vertices in $N(w_1) \setminus \{v_1\}$ can be colored, then the edges incident with the vertices in $\bigcup_{2\leq j\leq 3} N(w_j) \setminus \{v_j\}$ can also be colored. Next, we will first color the edges zt, where $z \in N(w_1) \setminus \{v_1\}$ and $t \in N(z) \setminus \{w_1\}$, then color the edges w_1z.

Case 1. $d(w_1) = 2$. Denote $N(w_1) = \{v_1, z_1\}$. Since H has no adjacent 2-vertices (by Claim 21), $d(z_1) = 3$. Denote $N(z_1) = \{z_{11}, z_{12}, w_1\}$. Since H is claw-free, $z_{11}z_{12} \in E(H)$. Now we color $z_1z_{11}, z_1z_{12}, z_1w_1$ in order. Note that since $\Delta(H) \leq 3$, there are at most 6 colored edges in $N''(z_1z_{11})$ including v_1w_1. Hence, we can color z_1z_{11} with a 2-color. Then there are at most 7 colored edges in $N''(z_1z_{12})$ including v_1w_1. If some edge incident with z_{12} is colored with 0, then we can color z_1z_{12} with a 2-color, otherwise we can color z_1z_{12} with color 0. Finally, for the edge z_1w_1, it can be seen that there are at most 6 colored edges in $N''(z_1w_1)$ including v_1w_1, hence we can color it with a 2-color.

Case 2. $d(w_1) = 3$. Denote $N(w_1) = \{v_1, z_1, z_2\}$. Then $z_1z_2 \in E(H)$ as H is claw-free. Since H has no 2-vertices in 3-cycle (by Claim 20), $d(z_1) = d(z_2) = 3$. Let z_{11} and z_{21} be the neighbors of z_1 and z_2 not in the 3-cycle $z_1z_2w_1$ respectively (see Figure 4). Next, we color $z_1z_{11}, z_2z_{21}, z_1z_2, z_1w_1$ and z_2w_1 in order. Note that there are at most 6 colored edges in $N''(z_1z_{11})$ including v_1w_1. Hence, we can color z_1z_{11} with a 2-color. Then there are at most 7 colored edges in $N''(z_2z_{21})$. If some edge incident with z_{21} is colored with 0, then we can color z_2z_{21} with a 2-color, otherwise we can color z_2z_{21} with color 0. For the edge z_1z_2, there are also at most 7 colored edges in $N''(z_1z_2)$ including v_1w_1. If z_1z_{11} or z_2z_{21} is colored with 0, then we can color z_1z_2 with a 2-color, otherwise we can color z_1z_2 with color 0. Now for the edge z_1w_1, observe that there are at most 6 colored edges in $N''(z_1w_1)$ including v_1w_1, hence we can color it with a 2-color. Finally, for the edge z_2w_1, there are at most 7 colored edges in $N''(z_2w_1)$. The only case in which z_2w_1 cannot be colored is when all the colored edges in $N''(z_2w_1)$, except w_1v_1, are colored with different 2-colors. In this case, we can erase the 2-color, say α, of z_1z_2. Then color z_1z_2 and z_2w_1 with color 0 and α, respectively.

Step 4. Color the edges in $E(C)$.

Denote by φ the coloring of H after steps 1–3. Observe that $|A^2_\varphi(v_iv_j)| \geq 2$ for $1 \leq i \neq j \leq 3$. If $w_1w_2 \in E(H)$ or there are two colored edges in $N'(v_1w_1) \cup N'(v_2w_2)$ that are colored with the same 2-color, then $|A^2_\varphi(v_1v_2)| \geq 3$. Hence, we can finish φ by coloring v_1v_3, v_2v_3, v_1v_2 in order, a contradiction. Therefore, we have $w_iw_j \notin E(H)$ for $1 \leq i \neq j \leq 3$ by symmetry, and all the six colored edges in $\bigcup_{1\leq i\leq 3} N'(v_iw_i)$ are colored with different 2-colors. In this case, we can color each v_iv_j with the same 2-color of a colored edge in $N'(v_tw_t)$, where $1 \leq i \neq j \leq 3$ and $t \in \{1, 2, 3\} \setminus \{i, j\}$, to obtain a $(1, 2^6)$-coloring of H, a contradiction.

The proof is complete.

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