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# ON S-PACKING EDGE-COLORING OF GRAPHS WITH GIVEN EDGE WEIGHT

#### JIAN LU

School of Statistics and Applied Mathematics Anhui University of Finance & Economics, Bengbu 233030, P.R. China

e-mail: lujianmath@163.com

## AND

#### XIANG-FENG PAN

School of Mathematical Sciences Anhui University, Hefei 230601, P.R. China e-mail: xfpan@ahu.edu.cn

#### Abstract

Let  $S = (s_1, \ldots, s_k)$  be a non-decreasing sequence of positive integers. A graph G = (V(G), E(G)) is said to be S-packing edge-colorable if E(G) can be decomposed into disjoint sets  $E_1, \ldots, E_k$  such that for every  $1 \le i \le k$ , the distance between any two distinct edges in  $E_i$  is at least  $s_i + 1$ . The edge weight of G is defined as  $ew(G) = \max\{d(u) + d(v) | uv \in E(G)\}$ . A fork is the graph obtained from  $K_{1,3}$  by subdividing an edge once. In 2023, Liu *et al.* proved that every subcubic multigraph is  $(1, 2^7)$ -packing edge-colorable. Based on the work of Liu *et al.*, we prove that every multigraph G with  $ew(G) \le 6$  is  $(1, 2^7)$ -packing edge-colorable, which confirms a conjecture of Yang and Wu (2022). In addition, we demonstrate that if G is a fork-free graph with  $ew(G) \le 6$ , then G is  $(1, 2^6)$ -packing edge-colorable.

Keywords: S-packing edge-coloring, edge weight, fork-free graphs.

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#### 1. INTRODUCTION

All graphs considered in this paper are finite and undirected. The *distance* between two vertices u and v in a graph G = (V(G), E(G)) is the length of a shortest path between u and v. For two edges  $e_1, e_2$  in E(G), the distance  $d(e_1, e_2)$  between  $e_1$  and  $e_2$  is the distance between the corresponding vertices of  $e_1$  and  $e_2$ in the line graph of G.

Let  $S = (s_1, s_2, \ldots, s_k)$  be a non-decreasing sequence of integers. An *S*packing edge-coloring of *G* is a partition  $E_1, E_2, \ldots, E_k$  of E(G) such that for  $1 \le i \le k$ ,  $d(e_1, e_2) \ge s_i + 1$  for any two edges  $e_1$  and  $e_2$  in  $E_i$ . Note that if all  $s_i = 1$  or all  $s_i = 2$ , then an *S*-packing edge-coloring is equivalent to a proper edge-coloring or a strong edge-coloring [2], respectively. In this paper, we are only concerned with the case where each  $s_i \in \{1, 2\}$ . For convenience, we use exponents to indicate identical components repeated in *S*, e.g., (1, 1, 2, 2, 2) = $(1^2, 2^3)$ . And we write the color set of the  $(1, 2^k)$ -packing edge-coloring of *G* as  $\{0, 1, 2, \ldots, k\}$ , where 0 is the color that allows edges with distance at least 2 to be colored, and we collectively refer to colors 1 through *k* as the 2-colors, which are the colors allow edges with distance at least 3 to be colored.

The concept of S-packing edge-colorings is derived from its corresponding vertex counterpart, which was first proposed by Gastineau and Togni [3] as a logical extension of the packing chromatic number [4]. Fouquet and Vanherpe [1] proved that any subcubic graph admits a  $(1^3, 2)$ -packing edge-coloring. A spanning subgraph G' of G is called a 2-factor of G if each component of G' is a cycle. Gastineau and Togni [3] demonstrated that for each cubic graph G with a 2-factor, G is  $(1^2, 2^5)$ -packing edge-colorable.

In light of Vizing's [10] work  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ , we know that if  $\chi'(G) = \Delta(G)$ , then G is said to be in class I; if  $\chi'(G) = \Delta(G) + 1$ , then G is said to be in class II, where  $\chi'(G)$  and  $\Delta(G)$  are the chromatic index and the maximum degree of G, respectively. Gastineau *et al.* [3] and Hocquard *et al.* [6] posed several conjectures of S-packing edge-coloring on subcubic graphs, especially in class I.

#### **Conjecture 1.** If G is a simple subcubic graph, then G is

- (a)  $(1^2, 2^4)$ -packing edge-colorable [3];
- (b)  $(1, 2^7)$ -packing edge-colorable [3];
- (c)  $(1^2, 2^3)$ -packing edge-colorable if G is in class I [3];
- (d)  $(1, 2^6)$ -packing edge-colorable if G is in class I [6].

In [6], Hocquard *et al.* made progress towards these conjectures by proving the following results.

**Theorem 2** [6]. If G is a simple subcubic graph, then G is

- (a)  $(1^2, 2^5)$ -packing edge-colorable;
- (b)  $(1, 2^8)$ -packing edge-colorable;
- (c)  $(1^2, 2^4)$ -packing edge-colorable if G is in class I;

(d)  $(1, 2^7)$ -packing edge-colorable if G is in class I.

Moreover, Hocquard *et al.* posed several problems of S-packing edge-coloring on planar graphs and bipartite graphs with  $ew(G) \leq 5$ , where  $ew(G) = \max\{d(u) + d(v) : uv \in E(G)\}$  is called the *edge weight* of G.

**Problem 3** [6]. If G is a simple bipartite graph with  $ew(G) \leq 5$ , then G is  $(1, 2^4)$ -packing edge-colorable.

Recently, Liu *et al.* [7, 8] proved Conjecture 1(a) and (b), and in particular, for Conjecture 1(b), they obtained a stronger conclusion.

**Theorem 4** [8]. If G is a connected subcubic graph with more than 70 vertices, then G is  $(1^2, 2^4)$ -packing edge-colorable.

**Theorem 5** [7]. If G is a subcubic multigraph, then G is  $(1, 2^7)$ -packing edgecolorable.

In [11], Yang and Wu solved Problem 3 and got a more favorable result. They showed that every simple graph G with  $ew(G) \leq 5$  is  $(1, 2^4)$ -packing edgecolorable. In addition, they proved that every simple graph G with  $ew(G) \leq 6$  is  $(1, 2^8)$ -packing edge-colorable and made the following conjecture.

**Conjecture 6** [11]. If G is a simple graph with  $ew(G) \leq 6$ , then G is  $(1, 2^7)$ -packing edge-colorable.

In this paper, by proving the following result and combining it with Theorem 5, we prove Conjecture 6.

**Theorem 7.** For every multigraph G with  $ew(G) \leq 6$  and  $\Delta(G) \geq 4$ , G is  $(1,2^7)$ -packing edge-colorable.

The graph obtained from  $K_{1,3}$  (usually called *claw*) by subdividing an edge once is called a *fork*. A graph is *H*-free if it does not contain *H* as an induced subgraph. In this paper, we also consider the *S*-packing edge-coloring on fork-free graphs with  $ew(G) \leq 6$  and get the following result.

**Theorem 8.** For every fork-free multigraph G with  $ew(G) \leq 6$ , G is  $(1, 2^6)$ -packing edge-colorable.

The graphs G' and G'' in Figure 1 show that our results in Theorems 7–8 are sharp. In [6], Hocquard *et al.* showed that G' is  $(1, 2^7)$ -packing edge-colorable but not  $(1, 2^6)$ -packing edge-colorable. For the graph G'', it can be seen that G''is  $(1, 2^6)$ -packing edge-colorable. Note that |E(G'')| = 9, the distance between any two edges in E(G'') is at most two, and the maximum size of a matching in G'' is three. Hence, at most three edges in G'' can be colored with color 0, and the remaining six edges must be colored with different 2-colors. Therefore, G'' is not  $(1, 2^5)$ -packing edge-colorable.



Figure 1. Sharpness examples of Theorems 7–8.

#### 2. Preliminaries

Let G = (V(G), E(G)) be a graph. For  $v \in V(G)$ ,  $N_G(v)$  is the set of all neighbors of v. We denote the degree of v by  $d_G(v) = |N_G(v)|$ . If  $d_G(v) = d$ , then v is called a d-vertex. When G is clear from the context, then we write N(v) and d(v)instead of  $N_G(v)$  and  $d_G(v)$ , respectively. Let  $\delta(G) = \min\{d(v)|v \in V(G)\}$ . We denote by N'(e) the set of edges with distance 1 to e and N''(e) the set of edges with distance at most 2 to e. Clearly,  $N'(e) \subseteq N''(e)$ . For  $V' \subseteq V(G)$ , the graph induced by  $V(G) \setminus V'$  is denoted as G - V'. If  $V' = \{v\}$ , we simplify  $G - \{v\}$  to G - v.

To prove Theorem 7, we need the following two lemmas. Let  $T_1, \ldots, T_n$  be n subsets of a set T. A subset  $\{t_1, t_2, \ldots, t_n\} \subseteq T$  is called the system of distinct representatives of  $\{T_1, \ldots, T_n\}$  if  $t_i \in T_i$  and  $t_i \neq t_j$  for  $1 \leq i, j \leq n$ .

**Lemma 9** (Hall's marriage theorem [5]). Let  $T_1, \ldots, T_n$  be n subsets of a set T. A system of distinct representatives of  $\{T_1, \ldots, T_n\}$  exists if and only if for all  $k, 1 \leq k \leq n$  and every subcollection of size  $k, \{T_{i_1}, \ldots, T_{i_k}\}$ , we have  $|T_{i_1} \cup \cdots \cup T_{i_k}| \geq k$ .

Recall that a strong k-edge-coloring is a  $(2^k)$ -packing edge-coloring. Due to the work of Nakprasit [9] on strong edge-coloring of bipartite graphs, we have the following lemma.

**Lemma 10** [9]. Let G be a simple bipartite graph in which the vertices in one part have maximum degree 2. Then G is  $(2^{2\Delta(G)})$ -packing edge-colorable.

From now on, a  $(1, 2^k)$ -coloring in this paper refers to  $(1, 2^k)$ -packing edgecoloring, where k = 6 or 7. A partial coloring of G is a coloring of any subset of E(G) using the colors  $\{0, 1, 2, \ldots, k\}$ , such that for any two colored edges  $e_1$  and  $e_2$ , if they are both colored with the color 0, then  $d(e_1, e_2) \ge 2$ ; if they are both colored with a same 2-color, then  $d(e_1, e_2) \ge 3$ .

Let  $G_1$  be a proper subgraph of G with a partial coloring  $\varphi$ . For any  $e \in E(G) \setminus E(G_1)$ , we denote by  $A_{\varphi}(e)$  the set of 0-color and 2-colors that can be used to color e, and by  $A_{\varphi}^2(e)$  the set of 2-colors that can be used to color e. And we write  $A_{\varphi}(e)$  and  $A_{\varphi}^2(e)$  as A(e) and  $A^2(e)$ , respectively, if it is clear from the context. Obviously, if no edges in N'(e) are colored 0 under  $\varphi$ , then  $A(e) = A^2(e) \cup \{0\}$ , and we can color e with color 0. Otherwise,  $A(e) = A^2(e)$ , we can only color e with a 2-color in  $A^2(e)$ . We say that  $\varphi$  is *finished* when we extend  $\varphi$  from  $G_1$  to G.

#### 3. Proof of Theorem 7

Assume that Theorem 7 fails, and H is a counterexample with |V(H)| + |E(H)|as small as possible. By the choice of H, H is a connected multigraph with  $ew(H) \leq 6$  and any proper subgraph of H has a  $(1, 2^7)$ -coloring.

Claim 11. *H* is simple.

**Proof.** Suppose that there are k multiple edges  $e_1, e_2, \ldots, e_k$  between two vertices u and v in H, where  $k \ge 2$ . Since  $ew(H) \le 6$ ,  $k \le 3$ . If k = 3, then H is the graph with 2 vertices and 3 edges. Obviously, H is  $(1, 2^7)$ -colorable, a contradiction. Hence k = 2. Let  $H_1 = H - e_1$ . Then  $H_1$  has a  $(1, 2^7)$ -coloring  $\varphi$  by the minimality of H. Note that there are at most 7 edges in  $N''(e_1)$  as  $ew(H) \le 6$ . Hence,  $|A(e_1)| \ge 1$ . If all the edges in  $N''(e_1)$  are colored with different 2-colors under  $\varphi$ , then we can finish  $\varphi$  by coloring  $e_1$  with color 0. Otherwise, we can finish  $\varphi$  by coloring  $e_1$  with a 2-color in  $A^2(e_1)$ , a contradiction.

Claim 12.  $\delta(H) \geq 2$ .

**Proof.** Suppose that v is a 1-vertex in H with a neighbor u. By the minimality of H,  $H_1 = H - v$  has a  $(1, 2^7)$ -coloring  $\varphi$ . Since  $ew(H) \leq 6$ , there are at most 6 edges in N''(uv). Hence,  $|A^2(uv)| \geq 1$ , and we can finish  $\varphi$  by coloring uv with a 2-color in  $A^2(uv)$ , a contradiction.

Claim 13. *H* has no adjacent 2-vertices.

**Proof.** Suppose that u and v are two adjacent 2-vertices in H. Denote  $N(u) = \{u_1, v\}$  and  $N(v) = \{v_1, u\}$ . Since  $ew(H) \leq 6$ ,  $d(v_1) \leq 4$  and  $d(u_1) \leq 4$ . Let  $H_1 = H - \{u, v\}$ . Then  $H_1$  has a  $(1, 2^7)$ -coloring  $\varphi$  by the minimality of H.

If  $0 \in A_{\varphi}(vv_1)$  and  $0 \in A_{\varphi}(uu_1)$ , then we first color  $vv_1$  and  $uu_1$  with color 0 and call this coloring  $\phi_1$ . Observe that  $|A_{\phi_1}^2(uv)| \ge 7 - ((d(v_1) - 1) + (d(u_1) - 1)) \ge 1$ . Hence, we can finish  $\phi_1$  by coloring uv with a 2-color in  $A_{\phi_1}^2(uv)$ , a contradiction.

If only one of  $A_{\varphi}(vv_1)$  and  $A_{\varphi}(uu_1)$ , say  $A_{\varphi}(uu_1)$ , does not contain color 0, then there must exist some edge  $e \in N'(uu_1) \setminus \{uv\}$  such that  $\varphi(e) = 0$ . We first color  $vv_1$  with 0 and call this coloring  $\phi_2$ . Observe that  $|A_{\phi_2}^2(uu_1)| \ge 2$  and  $|A_{\phi_2}^2(uv)| \ge 2$ . Hence, we can finish  $\phi_2$  by coloring uv and  $uu_1$  with different 2-colors, a contradiction. If  $0 \notin A_{\varphi}(vv_1)$  and  $0 \notin A_{\varphi}(uu_1)$ , then there are two edges  $e_1 \in N'(uu_1) \setminus \{uv\}$ and  $e_2 \in N'(vv_1) \setminus \{uv\}$  such that  $\varphi(e_1) = \varphi(e_2) = 0$ . We first color uv with color 0 and call this coloring  $\phi_3$ . Observe that  $|A_{\phi_3}^2(uu_1)| \ge 2$  and  $|A_{\phi_3}^2(vv_1)| \ge 2$ . Hence, we can finish  $\phi_3$  by coloring  $uu_1$  and  $vv_1$  with different 2-colors, a contradiction.

Claim 14. *H* has no 4-cycle that contains a 4-vertex.

**Proof.** Suppose that  $C_4 = v_1 v_2 v_3 v_4 v_1$  is a 4-cycle in H with  $d(v_1) = 4$ . Denote  $N(v_1) = \{w_1, w_2, v_2, v_4\}$ . Since  $ew(H) \le 6$  and  $\delta(H) \ge 2$  (by Claim 12),  $d(w_1) = d(w_2) = d(v_2) = d(v_4) = 2$ . By Claim 13,  $3 \le d(v_3) \le 4$ .



Figure 2. The reducible configurations of Claims 14–16.

Case 1.  $d(v_3) = 3$ . Denote  $N(v_3) = \{w, v_2, v_4\}$  (see Figure 2(a)). Let  $H_1 = H - v_1$ . Then  $H_1$  has a  $(1, 2^7)$ -coloring  $\varphi$  by the minimality of H. Next, we erase the color of  $v_3v_4$  and  $v_3v_2$  in  $H_1$ . If  $\varphi(v_3w) = 0$ , then we first color  $v_1v_4$  with color 0 and call this partial coloring  $\phi_1$ . Observe that  $|A_{\phi_1}^2(e)| \ge 5$  for  $e \in E(C_4) \setminus \{v_1v_4\}$  and  $|A_{\phi_1}^2(v_1w_i)| \ge 2$  for  $1 \le i \le 2$ . Hence, we can finish  $\phi_1$  by sequentially coloring  $v_1w_1, v_1w_2, v_1v_2, v_2v_3, v_3v_4$  with different 2-colors, a contradiction. If  $\varphi(v_3w) \ne 0$ , then we first color  $v_1v_2$  and  $v_3v_4$  with color 0 and call this partial coloring  $\phi_2$ . Observe that  $|A_{\phi_2}^2(v_1v_4)| \ge 4$ ,  $|A_{\phi_2}^2(v_2v_3)| \ge 4$  and  $|A_{\phi_2}^2(v_1w_i)| \ge 2$  for  $1 \le i \le 2$ . Hence, we can finish  $\phi_2$  by sequentially coloring  $v_1w_1, v_1w_2, v_2v_3, v_3v_4$  with different 2-colors, a  $|A_{\phi_2}^2(v_1w_i)| \ge 2$  for  $1 \le i \le 2$ . Hence, we can finish  $\phi_2$  by sequentially coloring  $v_1w_1, v_1w_2, v_2v_3, v_1v_4$  with different 2-colors, a contradiction.

Case 2.  $d(v_3) = 4$ . Denote  $N(v_3) = \{w_3, w_4, v_2, v_4\}$  (see Figure 2(b)). Let  $H_2 = H - \{v_1, v_3\}$ . Then  $H_2$  has a  $(1, 2^7)$ -coloring  $\psi$  by the minimality of H. Since  $ew(H) \leq 6$  and  $\delta(H) \geq 2$ ,  $d(w_3) = d(w_4) = 2$ . By Claim 13,  $w_3w_4 \notin E(H)$ .

Subcase 2.1.  $w_3 = w_1$ . We first color  $v_3w_1$  and  $v_1v_4$  with color 0 and call this partial coloring  $\psi_1$ . Observe that  $|A^2_{\psi_1}(v_1w_2)| \ge 3$ ,  $|A^2_{\psi_1}(v_3w_4)| \ge 3$ ,  $|A^2_{\psi_1}(v_1w_1)| \ge 6$ ,  $|A^2_{\psi_1}(v_1v_2)| \ge 6$ ,  $|A^2_{\psi_1}(v_2v_3)| \ge 6$  and  $|A^2_{\psi_1}(v_3v_4)| \ge 6$ . Hence, we can finish  $\psi_1$  by sequentially coloring  $v_1w_2, v_3w_4, v_1w_1, v_1v_2, v_2v_3, v_3v_4$  with different 2-colors, a contradiction. Subcase 2.2.  $w_3 \neq w_1$ . By the symmetry of  $w_1, w_2, w_3$  and  $w_4$ , we have  $w_i \neq w_j$  for  $1 \leq i \neq j \leq 4$ . Recall that  $d(w_i) = 2$  for  $1 \leq i \leq 4$ . Let  $z_i$  be the neighbor of  $w_i$  not in  $C_4$  for  $1 \leq i \leq 4$ .

Subcase 2.2.1.  $\varphi(w_i z_i) = 0$  for all  $1 \le i \le 4$ . We first color  $v_1 v_4$  with color 0 and call this partial coloring  $\psi_2$ . Since  $d(w_i) = 2$  and  $d(z_i) \le 4$  for  $1 \le i \le 4$ , we have  $|A^2_{\psi_2}(v_1 w_1)| \ge 4$ ,  $|A^2_{\psi_2}(v_1 w_2)| \ge 4$ ,  $|A^2_{\psi_2}(v_3 w_3)| \ge 4$ ,  $|A^2_{\psi_2}(v_3 w_4)| \ge 4$ ,  $|A^2_{\psi_2}(v_1 v_2)| \ge 7$ ,  $|A^2_{\psi_2}(v_2 v_3)| \ge 7$  and  $|A^2_{\psi_2}(v_3 v_4)| \ge 7$ . Hence, we can finish  $\psi_2$  by sequentially coloring  $v_1 w_1$ ,  $v_1 w_2$ ,  $v_3 w_3$ ,  $v_3 w_4$ ,  $v_1 v_2$ ,  $v_2 v_3$ ,  $v_3 v_4$  with different 2-colors, a contradiction.

Subcase 2.2.2.  $\varphi(w_j z_j) \neq 0$  for some  $1 \leq j \leq 4$ . Assume  $\varphi(w_3 z_3) \neq 0$ by symmetry. We first color  $v_3 w_3$  with color 0 and call this partial coloring  $\psi_3$ . Then we delete the color of  $w_2 z_2$  under  $\psi_3$ . Observe that  $|A^2_{\psi_3}(v_1 w_1)| \geq 3$ ,  $|A^2_{\psi_3}(v_1 w_2)| \geq 3$ ,  $|A^2_{\psi_3}(v_1 v_2)| \geq 6$ ,  $|A^2_{\psi_3}(v_1 v_4)| \geq 6$ ,  $|A^2_{\psi_3}(v_2 v_3)| \geq 5$ ,  $|A^2_{\psi_3}(v_3 v_4)| \geq 5$ ,  $|A^2_{\psi_3}(v_3 w_4)| \geq 2$  and  $|A_{\psi_3}(w_2 z_2)| \geq 2$ .

Subcase 2.2.2.1.  $0 \in A_{\psi_3}(w_2 z_2)$ . Note that  $d(w_2 z_2, v_1 v_4) = 2$ , hence we can color  $w_2 z_2$  and  $v_1 v_4$  with color 0 and then color  $v_3 w_4$ ,  $v_1 w_1$ ,  $v_1 w_2$ ,  $v_2 v_3$ ,  $v_3 v_4$ ,  $v_1 v_2$  with different 2-colors in order to finish  $\psi_3$ , a contradiction.

Subcase 2.2.2.2.  $0 \notin A_{\psi_3}(w_2z_2)$ . Then  $\psi_3(z_2z'_2) = 0$ , where  $z'_2 \in N(z_2) \setminus \{w_2\}$ . Hence,  $|A^2_{\psi_3}(w_2z_2)| \geq 2$ . We first color  $v_1w_2$  with color 0. Recall that for  $1 \leq i \leq 2$ ,  $w_i \neq w_4$ , and by Claim 13,  $w_iw_4 \notin E(H)$ . Thus  $d(v_1w_i, v_3w_4) > 2$  for  $1 \leq i \leq 2$  and  $d(w_2z_2, v_2v_3) > 2$ . If  $A^2_{\psi_3}(w_2z_2) \cap A^2_{\psi_3}(v_2v_3) \neq \emptyset$ , then we can color  $w_2z_2$  and  $v_2v_3$  with a same color in  $A^2_{\psi_3}(w_2z_2) \cap A^2_{\psi_3}(v_2v_3)$  and then finish  $\psi_3$  by sequentially coloring  $v_3w_4, v_1w_1, v_3v_4, v_1v_4, v_1v_2$  with different 2-colors, a contradiction. If  $A^2_{\psi_3}(w_2z_2) \cap A^2_{\psi_3}(v_2v_3) = \emptyset$ , then  $|A^2_{\psi_3}(w_2z_2) \cup A^2_{\psi_3}(v_2v_3)| \geq 7$ . Let  $T = \{w_2z_2, v_3w_4, v_1w_1, v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ . Observe that for any  $T' \subseteq T$ ,  $|\bigcup_{e \in T'} A^2_{\psi_3}(e)| \geq |T'|$ . Hence, by Lemma 9, we can finish  $\psi_3$  by coloring each edge in T with a different 2-color, which is a contradiction.

Claim 15. *H* has no 5-cycle that contains a 4-vertex.

**Proof.** Suppose otherwise that there is a 5-cycle  $v_1v_2v_3v_4v_5v_1$  in H with  $d(v_1) = 4$ . Denote  $N(v_1) = \{w_1, w_2, v_2, v_5\}$ . Since  $ew(H) \le 6$ ,  $d(w_1) = d(w_2) = d(v_2) = d(v_4) = 2$ . Denote  $N(w_1) = \{v_1, w_1'\}$  and  $N(w_2) = \{v_1, w_2'\}$ . Then  $d(w_1') \le 4$  and  $d(w_2') \le 4$  as  $ew(H) \le 6$ . By Claim 13,  $d(v_3) = d(v_4) = 3$  (see Figure 2(c)). Let  $H_1 = H - \{v_1, v_2, v_5\}$ . Then  $H_1$  has a  $(1, 2^7)$ -coloring  $\varphi$  by the minimality of H. We erase the color of  $v_3v_4$  in  $H_1$  under  $\varphi$ . Observe that  $|A_{\varphi}(v_1w_1)| \ge 3$ ,  $|A_{\varphi}(v_1w_2)| \ge 3$ ,  $|A_{\varphi}(v_1v_2)| \ge 5$ ,  $|A_{\varphi}(v_1v_5)| \ge 5$ ,  $|A_{\varphi}(v_5v_4)| \ge 4$ ,  $|A_{\varphi}(v_3v_4)| \ge 2$  and  $|A_{\varphi}(v_2v_3)| \ge 4$ .

Case 1.  $0 \notin A_{\varphi}(v_2v_3)$ . Then  $\varphi(v_3v'_3) = 0$ , where  $v'_3 \in N(v_3) \setminus \{v_2, v_4\}$ . Denote  $N(v_4) = \{v_3, v_5, v'_4\}$ . Since  $ew(H) \leq 6$ , there are at most 7 colored edges in  $N''(v_4v'_4)$ , including  $v_3v'_3$ . We first erase the color of  $v_4v'_4$  and recolor  $v_4v'_4$  with a 2-color not used on the edges in  $N''(v_4v'_4)$  to ensure that  $\varphi(v_4v'_4) \neq 0$ . Then color  $v_4v_5$  with color 0 and call this partial coloring  $\phi$ . Next, if  $0 \in A_{\phi}(v_1w_1)$ , then we color  $v_1w_1$  with color 0. Observe that  $|A^2_{\phi}(v_3v_4)| \geq 2$ ,  $|A^2_{\phi}(v_1w_2)| \geq$ 2,  $|A^2_{\phi}(v_2v_3)| \geq 4$ ,  $|A^2_{\phi}(v_1v_5)| \geq 4$  and  $|A^2_{\phi}(v_1v_2)| \geq 5$ . Hence, we can finish  $\phi$  by sequentially coloring  $v_3v_4, v_1w_2, v_2v_3, v_1v_5, v_1v_2$  with different 2-colors, a contradiction. Hence,  $0 \notin A_{\phi}(v_1w_1)$ , which implies  $\phi(w_1w'_1) = 0$ , where  $w'_1 \in$  $N(w_1) \setminus \{v_1\}$ . Then  $|A^2_{\phi}(v_3v_4)| \geq 2$ ,  $|A^2_{\phi}(v_1w_1)| \geq 3$ ,  $|A^2_{\phi}(v_1w_2)| \geq 3$ ,  $|A^2_{\phi}(v_2v_3)| \geq$ 4,  $|A^2_{\phi}(v_1v_5)| \geq 5$  and  $|A^2_{\phi}(v_1v_2)| \geq 6$ . Hence, we can finish  $\phi$  by sequentially coloring  $v_3v_4, v_1w_1, v_1w_2, v_2v_3, v_1v_5, v_1v_2$  with different 2-colors, a contradiction.

Case 2.  $0 \in A_{\varphi}(v_2v_3)$ . By symmetry of  $v_3$  and  $v_4$ , we also have  $0 \in A_{\varphi}(v_4v_5)$ . We first color  $v_2v_3$  and  $v_4v_5$  with color 0 and call this partial coloring  $\tau$ . If  $0 \in A_{\tau}(v_1w_1)$ , then we color  $v_1w_1$  with color 0. Observe that  $|A_{\tau}^2(v_3v_4)| \ge 1$ ,  $|A_{\tau}^2(v_1w_2)| \ge 2$ ,  $|A_{\tau}^2(v_1v_5)| \ge 4$  and  $|A_{\tau}^2(v_1v_2)| \ge 4$ . Hence, we can finish  $\tau$  by sequentially coloring  $v_3v_4$ ,  $v_1w_2$ ,  $v_1v_5$ ,  $v_1v_2$  with different 2-colors, a contradiction. Hence,  $0 \notin A_{\tau}(v_1w_1)$ , which implies  $\tau(w_1w_1') = 0$ , where  $w_1' \in N(w_1) \setminus \{v_1\}$ . Then  $|A_{\tau}^2(v_3v_4)| \ge 1$ ,  $|A_{\tau}^2(v_1w_1)| \ge 3$ ,  $|A_{\tau}^2(v_1w_2)| \ge 3$ ,  $|A_{\tau}^2(v_1v_5)| \ge 5$  and  $|A_{\tau}^2(v_1v_2)| \ge 5$ . Hence, we can finish  $\tau$  by sequentially coloring  $v_3v_4$ ,  $v_1w_1$ ,  $v_1w_2$ ,  $v_1v_5$ ,  $v_1v_2$  with different 2-colors, a contradiction.

#### Claim 16. No 2-vertex in H is adjacent to a 4-vertex and a 3-vertex.

**Proof.** Suppose that v is a 2-vertex in H adjacent to a 4-vertex u and a 3-vertex w. Denote  $N(u) = \{v, u_1, u_2, u_3\}$ . Since  $ew(H) \leq 6$ ,  $d(u_1) = d(u_2) = d(u_3) = 2$  by Claim 12. By Claim 13,  $vu_i \notin E(H)$  and  $u_iu_j \notin E(H)$  for  $1 \leq i \neq j \leq 3$ . Denote  $N(u_i) = \{u, z_i\}$  for  $1 \leq i \leq 3$ , then  $d(z_i) \leq 4$  as  $ew(H) \leq 6$ . By Claim 14,  $z_i \neq z_j$  and  $z_i \neq w$  for  $1 \leq i \neq j \leq 3$ . By Claim 15,  $z_iw \notin E(H)$  and  $z_iz_j \notin E(H)$  for  $1 \leq i \neq j \leq 3$ . Hence, the distance between any two edges in  $\{u_1z_1, u_2z_2, u_3z_3, wv\}$  is 3 (see Figure 2(d)). Let  $H_1 = H - N(u)$ , then  $H_1$  has a  $(1, 2^7)$ -coloring  $\varphi$  by the minimality of H. Observe that  $|A_{\varphi}(vw)| \geq 2$ ,  $|A_{\varphi}(uv)| \geq 6$ ,  $|A_{\varphi}(u_iz_i)| \geq 2$  and  $|A_{\varphi}(uu_i)| \geq 5$  for  $1 \leq i \leq 3$ . We first color  $uu_3$  with color 0.

Case 1.  $0 \in A_{\varphi}(wv)$ . Then we color wv with color 0 and call this partial coloring  $\phi$ .

Subcase 1.1.  $0 \in \bigcup_{1 \le i \le 3} A_{\phi}(u_i z_i)$ . By symmetry, let  $0 \in A_{\phi}(u_1 z_1)$  and we color  $u_1 z_1$  with color 0. Observe that  $|A^2_{\phi}(u_2 z_2)| \ge 1$ ,  $|A^2_{\phi}(u_3 z_3)| \ge 1$ ,  $|A^2_{\phi}(u u_1)| \ge 4$ ,  $|A^2_{\phi}(u u_2)| \ge 4$  and  $|A^2_{\phi}(u v)| \ge 5$ . Hence, we can finish  $\phi$  by sequentially coloring  $u_2 z_2, u_3 z_3, u u_1, u u_2, u v$ , a contradiction.

Subcase 1.2.  $0 \notin \bigcup_{1 \leq i \leq 3} A_{\phi}(u_i z_i)$ . It follows that for each  $1 \leq i \leq 3$ , there exists some edge  $z_i z_i^T$  with  $\phi(z_i z_i') = 0$ , where  $z_i' \in N(z_i) \setminus \{u_i\}$ . Hence,  $|A_{\phi}^2(uv)| \geq 5$ ,  $|A_{\phi}^2(uu_1)| \geq 5$ ,  $|A_{\phi}^2(uu_2)| \geq 5$  and  $|A_{\phi}^2(u_i z_i)| \geq 2$  for  $1 \leq i \leq 3$ .

If there exist two edges in  $\{u_1z_1, u_2z_2, u_3z_3\}$ , say  $u_1z_1$  and  $u_2z_2$ , that satisfy  $A_{\phi}^2(u_1z_1) \cap A_{\phi}^2(u_2z_2) \neq \emptyset$ , then we can color  $u_1z_1$  and  $u_2z_2$  with the same color in  $A_{\phi}^2(u_1z_1) \cap A_{\phi}^2(u_2z_2)$  and then finish  $\phi$  by sequentially coloring  $u_3z_3, uu_1, uu_2, uv$  with different 2-colors, a contradiction.

If  $A_{\phi}^2(u_i z_i) \cap A_{\phi}^2(u_j z_j) = \emptyset$  for  $1 \leq i \neq j \leq 3$ , then  $|\bigcup_{1 \leq i \leq 3} A_{\phi}^2(u_i z_i)| \geq 6$ . Let  $T = \{u_1 z_1, u_2 z_2, u_3 z_3, u u_1, u u_2, uv\}$ . Observe that for any  $T' \subseteq T$ ,  $|\bigcup_{e \in T'} A_{\phi}^2(e)| \geq |T'|$ . Hence, by Lemma 9, we can finish  $\phi$  by coloring each edge in T with a different 2-color, which is a contradiction.

Case 2.  $0 \notin A_{\varphi}(wv)$ . Then there exists some edge ww' with  $\varphi(ww') = 0$ , where  $w' \in N(w) \setminus \{v\}$ .

Subcase 2.1.  $0 \in \bigcup_{1 \leq i \leq 3} A_{\phi}(u_i z_i)$ . By symmetry, let  $0 \in A_{\phi}(u_1 z_1)$ . We color  $u_1 z_1$  with color 0 and call this partial coloring  $\tau$ .

If  $0 \notin A_{\tau}(u_2 z_2)$ , then there exists some edge  $z_2 z'_2$  with  $\tau(z_2 z'_2) = 0$ , where  $z'_2 \in N(z_2) \setminus \{u_2\}$ . Observe that  $|A^2_{\tau}(u_3 z_3)| \ge 1$ ,  $|A^2_{\tau}(u_2 z_2)| \ge 2$ ,  $|A^2_{\tau}(wv)| \ge 2$ ,  $|A^2_{\tau}(uu_1)| \ge 4$ ,  $|A^2_{\tau}(uu_2)| \ge 5$  and  $|A^2_{\tau}(uv)| \ge 6$ . Hence, we can finish  $\tau$  by sequentially coloring  $u_3 z_3, u_2 z_2, wv, uu_1, uu_2, uv$ , a contradiction.

If  $0 \in A_{\varphi}(u_2z_2)$ , then we color  $u_2z_2$  with color 0. Observe that  $|A_{\tau}^2(u_3z_3)| \ge 1$ ,  $|A_{\tau}^2(wv)| \ge 2$ ,  $|A_{\tau}^2(uu_1)| \ge 4$ ,  $|A_{\tau}^2(uu_2)| \ge 4$  and  $|A_{\tau}^2(uv)| \ge 6$ . Hence, we can finish  $\tau$  by sequentially coloring  $u_3z_3$ , wv,  $uu_1$ ,  $uu_2$ , uv with different 2-colors, a contradiction.

Subcase 2.2.  $0 \notin \bigcup_{1 \le i \le 3} A_{\phi}(u_i z_i)$ . It is followed that for each  $1 \le i \le 3$ , there exists some edge  $z_i z'_i$  with  $\varphi(z_i z'_i) = 0$ , where  $z'_i \in N(z_i) \setminus \{u_i\}$ . Hence,  $|A^2_{\varphi}(vw)| \ge 2$ ,  $|A^2_{\varphi}(uv)| \ge 6$ ,  $|A^2_{\varphi}(uu_1)| \ge 5$ ,  $|A^2_{\varphi}(uu_2)| \ge 5$  and  $|A^2_{\varphi}(u_i z_i)| \ge 2$  for  $1 \le i \le 3$ . Let  $T = \{u_1 z_1, u_2 z_2, u_3 z_3, wv\}$ . Note that if any two edges  $e_1$  and  $e_2$  in T have  $A^2_{\varphi}(e_1) \cap A^2_{\varphi}(e_2) = \emptyset$ , then  $|\bigcup_{1 \le i \le 3} A^2_{\varphi}(u_i z_i) \cup A^2_{\varphi}(wv)| \ge 8$ , a contradiction. Therefore, there must exist at least two edges of T, say  $u_1 z_1$  and wv, that satisfy  $A^2_{\varphi}(u_1 z_1) \cap A^2_{\varphi}(wv) \ne \emptyset$ . We can first color  $u_1 z_1$  and wv with the same color in  $A^2_{\varphi}(u_1 z_1) \cap A^2_{\varphi}(wv)$  and then finish  $\varphi$  by sequentially coloring  $u_2 z_2, u_3 z_3, uu_1, uu_2, uv$ , a contradiction.

Since  $ew(H) \leq 6$  and  $\delta(H) \geq 2$ , then  $\Delta(H) = 4$ . Let v be a 4-vertex in H. Then each neighbor of v is a 2-vertex as  $ew(H) \leq 6$  and  $\delta(H) \geq 2$ . Let w be a 2-vertex in N(v). Then by Claims 13 and 16, each neighbor of w is a 4-vertex. It follows that H is a bipartite graph with one vertex part consisting of 2-vertices and the other vertex part consisting of 4-vertices. By Lemma 10, there exists a  $2^8$ -coloring  $\phi$  of H. By replacing a 2-color of  $\phi$  to a 0-color, then we obtain a  $(1, 2^7)$ -coloring of H, a contradiction.

The proof is complete.

#### 4. Proof of Theorem 8

Assume that Theorem 8 fails, and H is a counterexample with |V(H)| + |E(H)| as small as possible. By the choice of H, H is a connected fork-free multigraph with  $ew(H) \leq 6$  and any proper subgraph of H has a  $(1, 2^6)$ -coloring.

Claim 17. *H* is simple.

**Proof.** Suppose that there are k multiple edges  $e_1, e_2, \ldots, e_k$  between two vertices u and v in H, where  $k \ge 2$ . Since  $ew(H) \le 6$ ,  $k \le 3$ . If k = 3, then H is the graph with 2 vertices and 3 edges. Obviously, H is  $(1, 2^6)$ -colorable, a contradiction. Hence, k = 2. Let  $H_1 = H - e_1$ . Then  $H_1$  has a  $(1, 2^6)$ -coloring  $\varphi$  by the minimality of H. Assume  $d(u) \le d(v)$ , then  $2 \le d(u) \le 3$  as  $ew(H) \le 6$ .

Case 1. d(u) = 2. Then  $d(v) \leq 4$  as  $ew(H) \leq 6$ . If d(v) = 4, denote  $N(v) = \{v_1, v_2, u\}$ . Then  $d(v_1) \leq 2$  and  $d(v_2) \leq 2$  as  $ew(H) \leq 6$ . Hence, there are at most 5 edges in  $N''(e_1)$ . If  $d(v) \leq 3$ , then there are at most 4 edges in  $N''(e_1)$ . Thus, we always have  $|A^2_{\varphi}(e_1)| \geq 1$  and we can finish  $\varphi$  by coloring  $e_1$  with a 2-color in  $A^2_{\varphi}(e_1)$ , a contradiction.

Case 2. d(u) = 3. Then d(v) = 3 as  $ew(H) \leq 6$ . Denote  $N(v) = \{u, v_1\}$ ,  $N(u) = \{v, u_1\}$ . Then  $d(v_1) \leq 3$  and  $d(u_1) \leq 3$ . If  $v_1 = u_1$ , then  $|A_{\varphi}^2(e_1)| \geq 2$ . Hence, we can finish  $\varphi$  by coloring  $e_1$  with a 2-color in  $A_{\varphi}^2(e_1)$ , a contradiction. Thus,  $v_1 \neq u_1$ . If  $u_1v_1 \in E(H)$ , then there are at most 6 edges in  $N''(e_1)$ . Hence,  $|A_{\varphi}(e_1)| \geq 1$  and we can finish  $\varphi$  by coloring  $e_1$  with the color 0 or some 2-color in  $A_{\varphi}^2(e_1)$ , a contradiction. Therefore,  $u_1v_1 \notin E(H)$ . It follows that there are at most 7 edges in  $N''(e_1)$ . Denote by  $N'(u_1)$  and  $N'(v_1)$  the edges incident with  $u_1$  and  $v_1$ , respectively.

If there exists two edges  $e' \in N'(v_1)$  and  $e'' \in N'(u_1)$  satisfy  $\varphi(e') = \varphi(e'') = 0$ , then  $|A^2_{\varphi}(e_1)| \ge 1$  and we can finish  $\varphi$  by coloring  $e_1$  with a 2-color in  $A^2_{\varphi}(e_1)$ , a contradiction.

If there is only one edge  $e' \in N'(v_1) \cup N'(u_1)$  that satisfies  $\varphi(e') = 0$  (assume  $e' \in N'(v_1)$  by symmetry), then we first erase the color of  $e_2$  and  $uu_1$  under  $\varphi$ . Next, we recolor  $uu_1$  with color 0 and call this partial coloring  $\psi$ . Observe that  $|A^2_{\psi}(e_i)| \geq 2$  for each  $1 \leq i \leq 2$ . Hence, we can finish  $\psi$  by coloring  $e_1$  and  $e_2$  with different 2-colors, a contradiction.

If there is no edge  $e' \in N'(v_1) \cup N'(u_1)$  that satisfies  $\varphi(e') = 0$ , then we first erase the color of  $vv_1$ ,  $uu_1$  and  $e_2$  under  $\varphi$ . Next, we recolor both  $vv_1$  and  $uu_1$  with color 0 and call this partial coloring  $\phi$ . Observe that  $|A^2_{\phi}(e_i)| \geq 2$  for each  $1 \leq i \leq 2$ . Hence, we can also finish  $\phi$  by coloring  $e_1$  and  $e_2$  with different 2-colors, a contradiction.

Claim 18.  $\delta(H) \geq 2$ .

**Proof.** Suppose that v is a 1-vertex in H with a neighbor u. By the minimality of H,  $H_1 = H - v$  has a  $(1, 2^6)$ -coloring  $\varphi$ . Since  $ew(H) \leq 6$ , there are at most 6 edges in N''(uv). Hence,  $|A(uv)| \geq 1$ . If all the edges in N''(uv) are colored different 2-colors under  $\varphi$ , then we can finish  $\varphi$  by coloring vu with color 0. Otherwise, we can finish  $\varphi$  by coloring vu with a 2-color in  $A^2(uv)$ , which is a contradiction.

# Claim 19. $\Delta(H) \leq 3$ .

**Proof.** Since  $ew(H) \leq 6$ , we have  $\Delta(H) \leq 4$  by Claim 18. Suppose to the contrary that v is a 4-vertex with  $N(v) = \{v_1, v_2, v_3, v_4\}$ . Then  $d(v_i) = 2$  for  $1 \leq i \leq 4$  as  $ew(H) \leq 6$  and  $\delta(H) \geq 2$ . Note that H is fork-free, we consider the following two cases.

Case 1.  $v_1v_2 \in E(H)$ . Let  $H_1 = H - v_1$ . Then  $H_1$  has a  $(1, 2^6)$ -coloring  $\varphi$  by the minimality of H. Observe that  $|A^2(vv_1)| \ge 1$  and  $|A^2(v_1v_2)| \ge 3$ , hence we can finish  $\varphi$  by coloring  $vv_1$  and  $v_1v_2$  with different 2-colors, a contradiction.

Case 2.  $v_1v_2 \notin E(H)$ . By symmetry, we have  $v_iv_j \notin E(H)$  for  $1 \leq i \neq j \leq 4$ . Denote  $N(v_1) = \{v, w\}$ . Then w is adjacent to at least two vertices in  $N(v) \setminus \{v_1\}$ . For otherwise there is a fork induced by  $v, v_1, w$  and the two vertices in  $N(v) \setminus \{v_1\}$  not adjacent to w. By symmetry, we may assume  $\{v_2w, v_3w\} \subseteq E(H)$ . If  $v_4w \notin E(H)$ , then there is a fork induced by  $v, v_1, v_2, v_4$  and z, where  $z \in N(v_4) \setminus \{v\}$ , a contradiction. Hence,  $v_4w \in E(H)$ . Therefore,  $H \cong G_1$  (see Figure 3), and it can be seen that there is a  $(1, 2^6)$ -coloring of H, which is a contradiction.

**Claim 20.** If  $C_3 = v_1 v_2 v_3 v_1$  is a 3-cycle in H, then each  $v_i$  is a 3-vertex.

**Proof.** Suppose that  $v_1$  is not a 3-vertex. Then by Claims 18 and 19,  $v_1$  is a 2-vertex. Let  $H_1 = H - v_1$ . Then  $H_1$  has a  $(1, 2^6)$ -coloring  $\varphi$  by the minimality of H. By Claim 19,  $d(v_2) \leq 3$  and  $d(v_3) \leq 3$ . If  $d(v_2) = 2$ , then  $|A^2(v_1v_3)| \geq 2$  and  $|A^2(v_1v_2)| \geq 4$ , hence we can finish  $\varphi$  by sequentially coloring  $v_1v_3$  and  $v_1v_2$  with different 2-colors, a contradiction.

Therefore,  $d(v_2) = 3$ , and by the symmetry of  $v_2$  and  $v_3$ ,  $d(v_3) = 3$ . Observe that  $|A(v_1v_2)| \ge 2$  and  $|A(v_1v_3)| \ge 2$ . If no edges in  $N'(v_1v_2)$  are colored 0 under  $\varphi$ , then we can finish  $\varphi$  by coloring  $v_1v_2$  with color 0 and coloring  $v_1v_3$  with a 2-color in  $A^2(v_1v_3)$ , a contradiction. If an edge in  $N'(v_1v_2)$  is colored 0 under  $\varphi$ , then  $|A^2(v_1v_2)| \ge 2$  and  $|A^2(v_1v_3)| \ge 2$ . Hence, we can finish  $\varphi$  by coloring  $v_1v_2$  and  $v_1v_3$  with different 2-colors, which is also a contradiction.

# Claim 21. *H* has no adjacent 2-vertices.

**Proof.** Suppose that u and v are two adjacent vertices in H. Denote by  $N(u) = \{u_1, v\}$  and  $N(v) = \{u, v_1\}$ . By Claim 20,  $u_1 \neq v_1$ , and by Claim 19,  $d(u_1) \leq 3$ ,  $d(v_1) \leq 3$ . Let  $H_1$  be the graph obtained from H by contracting uv. Obviously,

 $ew(H_1) \leq 6$  and  $H_1$  is simple and fork-free. Hence,  $H_1$  has a  $(1, 2^6)$ -coloring  $\varphi$  by the minimality of H. If  $uu_1$  and  $vv_1$  are not colored 0 under  $\varphi$ , then we can finish  $\varphi$  by coloring uv with color 0, a contradiction. Otherwise, we can extend  $\varphi$  to H by coloring uv with a 2-color in  $A^2(uv)$  as there are at most 6 edges in N''(uv), which is also a contradiction.

### Claim 22. *H* is claw-free.

**Proof.** Suppose to the contrary that there is a claw in H. By Claim 19, we may assume that the claw is induced by  $\{v\} \cup N(v)$ , where  $N(v) = \{v_1, v_2, v_3\}$ . By Claims 18–19,  $2 \leq d(v_i) \leq 3$  for  $1 \leq i \leq 3$ .

Case 1.  $d(v_1) = 2$ . Let  $N(v_1) = \{v, w\}$ . By Claim 21, d(w) = 3. Since H is fork-free,  $wv_j \in E(H)$  for some  $2 \leq j \leq 3$ . We assume  $wv_3 \in E(H)$  by symmetry. Denote  $N(w) = \{z, v_1, v_3\}$ . Then  $d(z) \leq 3$  by Claim 19.

Subcase 1.1.  $d(v_3) = 2$ . Let  $H_1 = H - \{v_1, v_3\}$ . Then  $H_1$  has a  $(1, 2^6)$ -coloring  $\varphi$  by the minimality of H. If  $\varphi(vv_2) = 0$ , then  $|A_{\varphi}(wv_1)| \ge 4$ ,  $|A_{\varphi}(wv_3)| \ge 4$ ,  $|A_{\varphi}(vv_1)| \ge 3$  and  $|A_{\varphi}(vv_3)| \ge 3$ . Hence, we can finish  $\varphi$  by sequentially coloring  $vv_1, vv_3, wv_1$  and  $wv_3$ , a contradiction. Therefore,  $\varphi(vv_2) \ne 0$ . By symmetry of  $vv_2$  and  $wz, \varphi(wz) \ne 0$ . Then we first color  $vv_1$  and  $wv_3$  with color 0 and call this partial coloring  $\tau$ . Observe that  $|A_{\tau}^2(vv_3)| \ge 2$  and  $|A_{\tau}^2(wv_1)| \ge 2$ . Thus, we can finish  $\tau$  by sequentially coloring  $v_1w, vv_3$  with different 2-colors, a contradiction.

Subcase 1.2.  $d(v_3) = 3$ . Denote  $N(v_3) = \{v, w, z_1\}$ . Then  $z_1v_2 \in E(H)$ , for otherwise there is a fork induced by  $\{v, z_1\} \cup N(v)$ . Recall that d(w) = 3 and  $N(w) = \{z, v_1, v_3\}$ .

Subcase 1.2.1.  $z = v_2$ . If  $d(z_1) = 3$ , then there is a fork induced by  $\{v, v_3, w, z_1, z'_1\}$ , where  $z'_1 \in N(z_1) \setminus \{v_3, v_2\}$ . Hence  $d(z_1) = 2$ . It follows that  $H \cong G_2$  (see Figure 3), and it can be seen that there is a  $(1, 2^6)$ -coloring of  $G_2$ , a contradiction.

Subcase 1.2.2.  $z = z_1$ . If  $d(v_2) = 3$ , then there is a fork induced by  $\{v'_2, v_2, v, z_1, w\}$ , where  $v'_2 \in N(v_2) \setminus \{v, z_1\}$ . Hence  $d(v_2) = 2$ . It follows that  $H \cong G_3$  (see Figure 3), and it can be seen that there is a  $(1, 2^6)$ -coloring of  $G_3$ , a contradiction.

Subcase 1.2.3.  $z \notin \{v_2, z_1\}$ . Then  $zz_1 \in E(H)$ , otherwise there is a fork induced by  $\{v, v_3, z_1, w, z\}$ . If d(z) = 3, we can find a fork induced by  $\{v_1, v_3, w, z, z'\}$ , where  $z' \in N(z) \setminus \{w, z_1\}$ , a contradiction. If  $d(v_2) = 3$ , we can find a fork induced by  $\{v_1, v_3, v, v_2, v'_2\}$ , where  $v'_2 \in N(v_2) \setminus \{v, z_1\}$ , a contradiction. Hence,  $d(z) = d(v_2) = 2$  and  $H \cong G_4$  (see Figure 3), and it can be seen that there is a  $(1, 2^6)$ -coloring of  $G_4$ , a contradiction.

Case 2.  $d(v_1) = 3$ . By symmetry of  $v_1, v_2$  and  $v_3, d(v_2) = d(v_3) = 3$ . Denote  $N(v_1) = \{u_1, u_2\}$ . Then  $u_2$  must be adjacent to  $v_2$  or  $v_3$ , for otherwise there is

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a fork induced by  $N(v) \cup \{v, u_2\}$ . We may assume  $u_2v_3 \in E(H)$  by symmetry. Similarly,  $u_1$  must be adjacent to  $v_2$  or  $v_3$ .



Figure 3. The graphs  $G_1$ - $G_{10}$  and their corresponding  $(1, 2^6)$ -coloring, where the red color represents color 0.

Subcase 2.1.  $u_1v_3 \in E(H)$ . Since  $d(v_2) = 3$ , we have  $v_2u_1 \in E(H)$  and  $v_2u_2 \in E(H)$ . For otherwise there is a fork induced by  $N(v) \cup \{v, v'_2\}$ , where  $v'_2 \in N(v_2) \setminus \{v, u_1, u_2\}$ . Therefore,  $H \cong G_5$  (see Figure 3), and it can be seen that there is a  $(1, 2^6)$ -coloring of  $G_5$ , a contradiction.

Subcase 2.2.  $u_1v_3 \notin E(H)$ . Then  $u_1v_2 \in E(H)$ . Since H has no adjacent 2-vertices (by Claim 21), if both  $u_1, u_2, v_2$  and  $v_3$  have no other neighbors except the vertices in  $N(v) \cup N(v_1)$ , then H is isomorphic to one of  $G_5 - G_7$  (see Figure 3), and it can be seen that each  $G_i$  is  $(1, 2^6)$ -colorable for  $5 \leq i \leq 7$ , a contradiction. Hence, there is a vertex in  $\{u_1, u_2, v_2, v_3\}$ , say  $v_2$ , that has a neighbor  $v'_2 \notin$  $N(v) \cup N(v_1)$ . Then  $v'_2v_3 \in E(H)$ , for otherwise there is a fork induced by  $N(v) \cup \{v, v'_2\}$ .

If  $v'_2u_1 \in E(H)$ , then  $d(u_2) = 2$  (as if  $d(u_2) = 3$ , there is a fork induced by  $N(u_2) \cup \{u_2, v'_2\}$ ). Hence,  $H \cong G_8$  (see Figure 3), and it can be seen that there is a  $(1, 2^6)$ -coloring of  $G_8$ , a contradiction.

Therefore,  $v'_2u_1 \notin E(H)$ . By symmetry of  $u_1$  and  $u_2$ ,  $v'_2u_2 \notin E(H)$ . If  $d(v'_2) = 3$ , then  $v''_2u_1 \in E(H)$  and  $v''_2u_2 \in E(H)$ , where  $v''_2 \in N(v'_2) \setminus \{v_2, v_3\}$ . For otherwise there is a fork induced by  $N(v'_2) \cup \{v'_2, u_1\}$  or  $N(v'_2) \cup \{v'_2, u_2\}$ . Hence,  $H \cong G_9$  (see Figure 3), and there is a  $(1, 2^6)$ -coloring of  $G_9$ , a contradiction. Thus,  $d(v'_2) = 2$ . By symmetry of  $v'_2, u_1$  and  $u_2, d(u_1) = d(u_2) = 2$ . Then  $H \cong G_{10}$  (see Figure 3), and it can be seen that there is a  $(1, 2^6)$ -coloring of  $G_{10}$ , a contradiction.

Claim 23. *H* has no two 3-cycles share one common edge.

**Proof.** Suppose that  $v_1v_2v_3v_1$  and  $v_2v_3v_4v_2$  are two 3-cycles in H with common edge  $v_2v_3$ . By Claim 20,  $d(v_i) = 3$  for  $1 \le i \le 4$ . Note that  $v_1v_4 \notin E(H)$ , otherwise  $H \cong K_4$  and hence H is  $(1, 2^6)$ -colorable, a contradiction. Denote  $N(v_1) = \{w_1, v_2, v_3\}$  and  $N(v_4) = \{w_2, v_2, v_3\}$ . By Claim 19,  $d(w_j) \le 3$  for  $1 \le j \le 2$ . Let  $H_1 = H - \{v_1, v_2, v_3, v_4\}$ . Then  $H_1$  has a  $(1, 2^6)$ -coloring  $\varphi$  by the minimality of H. We first color  $v_1v_3$  and  $v_2v_4$  with color 0 and call this partial coloring  $\psi$ .

Case 1.  $w_1 = w_2$ . Observe that  $|A_{\psi}^2(v_1w_1)| \ge 3$ ,  $|A_{\psi}^2(v_4w_1)| \ge 3$ ,  $|A_{\psi}^2(v_1v_2)| \ge 5$ ,  $|A_{\psi}^2(v_3v_4)| \ge 5$  and  $|A_{\psi}^2(v_2v_3)| \ge 6$ . Hence, we can color  $v_1w_1, v_4w_1, v_1v_2, v_3v_4, v_2v_3$  with different 2-colors in order to finish  $\psi$ , a contradiction.

Case 2.  $w_1 \neq w_2$ . Since  $w_1$  and  $w_2$  may be adjacent in H, we consider the following two subcases.

Subcase 2.1.  $w_1w_2 \notin E(H)$ . Since H is claw-free (by Claim 22), we can observe that there are at most five edges in  $N''(v_jw_j)$  that are colored 2-colors for  $1 \leq j \leq 2$ . Thus,  $|A_{\psi}^2(v_1w_1)| \geq 1$  and  $|A_{\psi}^2(v_4w_2)| \geq 1$ . Note that  $|A_{\psi}^2(v_1v_2)| \geq 4$ ,  $|A_{\psi}^2(v_3v_4)| \geq 4$  and  $|A_{\psi}^2(v_2v_3)| \geq 6$ . Therefore, we can finish  $\psi$  by sequentially coloring  $v_1w_1, v_4w_2, v_1v_2, v_3v_4, v_2v_3$  with different 2-colors, a contradiction.

Subcase 2.2.  $w_1w_2 \in E(H)$ . Since H has no adjacent 2-vertices (by Claim 21), we may assume  $d(w_2) = 3$ . Let  $z \in N(w_2) \setminus \{v_4, w_1\}$ . Then  $zw_1 \in E(H)$ , otherwise, there is a claw induced by  $N(w_2) \cup \{w_2\}$ , which contradicts to Claim 22. Hence,  $|A_{\psi}^2(v_1w_1)| \geq 2$ ,  $|A_{\psi}^2(v_4w_2)| \geq 2$ ,  $|A_{\psi}^2(v_1v_2)| \geq 4$ ,  $|A_{\psi}^2(v_3v_4)| \geq 4$  and  $|A_{\psi}^2(v_2v_3)| \geq 6$ . Therefore, we can color  $v_1w_1, v_4w_2, v_1v_2, v_3v_4, v_2v_3$  with different 2-colors in order to finish  $\psi$ , a contradiction.



Figure 4. The edge coloring sequence of H.

By Claims 18–19,  $2 \leq \delta(H) \leq \Delta(H) \leq 3$ . If  $\Delta(H) = 2$ , then H is a cycle. Hence, H is  $(1, 2^6)$ -colorable, a contradiction. Therefore,  $\Delta(H) = 3$ . Let  $v_1$  be a 3-vertex in H with three neighbors  $v_2, v_3$  and  $w_1$ . Since H is claw-free (by Claim On S-Packing Edge-Coloring of Graphs with Given Edge Weight 15

22), we may assume  $v_2v_3 \in E(H)$ . Then by Claim 20,  $d(v_2) = d(v_3) = 3$ . Let  $w_j$  be the neighbor of  $v_j$  that is not in the cycle  $C = v_1v_2v_3v_1$  for  $2 \leq j \leq 3$ . Then  $w_1 \neq w_2 \neq w_3$  by Claim 23. For a vertex  $x \in V(H) \setminus V(C)$ , the distance between x and the cycle C is denoted by  $d(C, x) = \min\{d(x, v_i)|1 \leq i \leq 3\}$ . For an edge  $e = uv \in E(H) \setminus E(C)$ , the distance between e and C is denoted by d(C, e) = d(C, u) + d(C, v). In the below, we will give a  $(1, 2^6)$ -coloring of H in four steps.

**Step 1.** Color  $v_1w_1$ ,  $v_2w_2$  and  $v_3w_3$  with the same color 0.

**Step 2.** Color the edges in  $E(H) \setminus E(C)$  except the edges incident with the vertices in  $\bigcup_{1 \le i \le 3} N(w_i)$  according to their distance from the cycle C from far to near. That is, for two uncolored edges  $e_1$  and  $e_2$ , if  $d(C, e_1) > d(C, e_2)$ , then we color  $e_1$  before  $e_2$ ; if  $d(C, e_1) = d(C, e_2)$ , then we randomly pick one of them to be colored first.

Denote by S the edge coloring sequence. Next, we will show that every edge in S can be colored. Let xy be an edge in S. Clearly, if there are at most 6 edges in N''(xy) colored before xy, then we can color xy with a 2-color or a 0-color. To illustrate this, we discuss it in the following two cases.

Case 1. d(C,x) = d(C,y). Then there are vertices  $x_1 \in N(x) \setminus \{y\}$  and  $y_1 \in N(y) \setminus \{x\}$  that satisfy  $d(C,x_1) < d(C,x)$  and  $d(C,y_1) < d(C,y)$ . For any edge  $x_1x_2$  incident with  $x_1$ , since  $d(C,x_2) \leq d(C,x)$ ,  $d(C,x_1x_2) = d(C,x_1) + d(C,x_2) < 2d(C,x) = d(C,x) + d(C,y) = d(C,xy)$ . Similarly, for any edge  $y_1y_2$  incident with  $y_1$ ,  $d(C,y_1y_2) < d(C,xy)$ . Hence, the edges incident with  $x_1$  and  $y_1$  are colored after xy in S, as they are closer to C than xy. Note that  $\Delta(H) \leq 3$  (by Claim 19), thus there are at most 6 edges in N''(xy) that are colored before xy.

Case 2. d(C, x) > d(C, y). Then there is at least one vertex  $y_1 \in N(y) \setminus \{x\}$ that satisfy  $d(C, y_1) < d(C, y)$ , and hence the edges incident with  $y_1$  are colored after xy in S. When d(y) = 2, obviously there are at most 6 edges in N''(xy)that are colored before xy since  $\Delta(H) \leq 3$ . Hence, we only need to consider d(y) = 3. Denote  $y_2 \in N(y) \setminus \{x, y_1\}$ . Since H is claw-free (by Claim 22),  $y_2$  is adjacent to  $y_1$  or x. If  $y_2$  is adjacent to x, then there are at most 6 edges that are colored before xy in S (three edges incident with  $y_2$  and three edges incident with the vertex  $t \in N(x) \setminus \{y, y_2\}$ ). Thus, we may assume  $y_2$  is adjacent to  $y_1$ . Then  $d(C, y_2) \leq d(C, y) < d(C, x)$ , which implies  $yy_2$  and the edges incident with  $y_1$  are colored after xy in S. Note that there are at most five edges incident with the vertices in  $N(x) \setminus \{y\}$ , as H is claw-free and  $\Delta(H) \leq 3$ . Therefore, there are at most 6 edges in N''(xy) that may be colored before xy in S (one edge incident with  $y_2$  and five edges incident with the vertices in  $N(x) \setminus \{y\}$ ).

**Step 3.** Color the edges incident with the vertices in  $\bigcup_{1 \le i \le 3} N(w_i) \setminus \{v_i\}$ .

Based on the symmetry of  $w_1, w_2$  and  $w_3$ , it is clear that if the edges incident with the vertices in  $N(w_1) \setminus \{v_1\}$  can be colored, then the edges incident with the vertices in  $\bigcup_{2 \le j \le 3} N(w_j) \setminus \{v_j\}$  can also be colored. Next, we will first color the edges zt, where  $z \in N(w_1) \setminus \{v_1\}$  and  $t \in N(z) \setminus \{w_1\}$ , then color the edges  $w_1z$ .

Case 1.  $d(w_1) = 2$ . Denote  $N(w_1) = \{v_1, z_1\}$ . Since H has no adjacent 2-vertices (by Claim 21),  $d(z_1) = 3$ . Denote  $N(z_1) = \{z_{11}, z_{12}, w_1\}$ . Since H is claw-free,  $z_{11}z_{12} \in E(H)$ . Now we color  $z_1z_{11}, z_1z_{12}, z_1w_1$  in order. Note that since  $\Delta(H) \leq 3$ , there are at most 6 colored edges in  $N''(z_1z_{11})$  including  $v_1w_1$ . Hence, we can color  $z_1z_{11}$  with a 2-color. Then there are at most 7 colored edges in  $N''(z_1z_{12})$  including  $v_1w_1$ . If some edge incident with  $z_{12}$  is colored with 0, then we can color  $z_1z_{12}$  with a 2-color, otherwise we can color  $z_1z_{12}$  with color 0. Finally, for the edge  $z_1w_1$ , it can be seen that there are at most 6 colored edges in  $N''(z_1w_1)$  including  $v_1w_1$ , hence we can color it with a 2-color.

Case 2.  $d(w_1) = 3$ . Denote  $N(w_1) = \{v_1, z_1, z_2\}$ . Then  $z_1 z_2 \in E(H)$  as H is claw-free. Since H has no 2-vertices in 3-cycle (by Claim 20),  $d(z_1) = d(z_2) = 3$ . Let  $z_{11}$  and  $z_{21}$  be the neighbors of  $z_1$  and  $z_2$  not in the 3-cycle  $z_1 z_2 w_1$  respectively (see Figure 4). Next, we color  $z_1z_{11}, z_2z_{21}, z_1z_2, z_1w_1$  and  $z_2w_1$  in order. Note that there are at most 6 colored edges in  $N''(z_1z_{11})$  including  $v_1w_1$ . Hence, we can color  $z_1 z_{11}$  with a 2-color. Then there are at most 7 colored edges in  $N''(z_2 z_{21})$ . If some edge incident with  $z_{21}$  is colored with 0, then we can color  $z_2 z_{21}$  with a 2-color, otherwise we can color  $z_2 z_{21}$  with color 0. For the edge  $z_1 z_2$ , there are also at most 7 colored edges in  $N''(z_1z_2)$  including  $v_1w_1$ . If  $z_1z_{11}$  or  $z_2z_{21}$  is colored with 0, then we can color  $z_1 z_2$  with a 2-color, otherwise we can color  $z_1 z_2$ with color 0. Now for the edge  $z_1 w_1$ , observe that there are at most 6 colored edges in  $N''(z_1w_1)$  including  $v_1w_1$ , hence we can color it with a 2-color. Finally, for the edge  $z_2w_1$ , there are at most 7 colored edges in  $N''(z_2w_1)$ . The only case in which  $z_2w_1$  cannot be colored is when all the colored edges in  $N''(z_2w_1)$ , except  $w_1v_1$ , are colored with different 2-colors. In this case, we can erase the 2-color, say  $\alpha$ , of  $z_1 z_2$ . Then color  $z_1 z_2$  and  $z_2 w_1$  with color 0 and  $\alpha$ , respectively.

**Step 4.** Color the edges in E(C).

Denote by  $\varphi$  the coloring of H after steps 1–3. Observe that  $|A_{\varphi}^2(v_iv_j)| \geq 2$ for  $1 \leq i \neq j \leq 3$ . If  $w_1w_2 \in E(H)$  or there are two colored edges in  $N'(v_1w_1) \cup$  $N'(v_2w_2)$  that are colored with the same 2-color, then  $|A_{\varphi}^2(v_1v_2)| \geq 3$ . Hence, we can finish  $\varphi$  by coloring  $v_1v_3, v_2v_3, v_1v_2$  in order, a contradiction. Therefore, we have  $w_iw_j \notin E(H)$  for  $1 \leq i \neq j \leq 3$  by symmetry, and all the six colored edges in  $\bigcup_{1 \leq i \leq 3} N'(v_iw_i)$  are colored with different 2-colors. In this case, we can color each  $v_iv_j$  with the same 2-color of a colored edge in  $N'(v_tw_t)$ , where  $1 \leq i \neq j \leq 3$ and  $t \in \{1, 2, 3\} \setminus \{i, j\}$ , to obtain a  $(1, 2^6)$ -coloring of H, a contradiction.

The proof is complete.

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#### References

- J.-L. Fouquet and J.-M. Vanherpe, On parsimonious edge-colouring of graphs with maximum degree three, Graphs Combin. 29 (2013) 475–487. https://doi.org/10.1007/s00373-012-1145-3
- [2] J.-L. Fouquet and J.L. Jolivet, Strong edge-colorings of graphs and applications to multi-k-gons, Ars Combin. 16A (1983) 141–150.
- [3] N. Gastineau and O. Togni, On S-packing edge-colorings of cubic graphs, Discrete Appl. Math. 259 (2019) 63–75. https://doi.org/10.1016/j.dam.2018.12.035
- W. Goddard and H. Xu, The S-packing chromatic number of a graph, Discuss. Math. Graph Theory 32 (2012) 795–806. https://doi.org/10.7151/dmgt.1642
- [5] P. Hall, On representatives of subsetes, J. Lond. Math. Soc. 10 (1935) 26–30. https://doi.org/10.1112/jlms/s1-10.37.26
- H. Hocquard, D. Lajou and B. Lužar, Between proper and strong edge-colorings of subcubic graphs, J. Graph Theory 101 (2022) 686–716. https://doi.org/10.1002/jgt.22848
- [7] X.J. Liu, M. Santana and T. Short, Every subcubic multigraph is (1,2<sup>7</sup>)-packing edge-colorable, J. Graph Theory 104 (2023) 851–885. https://doi.org/10.1002/jgt.23004
- [8] X. J. Liu and G. Yu, On the (1<sup>2</sup>, 2<sup>4</sup>)-packing edge-coloring of subcubic graphs (2024). arXiv:2402.18353v1
- K. Nakprasit, A note on the strong chromatic index of bipartite graphs, Discrete Math. 308 (2008) 3726-3728. https://doi.org/10.1016/j.disc.2007.07.034
- [10] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz. 3 (1964) 25–30, in Russian.
- W. Yang and B. Wu, On S-packing edge-colorings of graphs with small edge weight, Appl. Math. Comput. 418 (2022) 126840. https://doi.org/10.1016/j.amc.2021.126840

J. LU AND X.-F. PAN

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